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# A NOTE ON $p$-BOUNDED AND QUASI-$p$-BOUNDED SUBSETS

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Abstract. We discuss the relationship between $p$-boundedness and quasi-$p$-boundedness in the realm of GLOTS for $p \in \omega^*$. We show that $p$-pseudocompactness, $p$-compactness, quasi-$p$-pseudocompactness and quasi-$p$-compactness are equivalent properties for a GLOTS; that bounded subsets of a GLOTS are strongly-bounded; and $C$-compact subsets of a GLOTS are strongly-$C$-compact. We also show that a topologically orderable group is locally precompact if and only if it is metrizable. For bounded subsets of a GLOTS, a version of the classical Gilcksberg's Theorem on pseudocompactness is obtained: if $A_\alpha$ is a bounded subset of a GLOTS $X_\alpha$ for each $\alpha \in \Delta$, then $\mathrm{cl}_{\beta(\prod_{\alpha\in\Delta} X_\alpha)}(\prod_{\alpha\in\Delta} A_\alpha) = \prod_{\alpha\in\Delta} \mathrm{cl}_{\beta(X_\alpha)} A_\alpha$. Also we prove that there exists an ultrapseudocompact topological group which is not quasi-$p$-compact for any $p \in \omega^*$. To see this example, $p$-pseudocompactness and $p$-compactness are investigated in the field of $C_\pi$-spaces, proving that ultracompactness, quasi-$p$-compactness for a $p \in \omega^*$ and countable compactness (respectively, ultrapseudocompactness, quasi-$p$-pseudocompactness for a $p \in \omega^*$ and pseudocompactness) are equivalent properties in the class of spaces of the form $C_\pi(X, [0,1])$.

## 1. Introduction

In this article we will assume that all spaces are Tychonoff unless otherwise stated. The set of natural numbers will be denoted by $\omega$, and the Stone-Čech compactification of a space $X$ will be denoted as $\beta(X)$. The space $\beta(\omega)$ is identified with the set of ultrafilters on $\omega$, and $\omega^* = \beta(\omega) \setminus \omega$ is the set of free ultrafilters.

For $p \in \omega^*$, Bernstein [B] introduced and investigated the concept of $p$-limit in connection with some problems in the theory of nonstandard analysis. Independently, Frólik [F] and Katětov [K1], [K2] introduced this concept in a different form, and Ginsburg and Saks [GS] generalized this notion as follows:

**1.1. Definition.** Let $p \in \omega^*$ and let $(S_n)_{n<\omega}$ be a sequence of nonempty subsets of a space $X$. A point $x \in X$ is a $p$-limit point of the sequence $(S_n)_{n<\omega}$, in symbols $x = p - lim(S_n)$, if for every $V \in \mathcal{N}(x)$, $\{n < \omega : V \cap S_n \neq \emptyset\} \in p$.

If $x_n \in X$ and $S_n = \{x_n\}$ for each $n < \omega$, then a $p$-limit point of $(S_n)_{n<\omega}$ is a Bernstein's $p$-limit point of the sequence $(x_n)_{n<\omega}$. Note that if there exists a $p$-limit point of a sequence $(x_n)_{n<\omega}$, this has to be unique, since $X$ is Hausdorff;

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but, in general, a sequence $(S_n)_{n<\omega}$ of nonempty subsets of a space $X$ could have more than one point. For instance, if $S_n = \{\frac{1}{n}\} \times \mathbb{R}$ for each $n < \omega$, then each point $(0, r) \in \mathbb{R}^2$ is a $p$-limit point of $(S_n)_{n<\omega}$, for each $p \in \omega^*$.

Let us recall that, for $p \in \omega^*$, a space $X$ is said to be *$p$-compact* if every sequence $(x_n)_{n<\omega}$ in $X$ has a $p$-limit. Spaces which are $p$-compact for every $p \in \omega^*$ are called *ultracompact*. Every compact space is ultracompact, and Vaughan proved in [V, Theorem 4.9], that in the class of regular spaces, $X$ is ultracompact if, and only if, $X$ is *$\omega$-bounded* (that is, the closure of each countable subset of $X$ is compact). By using Definition 1.1, Ginsburg and Saks introduced in [GS] the concept of p-pseudocompactness, and later García-Ferreira [GF1] defined the relative version of this concept:

**1.2. Definition.** Let $p \in \omega^*$. A subet $A$ of a space $X$ is said to be *$p$-bounded* in $X$ if for every sequence $(V_n)_{n<\omega}$ of nonempty open subsets of $X$ with $A \cap V_n \neq \emptyset$, for all $n < \omega$, there is $x \in X$ which is a $p$-limit point of the sequence $(V_n)_{n<\omega}$.

A space $X$ is called *$p$-pseudocompact* if $X$ is $p$-bounded in itself and $X$ is called *ultrapseudocompact* if it is $p$-pseudocompact for every $p \in \omega^*$ .

In this article, we are not going to engage only in $p$-bounded spaces but also in a weaker concept which has been introduced and analyzed in [STM2]:

**1.3. Definition.** Let $p \in \omega^*$ and let $A$ be a subset of a space $X$. We say that $A$ is a *quasi-$p$-bounded* subset of $X$ if for every sequence $(U_n)_{n<\omega}$ of nonempty open subsets of $X$ whose elements meet $A$, there exist $x \in X$ and $f : \omega \to \omega$ such that $|f(B)| = \aleph_0$ for every $B \in p$ and $x = p - lim(U_{f(n)})$.

It is worth to mention that quasi-$p$-boundedness was defined in [STM2] by using the word "subsequence" which produces confusion since subsequence requires strictly increasing functions. Here we present this notion in such a way a missunderstanding is not possible.

Given $p \in \omega^*$, a space $X$ is called *quasi-$p$-pseudocompact* if it is quasi-$p$-bounded in itself. In a similar way, we say that a space $X$ is *quasi-$p$-compact* if for each infinite sequence $(x_n)_{n<\omega}$ of points in $X$, there exist $x \in X$ and $f : \omega \to \omega$ such that $|f(B)| = \aleph_0$ for every $B \in p$ and $x = p - lim(x_{f(n)})$. This notion is implicit in [STM2, Theorem 2.8].

The concepts of quasi-$p$-compactness and quasi-$p$-pseudocompactness coincide with those of quasi-$M$-compactness and $M$-pseudocompactness, respectively, introduced in [GF2] and [GF1], where $M = P_{RK}(p) \setminus \omega$ and $P_{RK}(p)$ is the set of Rudin-Keisler predessesors of $p$. We state these assertions, in a more formal way, in the following theorem (the proof is left to the reader):

**1.4. Theorem.** *Let $p \in \omega^*$, let $X$ be a space and $A \subset X$. Then $A$ is quasi-$p$-bounded in $X$ (resp., quasi-$p$-compact) if and only if for every infinite sequence $(U_n)_{n<\omega}$ (resp., $(x_n)_{n<\omega}$) of nonempty open subsets of $X$ whose elements meet $A$ (resp., of points in $A$), there are $x \in X$ (resp., $x \in A$) and $r \in P_{RK}(p) \setminus \omega$ such that $x = r - lim(U_n)$ (resp., $x = r - lim(x_n)$).*

At any rate, theorems 2.2, 3.2, 3.4, 3,5 and 3.8 and Corollary 2.3, below, remain true if we replace "quasi-$p$-compact", "quasi-$p$-pseudocompact" and "quasi-$p$-bounded" by "quasi-$M$-compact", "$M$-pseudocompact" and "$M$-bounded" (the definition of this last concept must be clear), respectively, for a $M \subset \omega^*$ such that $p \in M \subset P_{RK}(p)$.

The properties $p$-pseudocompactness and ultrapseudocompactness are productive and preserved by continuous functions (see [GF1], [GS]). Besides, if $A$ is a regular closed subset of a $p$-pseudocompact space $X$, then $A$ itself is $p$-pseudocompact; and if $A$ is $p$-bounded in $X$, then $cl_X A$ is $p$-bounded in $X$ too. The property of being $p$-bounded is monotone with respect to the Rudin-Keisler pre-order, that is, if $q \leq_{RK} p$ and $A$ is $p$-bounded in $X$, then $A$ is $q$-bounded in $X$ (see [GF1] for details). Some of these properties remain true for quasi-$p$-pseudocompact spaces (see [STM2, Lemma 2.4]) and some of them do not hold (see [STM2, Example 3.8]). It is well known that a subset $A$ of a space $X$ is *bounded* (in $X$) if every sequence $(U_n)_{n<\omega}$ of open subsets of $X$ meeting $A$ admits a cluster point. So, if $A$ is $p$-bounded for some $p \in \omega^*$, then $A$ is quasi-$p$-bounded, and this implies that $A$ is bounded (in general, none of these implications can be reversed, see [STM2]). In particular, $p$-pseudocompactness implies quasi-$p$-pseudocompactness and this implies pseudocompactness.

In this article, we discuss the notions of boundedness in GLOTS and $C_\pi$-spaces. In the first section, we study the relationship among boundedness, (quasi)-$p$-boundedness, (quasi)-$p$-pseudocompactness and (quasi)-$p$-compactness in GLOTS, and a version of the classical Glicksberg's Theorem on pseudocompactness is given in the realm of GLOTS. These results are applied in the realm of topologically ordered groups. In particular, we prove that locally precompact topologically ordered groups are metrizable. In the second section we prove that $\sigma$-$p$-pseudocompactness, $\sigma$-quasi-$p$-pseudocompactness and $\sigma$-pseudocompactness on the one hand, and $\sigma$-$p$-compactness, $\sigma$-quasi-$p$-compactness and $\sigma$-countable compactness, on the other, are equivalent in the class of $C_\pi$-spaces (Recall that for a topological property $\mathcal{P}$, a space $X$ is $\sigma$-$\mathcal{P}$ if $X$ is the union of a countable family of subspaces having $\mathcal{P}$). These results permit us to show that there exists an ultrapseudocompact topological group which is not quasi-$p$-compact for any $p \in \omega^*$. Our terminology and notation are standard. For example, $cl_X A$ stands for the closure of $A$ in $X$, $\mathbb{R}$ means for the real line endowed with the usual topology and a subset $A$ of a space $X$ is said to be $C^*$-*embedded* in $X$ if every bounded real-valued continuous function on $A$ admits a continuous extension to $X$.

## 2. Bounded subsets in GLOTS

A *Linearly Ordered Topological Space (LOTS)* is a space $(X, \tau)$ where the topology $\tau$ is generated by the initial and final segments, as a subbase, of a linear order $\leq$ defined on $X$. A *Generalized Linearly Ordered Topological Space (GLOTS)* is a space homeomorphic to some subspace of a LOTS. Let $(X, \leq)$ be a LOTS and let $x \in X$. A subset $K$ of $X$ is *left* (resp., *right*) *cofinal* at $x$ in $X$ if the set of elements strictly less than $x$ is not empty and for every $a \in X$ with $a < x$ (resp., the set of elements strictly bigger than $x$ is not empty, and for every $a > x$) there is $y \in K$ such that $a < y < x$ (resp., $x < y < a$). It is well known that every GLOTS is collectionwise normal (see [St]). On the other hand, if $X$ is a normal space, it is easy to check that $cl_X A$ is a pseudocompact space (so, countably compact) whenever $A$ is a bounded subset of $X$. The following theorem improves this result in the realm of GLOTS. First a lemma.

**2.1. Lemma.** *Let $X$ be a LOTS. Let $x \in X$ and let $F \subset X$ be a countable left (resp., right) cofinal at $x$ subset in $X$. Then, there exists a sequence $y_0 > y_1 > \cdots > y_n > \ldots$ (resp., $y_0 < y_1 < \cdots < y_n < \ldots$) of elements that belong to $F$ converging to $x$ in $X$.*

*Proof.* Assume that the set $F$ is countable left cofinal at $x$ in $X$. Enumerate $F$ as $F = \{z_n : 0 < n < \omega\}$. We can assume, without loss of generality, that $z_n < x$ for every $n < \omega$. Let $y_1 = z_1$; $y_2 = z_{n_2}$ where $n_2 =$ the first natural number such that $z_{n_2} > max\{y_1, z_2\}$. If we have chosen $y_1 < y_2 < \cdots < y_k$, let $n_{(k+1)}$ be the first natural number such that $z_{n_{(k+1)}} > max\{y_k, z_{k+1}\}$. Let $y_{k+1} = z_{n_{(k+1)}}$. We have that $y_1 < y_2 < \cdots < y_n < \ldots$ and $\sup\{y_n : 0 < n < \omega\} = x$. It is easy now to verify that $(y_n)_{n<\omega}$ converges to $x$ in $X$. The proof is complete. $\square$

**2.2. Theorem.** *Let $X$ be a GLOTS. For a subset $A$ of $X$, the following conditions are equivalent:*

(1) *$A$ is bounded in $X$;*
(2) *$cl_X A$ is sequentially compact;*
(3) *$cl_X A$ is ultracompact;*
(4) *$cl_X A$ is $p$-compact for some $p \in \omega^*$;*
(5) *$cl_X A$ is quasi-$p$-compact for some $p \in \omega^*$.*

*Proof.* (1) $\Longrightarrow$ (2) Since $A$ is bounded in $X$, $cl_X A$ is a countably compact GLOTS. Now, by using Lemma 2.1 it can be proven that $cl_X A$ is sequentially compact.

(2) $\Longrightarrow$ (3) Let $p \in \omega^*$. There exists a LOTS $(Y, \leq)$ such that $cl_X A$ can be considered as a dense subspace of $Y$. Without loss of generality we can assume that $Y$ is compact. Let $(x_n)_{n<\omega}$ be a sequence in $cl_X A$. Since $Y$ is compact, there exists a $p$-limit point $y \in Y$ of $(x_n)_{n<\omega}$. Observe that, in order to establish what we want, it is enough to show that $y$ is an element of $cl_X A$. So, assume that $y \notin \{x_n : n < \omega\} = F$. Either for every $a \in Y$ with $a < y$, the set $\{n < \omega : x_n \in (a, y]\}$ belongs to $p$, or for every $b \in Y$ with $x < b$, the set $\{n < \omega : x_n \in [y, b)\}$ belongs to $p$. Without loss of generality, assume that the first case holds. So $F$ is left cofinal at $y$ in $Y$. By Lemma 2.1, there exists a strictly increasing sequence $(z_n)_{n<\omega}$ of $F$ converging to $y$. But $\{z_n : n < \omega\} \subset cl_X A$ and $cl_X A$ is sequentially compact, then $y$ must belong to $cl_X A$, and, by assumption, $y$ is the $p$-limit point of $(x_n)_{n<\omega}$ in $cl_X A$.

(3) $\Longrightarrow$ (4) , (4) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (1) are clear. $\square$

Let $X$ be a topological space. A subset $A$ of $X$ is said to be *strongly bounded* in X if for each infinite family of pairwise disjoint open subsets of $X$ meeting $A$ contains an infinite subfamily $\{U_n : n < \omega\}$ such that for each filter $\mathcal{G}$ of infinite subsets of $\omega$,

$$\bigcap_{F \in \mathcal{G}} cl_X \left( \bigcup_{n \in F} U_n \right) \neq \emptyset.$$

This concept was introduced by Tkačenko in [Tk], and in [BS] it was proved that a subset $A$ of a space $X$ is strongly bounded if and only if for each space $Y$ and each bounded subset $B$ of $Y$, the subset $A \times B$ is bounded in $X \times Y$. According to the definition, it is clear that if $cl_X A$ is ultracompact, then $A$ is strongly bounded in $X$. From this and the previous theorem we have:

**2.3. Corollary.** *Let $X$ be a GLOTS. For a subset $A$ of $X$, the following conditions are equivalent:*

(1) *$A$ is bounded in $X$;*
(2) *$A$ is $p$-bounded in $X$ for every $p \in \omega^*$;*
(3) *$A$ is $p$-bounded in $X$ for some $p \in \omega^*$;*
(4) *$A$ is quasi-$p$-bounded in $X$ for some $p \in \omega^*$;*
(5) *$A$ is quasi-$p$-bounded in $X$ for every $p \in \omega^*$;*
(6) *$A$ is strongly bounded in $X$.*

Notice that the previous results imply that, in the field of GLOTS, ultrapseudocompactness, ultracompactness, pseudocompactness, quasi-$p$-pseudocompactness (for some $p \in \omega^*$ ), $p$-pseudocompactness (for some $p \in \omega^*$ ) and quasi-$p$-compactness (for some $p \in \omega^*$ ) are equivalent properties. Since ultracompactness is a productive property we can obtain:

**2.4. Corollary.** *Let $\{X_\alpha\}_{\alpha\in\Delta}$ be a family of pseudocompact GLOTS. Then the product space $\prod_{\alpha\in\Delta} X_\alpha$ is an ultracompact space.*

From now on, we are concerned with $C$-compact subsets and with a version, in GLOTS, of the classical Gilcksberg's Theorem on pseudocompactness. A subset $A$ of a space $X$ is called *$C$-compact* (in $X$) if $f(A)$ is a compact subset of $\mathbb{R}$ for every real-valued continuous function on $X$, (equivalently, $A$ is $G_\delta$-dense in $cl_{\beta(X)}A$). A $C$-compact subset $A$ of a space $X$ is said to be *strongly $C$-compact* if, for each space $Y$ and each $C$-compact subset $B$ of $Y$, the subset $A \times B$ is $C$-compact in $X \times Y$. Every $C$-compact subset is bounded but the converse fails to be true. For general background on $C$-compact subsets, related topics, and the relationship between strongly $C$-compactness and strongly boundedness, the reader may see [GFG], [GFS] and [GFST].

The classical Gilsckberg's Theorem on pseudocompactness says that on spaces $\{X_\alpha\}_{\alpha\in\Delta}$ the condition for the product space $\prod_{\alpha\in\Delta} X_\alpha$ to be pseudocompact is equivalent to the condition that $\beta(\prod_{\alpha\in\Delta} X_\alpha)$ and $\prod_{\alpha\in\Delta} \beta(X_\alpha)$ be equivalent compactifications of $\prod_{\alpha\in\Delta} X_\alpha$.

Let $p \in \omega^*$. It is known that the product of $p$-bounded subsets satisfies a relativized version of Glicksberg' Theorem [STM1, Corollary 4.14]. Therefore, condition (3) in Corollary 2.3 implies that

**2.5. Corollary.** *Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of GLOTS and let $A_\alpha$ be a bounded subset of $X_\alpha$ for all $\alpha \in \Delta$. Then*

$$cl_{\beta(\prod_{\alpha\in\Delta} X_\alpha)}(\prod_{\alpha\in\Delta} A_\alpha) = \prod_{\alpha\in\Delta} cl_{\beta(X_\alpha)} A_\alpha.$$

**2.6. Corollary.** *Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of GLOTS and let $A_\alpha$ be a $C$-compact subset of $X_\alpha$ for all $\alpha \in \Delta$. Then $\prod_{\alpha\in\Delta} A_\alpha$ is $C$-compact in $\prod_{\alpha\in\Delta} X_\alpha$.*

*Proof.* Since every $C$-compact subset is bounded, by Corollary 2.5 we have

$$cl_{\beta(\prod_{\alpha\in\Delta} X_\alpha)}(\prod_{\alpha\in\Delta} A_\alpha) = \prod_{\alpha\in\Delta} cl_{\beta(X_\alpha)} A_\alpha.$$

The result now follows from the fact that $A_\alpha$ is $G_\delta$-dense in $cl_{\beta(X_\alpha)}A_\alpha$ for all $\alpha \in \Delta$ if and only if $\prod_{\alpha\in\Delta} A_\alpha$ is $G_\delta$-dense in $\prod_{\alpha\in\Delta} cl_{\beta(X_\alpha)}A_\alpha$ [GFST, Lemma 3.3]. $\square$

In the Corollary 2.9 we shall prove that every $C$-compact subset of a GLOTS is strongly-$C$-compact. For this in turn, we shall show a version of the classical Glicksberg's Theorem on pseudocompactness when a factor is a GLOTS. We need the following theorem taken from [GFSW, Theorem 2.8].

**2.7. Theorem.** *Let $A$ and $B$ be two subsets of a space $X$ and of a space $Y$, respectively. If $A \times B$ is bounded in $X \times Y$ and $A$ is pseudocompact, then $cl_{\beta(X\times Y)}(A \times B) = cl_{\beta(X)}A \times cl_{\beta(Y)}B$.*

**2.8. Theorem.** *Let $A$ and $B$ be two subsets of a space $X$ and of a space $Y$, respectively. If $A\times B$ is bounded in $X\times Y$ and $X$ is normal, then $cl_{\beta(X\times Y)}(A\times B) = \beta(cl_X A) \times cl_{\beta(Y)}B$.*

*Proof.* Since $A \times B$ is bounded in $X \times Y$, we have that $cl_X A \times B$ is also bounded in $X \times Y$. Because of normality of $X$, $cl_X A$ is pseudocompact. So, by Theorem 2.7, $cl_{\beta(X\times Y)}(A \times B) = cl_{\beta(X)}A \times cl_{\beta(Y)}B$. On the other hand, normality of $X$ implies that $cl_X A$ is $C^*$-embedded in $X$ [GJ, 3D(3)] and, consequently, $\beta(cl_X A) = cl_{\beta(X)}(cl_X A) = cl_{\beta(X)}A$ [GJ, 6.9(a)]. This completes the proof. $\square$

The following corollary is an easy consequence of Theorem 2.8.

**2.9. Corollary.** *Let $X$ be a GLOTS. If $A$ is a bounded subset of $X$, then, for each space $Y$ and each bounded subset $B$ of $Y$, $cl_{\beta(X\times Y)}(A \times B) = \beta(cl_X A) \times cl_{\beta(Y)}B$. Consequently, if $A$ and $B$ are $C$-compact subsets of $X$ and $Y$, respectively, then $A \times B$ is $C$-compact in $X \times Y$.*

Only special classes of spaces whose bounded subsets are strongly-bounded and whose $C$-compact subsets are strongly-$C$-compact, are known; one of the most interesting is the class of (Hausdorff) Topological Groups. In addition, it was proved in [GFS] that a topological group $G$ is pseudocompact if and only if $G$ is $p$-pseudocompact for some $p \in \omega^*$ if and only if $G$ is ultrapseudocompact. However, in contrast with GLOTS, we can prove that $p$-compactness and $p$-pseudocompactness are not equivalent properties in the realm of topological groups. And the same for quasi-$p$-compactness and quasi-$p$-pseudocompactness. To see this we will need several results on $C_\pi$-spaces which are given in the next section.

Next, we shall apply the previous results in the field of *topologically orderable groups*. A topological group $(G, \tau)$ is called a *topologically orderable group* if the topology $\tau$ is induced by a linearly (totally) order; that is, if $(G, \tau)$ as a topological space is a LOTS. The following theorem will be useful.

**2.10. Theorem [NR, Theorem 6].** *Let $(G, \tau)$ be a topological group which is not metrizable. The following conditions are equivalent:*

1. *$(G, \tau)$ is topologically orderable;*
2. *the identity element of $G$ has a totally ordered local base.*

Since topologically orderable groups that are not metrizable are $P$-spaces (see [NR, Remark 10], we have:

**2.11. Theorem.** *A non discrete topologically orderable group* $(G, \tau)$ *is metrizable if and only if* $G$ *contains a non trivial convergent sequence.*

Given a topological group $(G, \tau)$, $\mathcal{L}$ (respectively, $\mathcal{R}$, $\mathcal{L} \vee \mathcal{R}$) stands for the *left* (respectively, *right*, *bilateral*) uniformity on $(G, \tau)$. A subset $A$ of $G$ is said to be *precompact* if it is precompact for the left uniformity on $(G, \tau)$. A topological group $(G, \tau)$ is called a *locally precompact topological group* if there exits a precompact neighborhood of the identity. $(G, \tau)$ is called *precompact* if $G$ is precompact in itself. It is well-known that a symmetric subset $A$ of $(G, \tau)$ is precompact for the left uniformity if and only if $A$ is precompact for the right uniformity if and only if $A$ is precompact for the bilateral uniformity (see, [RD, Lemma 9.12]). So, we can replace the uniformity $\mathcal{L}$ either by $\mathcal{R}$ or by $\mathcal{L} \vee \mathcal{R}$ in the definition of precompact (respectively, locally precompact) group.

**2.12. Remark**. In some contexts, precompact subsets of topological groups are called bounded and precompact groups are called totally bounded groups. Hereon we prefer the uniform notation in order to differentiate between the *uniform* concept of *precompactness* and the *topological* concept of *boundedness* which is used in this paper.

The following theorem characterizes metrizable topologically orderable groups by means of precompact subsets.

**2.13. Theorem.** *Let* $(G, \tau)$ *be a nondiscrete topologically orderable group. The following conditions are equivalent:*

(1) $(G, \tau)$ *is metrizable;*
(2) $(G, \tau)$ *contains an infinite countable bounded set;*
(3) $(G, \tau)$ *contains an infinite countable precompact subset for the* $\mathcal{L} \vee \mathcal{R}$ *uniformity.*

*Proof.* (1) $\Longrightarrow$ (2) Since $(G, \tau)$ is an infinite non discrete metrizable group, $(G, \tau)$ contains a non trivial covergent sequence $(x_n)_{n<\omega}$. Obviously, $(x_n)_{n<\omega}$ is bounded in $(G, \tau)$.

(2) $\Longrightarrow$ (3) Let $A$ be an infinite countable bounded subset of $(G, \tau)$. By Theorem 2.2, $cl_G A$ is sequentially compact. So, there exists an infinite countable compact subset $K \subset cl_G A$. It is clear that $K$ is precompact for the $\mathcal{L} \vee \mathcal{R}$ uniformity.

(3) $\Longrightarrow$ (1) Let $G$ be a non-metrizable topologically orderable group. By Fox's Theorem [Fo], the completion $\widehat{G}$ of the uniform space $(G, \mathcal{L} \vee \mathcal{R})$ is a topologically orderable, non-metrizable group. By the Remark 10 from [NR], $\widehat{G}$ is a $P$-space and hence every pseudocompact subspace of $\widehat{G}$ is finite. Hence, if $A$ is a precompact subset of $G$, then $cl_{\widehat{G}} A$ is compact and so $cl_{\widehat{G}} A$ is finite. This shows that $A$ is finite. $\square$

**2.14. Remark.** Given a topological group $(G, \tau)$, a subset $A$ is precompact for $\mathcal{U} \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$ if and only if the symmetric subset $A \cup A^{-1}$ is precompact for $\mathcal{U} \in \{\mathcal{L}, \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$. So, the $\mathcal{L} \vee \mathcal{R}$ *uniformity* can be replaced either by the $\mathcal{L}$ *uniformity* or by the $\mathcal{R}$ *uniformity* in Theorem 2.13.

**2.15. Corollary.** *Every locally precompact topologically orderable group* $(G, \tau)$ *is metrizable.*

*Proof.* We may assume that $(G,\tau)$ is not discrete. Let $U$ be a precompact symmetric neighborhood of the identity of $(G,\tau)$. Since $U$ is infinite, the result follows from Theorem 2.13. $\square$

**2.16. Corollary.** *Every precompact (so, every compact) topologically orderable group is metrizable.*

**2.17. Corollary.** *Every pseudocompact topologically orderable group is compact.*

**2.18. Remark.** (a) Corollary 2.17 also follows from the Scott-Watson's Theorem which states that a paracompact pseudocompact space is compact [En, Theorem 5.1.20].

(b) Notice that every non discrete, locally compact, totally disconnected topologically orderable group contains an open subgroup homeomorphic to the Cantor set ([VRS, Theorem 2.5]).

(c) An *algebraic ordered group* is an abstract group $G$ with a subset $P$ closed under the binary group operation (called the set of positive elements) such that $G$ is the disjoint union of $P$, $P^{-1}$ and $\{e\}$ (here $e$ stands for the identity element of $G$). Every algebraic ordered group $G$ is a topologically ordered group via the topology $\tau_P$ induced by the natural order $x<y$ whenever $yx^{-1}\in P$. The topological group $(G,\tau_P)$ is never precompact. In fact, suppose that $(G,\tau_P)$ is precompact. Then, by Fox's Theorem [Fo], the completion $\widehat{G}$ of the uniform space $(G,\mathcal{L}\vee\mathcal{R})$ is a compact topologically ordered group and, consequently, we can find $m\in\widehat{G}$ such that $g\leq m$ for each $g\in\widehat{G}$ ([En, 3.12.3]). Since every element of an algebraic ordered group other than the identity is of infinite order, $\widehat{G}$ is a non-discrete group. So, there exists an increasing net $\{x_\lambda:\lambda<\alpha\}$ in $G$ converging to $m$ such that $e<x_\lambda$ for each $\lambda<\alpha$. Since translation (in $G$) is order-preserving, for each $\lambda_0<\alpha$, $x_{\lambda_0}<x_{\lambda_0}x_\lambda$ whenever $\lambda<\alpha$. Then, for each $\lambda_0<\alpha$, $x_{\lambda_0}\leq x_{\lambda_0}m$. Consequently, $m\leq m^2$ which implies $m=e$, a contradiction.

## 3. Boundedness in $C_\pi$-spaces

If $X$ and $Y$ are two spaces, we will denote by $C(X,Y)$ the set of continuous functions defined on $X$ and with values in $Y$. If $Y=\mathbb{R}$, then we will write $C(X)$ instead of $C(X,\mathbb{R})$. The set of bounded real-valued continuous functions defined on $X$ is denoted by $C^*(X)$. We will write $C_\pi(X,Y)$, $C_\pi(X)$ and $C^*_\pi(X)$ in order to symbolize the sets $C(X,Y)$, $C(X)$ and $C^*(X)$ equipped with the pointwise convergence topology.

In this section we are going to give necessary and sufficient conditions in the space $X$ in order to have $\mathcal{P}$ and $\sigma$-$\mathcal{P}$ for $C_\pi(X,[0,1])$ and $C_\pi(X)$ where $\mathcal{P}\in\{p$-pseudocompactness, $p$-compactness, quasi-$p$-pseudocompactness, quasi-$p$-compactness, and $p$-boundedness$\}$. As usual, if $\mathcal{P}$ is a topological property, then a space $X$ is $\sigma$-$\mathcal{P}$ if $X$ is the countable union of subspaces having $\mathcal{P}$. A space $X$ is a *$P$-space* if every $G_\delta$-set in $X$ is open. For a cardinal number $\alpha$, a space $X$ is *$\alpha$-$b$-discrete* if every subset $Y$ of $X$ of cardinality $\leq\alpha$ is discrete and $C^*$-embedded in $X$. A space $X$ is *$b$-discrete* if $X$ is $\omega$-$b$-discrete. A subset $Y$ of a product $X=\prod_{j\in J}X_j$ is said to be *$\alpha$-dense* in $X$ if for every $K\subset J$ of cardinality $\leq\alpha$ we have $\pi_K(Y)=\prod_{k\in K}X_k$, where $\pi_K:X\to\prod_{k\in K}X_k$ is the $K$-projection. Observe that if $\gamma<\alpha$ and $Y$ is $\alpha$-dense in $X$, then $Y$ is $\gamma$-dense and dense in $X$. For $k\in J$ and $A\subset X_k$, we will denote by $[k; A]$ the set $\{f \in \prod_{j \in J} X_j : f(k) \in A\} = \pi_k^{-1}(A)$; and $[k_1, ..., k_n; A_1, ..., A_n]$ will be the intersection of the sets $[k_1; A_1]$, $[k_2; A_2]$,... and $[k_n; A_n]$.

**3.1. Notation.** *In order to be brief, we will denote by $\mathcal{Q}$ one of the following properties: ultracompactness, $p$-compactness for some $p \in \omega^*$, quasi-$p$-compactness for every $p \in \omega^*$, quasi-$p$-compactness for some $p \in \omega^*$, countable compactness. We will denote by $\mathcal{S}$ one of the following properties: ultrapseudocompactness, $p$-pseudocompactness for some $p \in \omega^*$, quasi-$p$-pseudocompactness for every $p \in \omega^*$, quasi-$p$-pseudocompactness for some $p \in \omega^*$, pseudocompactness. And $\mathcal{T}$ will be one of the following: $p$-boundedness for every $p \in \omega^*$, $p$-boundedness for some $p \in \omega^*$, quasi-$p$-boundedness for every $p \in \omega^*$, quasi-$p$-boundedness for some $p \in \omega^*$, boundedness.*

Our first result in this section generalizes Theorem 1 in [ST].

**3.2. Theorem.** *Let $X$ be a space. Then, the following conditions are equivalent:*

(1) *$X$ is a $P$-space;*
(2) *$C_\pi(X, [0, 1])$ has $\mathcal{Q}$;*
(3) *$C_\pi(X, [0, 1])$ has $\sigma$-$\mathcal{Q}$;*
(4) *$C^*_\pi(X)$ has $\sigma$-$\mathcal{Q}$.*

*Proof.* Property $\mathcal{Q}$ implies trivially $\sigma$-$\mathcal{Q}$, and if the space $C_\pi(X, [0, 1])$ has $\mathcal{Q}$, then $C^*_\pi(X)$ has $\sigma$-$\mathcal{Q}$ because $C^*_\pi(X) = \bigcup_{n<\omega} C_\pi(X, [-n, n])$. Besides, property $\mathcal{Q}$ implies countable compactness, so $(i) \Rightarrow (1)$ for all $i \in \{2, .., 4\}$ ([ST]). And since ultracompactness implies $\mathcal{Q}$, we need only to prove that the assertion in (1) implies that $C_\pi(X, [0, 1])$ is ultracompact:

Let $p$ be an arbitrary element in $\omega^*$, and assume that $C_\pi(X, [0, 1])$ is not $p$-compact. Then there exists a sequence $(f_n)_{n<\omega}$ in $C_\pi(X, [0, 1])$ which does not have a $p$-limit point. Since $[0, 1]^X$ is a compact space, then there is $f \in [0, 1]^X \setminus C_\pi(X, [0, 1])$ that is a $p$-limit point of $(f_n)_{n<\omega}$. Since $f$ is not a continuous function, there exist $x \in X$ and an open set $A$ of $[0, 1]$ such that $f(x) \in A$ and for every $V \in \mathcal{N}(x)$ we have $f(V) \cap ([0, 1] \setminus A) \neq \emptyset$. Let $B$ be an open set in $[0, 1]$ such that $f(x) \in B \subset cl_{[0,1]}B \subset A$. Since $f$ is a $p$-limit point of $(f_n)_{n<\omega}$ and $f \in [x, B]$, $F = \{k \in \omega : f_k \in [x, B]\} \in p$. So $x$ belongs to the $G_\delta$-set $C = \bigcap_{k \in F} f_k^{-1}(B)$. Since $X$ is a $P$-space, $C$ is open in $X$. Besides, $f_k(C) \subset B \subset A$ for every $k \in F$. On the other hand, there exists $c \in C$ such that $f(c) \notin A$. Let $W = [0, 1] \setminus \mathrm{cl}_{[0,1]}B$. Consider the open set $T = [x, B] \cap [c, W]$. We have that $f \in T$, $f_k \notin T$ for all $k \in F$ (because $f_k(c) \in B$ and $B \cap W = \emptyset$), and that $G = \{n < \omega : f_n(x) \in B$ and $f_n(c) \in W\} \in p$. Then $\emptyset = F \cap G \in p$, which is a contradiction. $\square$

**3.3. Remark.** If $K$ is a compact space, then $K$ is homeomorphic to a closed subset of $[0, 1]^{w(K)}$, where $w(K)$ is the weight of $K$. So, $C_\pi(X, K)$ is homeomorphic to a closed subset of $C_\pi(X, [0, 1]^{w(K)}) \cong C_\pi(X, [0, 1])^{w(K)}$. Besides, if $K$ contains a nontrivial path, then $K$ contains a closed copy of $[0, 1]$. In this case, $C_\pi(X, [0, 1])$ is homeomorphic to a closed set of $C_\pi(X, K)$. Since $p$-compactness is productive and inherited by closed subsets, then, for a compact space $K$, each one of the assertions (1)-(4) in Theorem 3.2 implies that $C_\pi(X, K)$ is ultracompact. Furthermore, if $K$ contains a nontrivial path, then $C_\pi(X, K)$ is ultracompact if and only if it is

$p$-compact for some $p \in \omega^*$; if and only if it is quasi-$p$-compact for some $p \in \omega^*$; if and only if $X$ is a $P$-space.

**3.4. Theorem.** *Let $J$ be a set, let $X = \prod_{j\in J} X_j$ be a product of compact metrizable spaces and let $Y$ be a dense subset of $X$. For $p \in \omega^*$, the following conditions are equivalent:*

(1) *$Y$ is $\omega$-dense in $X$;*
(2) *$Y$ is $G_\delta$-dense in $X$;*
(3) *$Y$ is $C$-compact in $X$;*
(4) *$Y$ has $\mathcal{S}$;*
(5) *$Y$ is ultrapseudocompact.*

*Proof.* The equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ $Y$ is pseudocompact, are already known (see [GFST]), and (5) $\Rightarrow$ (4) $\Rightarrow$ $Y$ is pseudocompact are trivial. It remains to prove (2) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (2). The condition of metrizability of the spaces $X_j$ is require to prove the implications (2) $\Rightarrow$ (1) (see below) and (3) $\Rightarrow$ (1) (see the proof of (2) $\Longrightarrow$ (3) of Lemma 4.7 in [GFST]).

(2) $\Longrightarrow$ (1) Let $N$ be a countable subset of $J$, and let $x = (x_j)_{j\in N}$ be an element of $\prod_{j\in N} X_j$. We are going to find a $y \in Y$ for which $\pi_N(y) = x$. For each $j \in N$, let $(U_n^j)_{n<\omega}$ be a local base at $x_j$. The set $U = \bigcap_{j\in N}\bigcap_{n<\omega}\pi_j^{-1}(U_n^j)$ is a nonempty $G_\delta$-set in $X$. So, there is $y \in Y \cap U$. We have that $\pi_N(y) = x$.

(1) $\Longrightarrow$ (5) Let $p$ be an element in $\omega^*$. Consider a sequence $(U_n)_{n<\omega}$ of nonempty open subsets of $Y$. For each $n < \omega$, there exists a basic nonempty open set $A_n = [j_1^n, ..., j_{k_n}^n; A_1^n, ..., A_{k_n}^n]$ satisfying $V_n = A_n \cap Y \subset U_n$. We will prove that there exists a $p$-limit point of $(V_n)_{n<\omega}$ in $Y$. We take the countable set $N = \{j_i^n : n < \omega, 1 \le i \le k_n\}$ and let $\pi_N : X \to \prod_{j\in N} X_j$ be the projection.

**Claims:**

(1) $\pi_N(V_n) = \pi_N(A_n)$.
(2) $\pi_N^{-1}\pi_N(A_n) = A_n$.

1. Indeed, it is clear that $\pi_N(V_n) \subset \pi_N(A_n)$. Now, let $a \in A_n$. We want to prove that there is $b \in V_n$ for which $\pi(b) = \pi_N(a)$. Since $Y$ is $\omega$-dense in $X$, there is $b \in Y$ such that $\pi_N(b) = \pi_N(a)$. This means that, for every $j \in N$, $a(j) = b(j)$. In particular, $a(j_i^n) = b(j_i^n)$ for all $1 \le i \le k_n$. So, $b$ belongs to $A_n$, and therefore $b \in A_n \cap Y = V_n$.

2. The relation $A_n \subset \pi_N^{-1}\pi_N(A_n)$ is trivial. If $x \in \pi_N^{-1}\pi_N(A_n)$, then there is $a \in A_n$ with the property $\pi_N(x) = \pi_N(a)$. But this implies that $x$ belongs to $A_n$.

Now we continue the proof of (1) $\Longrightarrow$ (5). The sequence $(B_n)_{n<\omega}$, where $B_n = \pi_N(V_n) = \pi_N(A_n)$ for each $n < \omega$, is a sequence of nonempty open sets in $\prod_{j\in N} X_j = X_N$. The space $X_N$ is compact, so it is $p$-pseudocompact. Hence, there exists $y_0 \in X_N$ which is a $p$-limit point of $(B_n)_{n<\omega}$ .

**Claim:** Each point in $\pi_N^{-1}(y_0)$ is a $p$-limit point of the sequence $(A_n)_{n<\omega}$.

In order to prove this claim, we take a point $x \in \pi_N^{-1}(y_0)$ and a basic open set $A = [t_1, ..., t_m; T_1, ..., T_m]$ in $X$ containing $x$. The set $\pi_N(A)$ is open, and $y_0 \in \pi_N(A)$. Thus, $P = \{n < \omega : \pi_N(A) \cap B_n \neq \emptyset\} \in p$. Take $n_0 \in P$. We are going to prove that $A \cap A_{n_0} = A \cap \pi_N^{-1}(B_{n_0}) \neq \emptyset$. Let $z$ be an element of $\pi_N(A) \cap B_{n_0} = \pi_N(A) \cap \pi_N(A_{n_0})$. There are $a \in A$ and $a_{n_0} \in A_{n_0}$ satisfying

$\pi_N(a) = z = \pi_N(a_{n_0})$. But this last equality implies that $a$ belongs to $A_{n_0}$ too. So, $\{n < \omega : A \cap A_n \neq \emptyset\} \in p$. That is, $x$ is a $p$-limit point of $(A_n)_{n<\omega}$.

Since $Y$ is $\omega$-dense, we can take a point $l$ in $Y \cap \pi_N^{-1}(y_0)$. Because of the last claim, $l$ is a $p$-limit of $(A_n)_{n<\omega}$. We will finish the proof of the implication (1) $\Rightarrow$ (5), by proving that $l$ is a $p$-limit point of $(V_n)_{n<\omega}$:

Let $L$ be an open neighborhood of $l$ in $Y$. The set $L$ is of the form $\widetilde{L} \cap Y$, where $\widetilde{L}$ is an open set in $X$ which contains $l$. Since $l$ is a $p$-limit point of the sequence $(A_n)_{n<\omega}$, $P_0 = \{n < \omega : \widetilde{L} \cap A_n \neq \emptyset\} \in p$. Now, for each $n \in P_0$, $\widetilde{L} \cap A_n$ is an open set of $X$. But $Y$ is dense in $X$, so $L \cap V_n = (\widetilde{L} \cap Y) \cap (A_n \cap Y) = (\widetilde{L} \cap A_n) \cap Y \neq \emptyset$. Therefore, $\{n < \omega : L \cap V_n \neq \emptyset\} \in p$. This implies that $l$ is a $p$-limit point of $(V_n)_{n<\omega}$; that is, $Y$ is $p$-pseudocompact. Since $p$ was arbitrarily selected, we conclude that $Y$ is ultrapseudocompact.

(5) $\Longrightarrow$ (2) Since $Y$ is ultrapseudocompact, $Y$ is pseudocompact. So, it is $G_\delta$-dense in its Stone-Čech-compactification ([H]). This implies that $Y$ is $G_\delta$-dense in all its compactifications. In particular, $Y$ is $G_\delta$-dense in $X$. □

It was proved in [Tka] that $X$ is $b$-discrete if and only if $C_\pi(X, [0,1])$ is pseudocompact if and only if $C_\pi(X, [0,1])$ is $\sigma$-bounded if and only if $C^*_\pi(X)$ is $\sigma$-bounded. These facts and Theorem 3.4 imply the following:

**3.5. Corollary.** *Let $X$ be a space. Then, the following are equivalent*

(1) *$X$ is $b$-discrete;*
(2) *$C_\pi(X, [0,1])$ is $C$-compact in $[0,1]^X$;*
(3) *$C_\pi(X, [0,1])$ has $\mathcal{S}$;*
(4) *$C_\pi(X, [0,1])$ has $\sigma$-$\mathcal{S}$;*
(5) *$C^*_\pi(X)$ has $\sigma$-$\mathcal{S}$;*
(6) *$C_\pi(X, [0,1])$ has $\sigma$-$\mathcal{T}$;*
(7) *$C^*_\pi(X)$ has $\sigma$-$\mathcal{T}$.*

We recall that a metric continuum $K$ (that is, a compact, connected metric space) is said to be *a Peano's Continuum* if $K$ is the continuous image of the unit interval $[0,1]$. The well-known Hahn-Mazurkiewicz's Theorem asserts that Peano's Continuum agrees with metric continuum locally connected spaces ([En, 6.3.14]. We have the following result

**3.6. Corollary.** *If $X$ is a $b$-discrete space and $K$ is a Peano's Continuum, then $C_\pi(X, K)$ is ultrapseudocompact.*

*Proof.* Let $h : [0,1] \to K$ be a continuous and onto function. We have that the function $h_* : C_\pi(X, [0,1]) \to C_\pi(X, K)$ defined by $h_*(F) = f \circ h$ is continuous. Let $W = [x_1, ..., x_n; A_1, ..., A_n]$ be a canonical open set in $C_\pi(X, K)$. The set $h^{-1}(A_i)$ is nonempty for every $i = 1, ..., n$. But $X$ is a Tychonoff space, so there is a continuous function $f : X \to [0,1]$ such that $f(x_i) \in h^{-1}(A_i)$. So, $h \circ f \in W$. We conclude that $h_*(C_\pi(X, [0,1]))$ is a dense subset of $C_\pi(X, K)$. Since $X$ is a $b$-discrete space, $C_\pi(X, [0,1])$ is ultrapseudocompact (Corollary 3.5). Besides, ultrapseudocompactness is preserved by continuous functions and every space containing an ultrapseudocompact dense subspace satisfies this property too. Therefore, $C_\pi(X, K)$ is ultrapseudocompact. □

It was proved in [GFS] that a topological group $G$ is pseudocompact if and only if $G$ is $p$-pseudocompact for some $p \in \omega^*$, if and only if $G$ is ultrapseudocompact. Using the previous results we can prove that $p$-compactness and $p$-pseudocompactness are different properties even in the class of Topological Groups. And the same for quasi-$p$-compactness and quasi-$p$-pseudocompactness.

**3.7. Example.** *There exists an ultrapseudocompact Topological Group which is not quasi-$p$-compact for any $p \in \omega^*$.*

*Proof.* D.B. Shakhmatov constructed in [Sh] an example of a pseudocompact $b$-discrete space $Z$ which, of course, is not a $P$-space. (Another example of a pseudocompact $b$-discrete space is given, as the referee pointed out to the authors, by the subspace $T(q)$ of $\omega^*$ of all the ultrafilters equivalent to $q$, where $q$ is a weak $P$-point and is not a $P$-point.) So, for this $Z$, $C_\pi(Z,[0,1])$ is ultrapseudocompact and it is not quasi-$p$-compact for any $p \in \omega^*$ (Theorem 3.2 and Corollary 3.5). Because of Remark 3.3, $C_\pi(Z,K)$ is not quasi-$p$-compact for any $p \in \omega^*$ and for every compact space $K$ containing a nontrivial path. Hence, $C_\pi(Z,\mathbb{S}^1)$ is not quasi-$p$-compact for any $p \in \omega^*$ where $\mathbb{S}^1$ is the unit circle. On the other hand, by Corollary 3.6, it is ultrapseudocompact and the proof is complete. □

We finish this section with the following result which is a consequence of Theorem 1.5.4 in [Tka] and Corollary 3.5.

**3.8. Theorem.** *Let $X$ be a space. Then the following assertions are equivalent.*

(1) *$X$ is pseudocompact and $b$-discrete.*
(2) *$C_\pi(X)$ has $\sigma$-$\mathcal{S}$.*
(3) *$C_\pi(X)$ has $\sigma$-$\mathcal{T}$.*

## 4. Open questions

By Theorem 6 from [NR], totally disconnected topologically orderable groups belong to the class of $\omega_\mu$-metrizable spaces [ST], which were originally defined by Sikorski as those spaces $X$ with a distance function $d : X \times X \longrightarrow (G, \leq)$, where $G$ is an algebraically ordered group, the distance function being required to satisfy all the formal properties of a metric. The class of $\omega_\mu$-metrizable spaces is identical with the class of those spaces which admit a uniformity with a totally ordered base. Beginning from this facts and the previous results, the following two specific questions seem worthy of study:

**4.1. Question.** *Let $X$ be an $\omega_\mu$-metrizable space. If $X$ contains an infinite countable bounded subset, is $X$ metrizable?*

**4.2. Question.** *Is every precompact $\omega_\mu$-metrizable space metrizable?*

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