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# $p$-pseudocompactness and related topics in topological spaces

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**Abstract**

We prove some basic properties of $p$-bounded subsets ($p \in \omega^*$) in terms of $z$-ultrafilters and families of continuous functions. We analyze the relations between $p$-pseudocompactness with other pseudocompact like-properties as $p$-compactness and $\alpha$-pseudocompactness where $\alpha$ is a cardinal number. We give an example of a sequentially compact ultrapseudocompact $\alpha$-pseudocompact space which is not ultracompact, and we also give an example of an ultrapseudocompact totally countably compact $\alpha$-pseudocompact space which is not $q$-compact for any $q \in \omega^*$, answering affirmatively to a question posed by S. García-Ferreira and Kočinac (1996). We show the distribution law $\mathrm{cl}_{\gamma(X\times Y)}(A \times B) = \mathrm{cl}_{\gamma X} A \times \mathrm{cl}_{\gamma Y} B$, where $\gamma Z$ denotes the Dieudonné completion of $Z$, for $p$-bounded subsets and we generalize the classical Glisckberg Theorem on pseudocompactness in the realm of $p$-boundedness. These results are applied to study the degree of pseudocompactness in the product of $p$-bounded subsets. © 1999 Published by Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In this article we will assume that all spaces are Tychonoff spaces unless otherwise stated. If $X$ is a space and $A \subset X$, then $\mathrm{cl}_X(A)$ (or simply $\mathrm{cl}(A)$) denotes the closure of $A$ in $X$. For $x \in X$, let $\mathcal{N}(x)$ denote the set of all neighborhoods of $x$. For a set $X$, $\mathcal{P}(X)$ is the collection of subsets of $X$, and $|X|$ denotes the cardinality of $X$. The Greek letters will stand for infinite ordinal numbers. For an ordinal number $\kappa$, $[0,\kappa)$ will indicate the space of ordinal numbers $\lambda < \kappa$ endowed with the order topology. The cardinal number $2^\omega$ will also be denoted by $\mathfrak{c}$. If $\alpha$ is a cardinal number, then $\mathrm{cf}(\alpha)$ is the cofinality of $\alpha$; besides, $[X]^{<\alpha}$ (respectively, $[X]^{\leqslant\alpha}$, $[X]^{\alpha}$) will stand for the family of subsets of $X$ of cardinality $< \alpha$ (respectively, $\leqslant \alpha$, $= \alpha$); and ${}^{\alpha}X$ is the set of functions from $\alpha$ to $X$. The set of natural numbers will be denoted by $\omega$. The Stone–Čech compactification of a space $X$ will be denoted as $\beta(X)$. The space $\beta(\omega)$ is identified with the set of ultrafilters on $\omega$, and $\omega^*$ is the set of free ultrafilters, that is $\omega^* = \beta(\omega) \setminus \omega$. If $f : X \to Y$ is a continuous function, then $f^\beta : \beta(X) \to \beta(Y)$ denotes the Stone–Čech extension of $f$. The *Rudin–Keisler* (*pre-*)*order* on $\beta(\omega)$ is defined by $p \leqslant_{RK} q$ if there is a function $f : \omega \to \omega$ such that $f^\beta(q) = p$, for $p, q \in \beta(\omega)$. Observe that $n \leqslant_{RK} p$ for every $n < \omega$ and $p \in \beta(\omega)$, and if $p \leqslant_{RK} q$, then there exists a surjection $f : \omega \to \omega$ such that $f^\beta(q) = p$. For $p \in \omega^*$, we set

$$P_{RK}(p) = \bigl\{r \in \beta(\omega):\ r \leqslant_{RK} p\bigr\}.$$

If $p \leqslant_{RK} q$ and $q \leqslant_{RK} p$, for $p, q \in \omega^*$, then we say that $p$ and $q$ are equivalent and we write $p \approx_{RK} q$. It is not difficult to verify that $p \approx_{RK} q$ iff there exists a permutation $\sigma : \omega \to \omega$ such that $\sigma^\beta(p) = q$. The *type* of $p \in \omega^*$ is $T(p) = \{q \in \omega^*:\ p \approx_{RK} q\}$.

The concept of $p$-limit, for $p \in \omega^*$, was discovered and investigated by Bernstein [1] in connection with some problems in the theory of nonstandard analysis. Independently, Frólik [10] and Katětov [20,21] introduced this concept in a different form, and Ginsburg and Saks [17] generalized this notion as follows:

**Definition 1.1.** Let $p \in \omega^*$ and $(S_n)_{n<\omega}$ be a sequence of nonempty subsets of a space $X$. A point $x \in X$ is a *p-limit* point of the sequence $(S_n)_{n<\omega}$ if for every $V \in \mathcal{N}(x)$, $\{n < \omega:\ V \cap S_n \neq \emptyset\} \in p$.

If $x_n \in X$ and $S_n = \{x_n\}$ for each $n < \omega$, then a $p$-limit point of $(x_n)_{n<\omega}$ is Bernstein's $p$-limit point of the sequence $(x_n)_{n<\omega}$. Note that if there exists a $p$-limit point of a sequence $(x_n)_{n<\omega}$, this has to be unique, since $X$ is Hausdorff; but, in general, a sequence $(S_n)_{n<\omega}$ of nonempty subsets of a space $X$ could have more than one point. For instance, if $S_n = \{1/n\} \times \mathbb{R}$ for each $n < \omega$, then each point $(0, r) \in \mathbb{R}^2$ is a $p$-limit point of $(S_n)_{n<\omega}$, for each $p \in \omega^*$.

**Definition 1.2** (Bernstein, [1]).
(1) Let $p \in \omega^*$. A space $X$ is *p-compact* if every sequence $(x_n)_{n<\omega}$ has a $p$-limit.
(2) A space $X$ is ultracompact if $X$ is $p$-compact for every $p \in \omega^*$.

Every compact space is ultracompact, and Vaughan proved [27, Theorem 4.9], that in the class of regular spaces, $X$ is ultracompact iff $X$ is *$\omega$-bounded* (= the closure of each countable subset of $X$ is compact). Ginsburg and Saks introduced in [17] the concept of $p$-pseudocompactness and later García-Ferreira defined the relative version of this concept [11]:

**Definition 1.3.**

(1) Let $p \in \omega^*$. A subspace $Y$ of a space $X$ is said to be *$p$-bounded* in $X$ if for every sequence $(V_n)_{n<\omega}$ of nonempty open subsets of $X$ with $Y \cap V_n \neq \emptyset$, for all $n < \omega$, there is $x \in X$ which is a $p$-limit point of the sequence $(V_n)_{n<\omega}$.

(2) If $X$ is p-bounded in itself, then $X$ is called *$p$-pseudocompact*.

These properties are productive and preserved by continuous functions (see [11,17]). Besides, if $Y$ is a regular closed subset of a $p$-pseudocompact space $X$, then $Y$ itself is $p$-pseudocompact; and if $Y$ is $p$-bounded in $X$, then $\mathrm{cl}_X Y$ is $p$-bounded in $X$ too. The property of being $p$-bounded is monotone with respect to the Rudin–Keisler pre-order, that is, if $q \leqslant_{RK} p$ and $Y$ is $p$-bounded in $X$, then $Y$ is $q$-bounded in $X$. It is well known that a space $X$ is pseudocompact if for every sequence $(U_n)_{n<\omega}$ of open subsets of $X$ there is a point $x \in X$ such that for every $V \in \mathcal{N}(x)$,

$$\bigl|\{n < \omega\colon\ V \cap U_n \neq \emptyset\}\bigr| \geqslant \aleph_0.$$

So, if $X$ is $p$-pseudocompact for some $p \in \omega^*$, then $X$ is pseudocompact.

The next concept and some of its properties were analyzed in [14]. Let $\alpha$ and $\gamma$ be cardinal numbers. A subset $B$ of $X$ is said to be *$C_\alpha$-compact in $X$* if $f[B]$ is a compact subset of $\mathbb{R}^\alpha$ for every continuous function $f : X \to \mathbb{R}^\alpha$. If $\alpha < \gamma$, then every $C_\gamma$-compact subset of $X$ is $C_\alpha$-compact; and if $X$ is $C_\alpha$-compact in itself, then we say that $X$ is *$\alpha$-pseudocompact*. A set $Y \subset X$ is a *$G_\delta$-set* in $X$ if there is a sequence $(U_n)_{n<\omega}$ of nonempty open sets in $X$ such that $Y = \bigcap_{n<\omega} U_n$. A subset $Y$ of $X$ is $G_\delta$-dense in $X$ if each nonempty $G_\delta$-set in $X$ has a nonempty intersection with $Y$. A space $X$ is pseudocompact iff $X$ is $\aleph_0$-pseudocompact; and $B$ is $C_\alpha$-compact (in $X$) iff $B$ is $G_\alpha$-dense in $\mathrm{cl}_{\beta(X)}(B)$. For each $\alpha < \gamma$ there exists a space $X$ which is $\alpha$-pseudocompact and is not $\gamma$-pseudocompact. In fact, the space of ordinal numbers $[0, \alpha^+)$ with its order topology is $\alpha$-pseudocompact but is not $\gamma$-pseudocompact.

Recall that a space $X$ is *sequentially compact* if every sequence in $X$ has a convergent subsequence. A space $X$ is *totally countably compact* if every sequence in $X$ has a subsequence contained in a compact subset of $X$. Every sequentially compact space is totally countably compact, and if a space $X$ has this latter property, then $X$ is countably compact. Recall also that, for a topological property $\mathcal{P}$, a space $X$ is $\sigma$-$\mathcal{P}$ if $X$ is the union of a countable family of subspaces having $\mathcal{P}$.

If $X$ and $Y$ are two spaces, we will denote by $C(X, Y)$ the set of continuous functions defined on $X$ and with values in $Y$. If $Y = \mathbb{R}$, then we will write $C(X)$ instead of $C(X, \mathbb{R})$. The set of real bounded continuous functions defined on $X$ is denoted by $C^*(X)$.

A subspace $Y$ of a space $X$ is $C^*$-embedded in $X$ if for every $f \in C^*(Y)$ there is $g \in C^*(X)$ such that $g|_Y = f$; and it is a zero-set (respectively, cozero-set) if there is $f \in C(X)$ such that $Y = f^{-1}\{0\}$ (respectively, $f^{-1}(\mathbb{R} \setminus \{0\})$).

In this article we give some basic properties of $p$-bounded subsets ($p \in \omega^*$) in terms of $z$-ultrafilters and families of continuous functions (Section 2). In Section 3, we analyze the relations between $p$-pseudocompactness with other pseudocompact-like properties such as $p$-compactness and $\alpha$-pseudocompactness where $\alpha$ is a cardinal number, we give an example of a sequentially compact ultrapseudocompact $\alpha$-pseudocompact space which is not ultracompact, and an example of an ultrapseudocompact totally countably compact $\alpha$-pseudocompact space which is not $q$-compact for any $q \in \omega^*$, answering affirmatively a question posed by S. García-Ferreira and Kočinac in [12], and we discuss the relation between $p$-pseudocompactness and $p$-compactness in normal and first countable spaces. Section 4 is dedicated to the product of two $p$-bounded subsets; we show the distribution law $\mathrm{cl}_{\gamma(X\times Y)}(A \times B) = \mathrm{cl}_{\gamma X} A \times \mathrm{cl}_{\gamma Y} B$, where $\gamma Z$ denotes the Dieudonné completion of $Z$, for $p$-bounded subsets, and we generalize the classical Glisckberg Theorem on pseudocompactness in the realm of $p$-boundedness. These results are applied to study the degree of pseudocompactness in the product of $p$-bounded subsets.

## 2. $p$-boundedness

In this section we are going to give some alternative descriptions of $p$-boundedness in terms of $z$-ultrafilters and families of continuous functions.

**Definition 2.1.** Let $X$ be a space, $Y \subset X$ and let $p \in \omega^*$.

(1) A family $\mathcal{A} = \{A_j\colon j \in J\}$ of subsets of a space $X$ is *$p$-generated relative to $Y$* if there exists a collection $\{U_n\colon n < \omega\}$ of nonempty open subsets of $X$ such that $Y \cap U_n \neq \emptyset$ for each $n < \omega$, and for each $j \in J$, the set $\{n \in \omega\colon U_n \subset A_j\}$ belongs to $p$ (that is, for each $j \in J$, there is $F_j \in p$ satisfying: $A_j \supset \bigcup_{n\in F_j} U_n$). We simply say that $\mathcal{A}$ is $p$-generated when $\mathcal{A}$ is $p$-generated relative to $X$.

(2) A collection $\mathcal{U}$ of subsets of $X$ with the finite intersection property is *$p$-real relative to $Y$* if each collection $\{A_j\colon j \in J\} \subset \mathcal{U}$ which is $p$-generated relative to $Y$, has nonempty intersection. If $Y = X$ we simply say that $\mathcal{U}$ is *$p$-real*.

(3) A collection $\mathcal{A} = \{A_n\colon n < \omega\}$ of subsets of $X$ is *locally $p$-finite* if for each $x \in X$ there is a neighborhood $V$ of $x$ such that $\{n < \omega\colon V \cap A_n \neq \emptyset\} \notin p$, that is, $\mathcal{A}$ does not admit $p$-limit points.

Observe that each $p$-generated family relative to $Y \subset X$ has the finite intersection property, and a locally $p$-finite sequence of nonempty open sets of a space $X$ cannot be a finite set because $p$ is an ultrafilter.

**Theorem 2.2.** *Let $X$ be a space, $Y \subset X$ and $p \in \omega^*$. Then, the following assertions are equivalent.*

(1) *$X$ is $p$-pseudocompact (respectively, $Y$ is $p$-bounded in $X$).*

(2) *For every sequence of nonempty open sets* $\{U_n\colon n<\omega\}$ *in* $X$ (*respectively, such that* $Y\cap U_n\neq\emptyset$ *for each* $n<\omega$) *we have*

$$\bigcap_{F\in p}\mathrm{cl}_X\left(\bigcup_{n\in F}U_n\right)\neq\emptyset.$$

(3) *Each* $p$*-generated family* $\{A_j\colon j\in J\}$ *in* $X$ (*respectively, relative to* $Y$) *satisfies that* $\bigcap_{j\in J}\mathrm{cl}(A_j)\neq\emptyset$.

(4) *Every z-ultrafilter on* $X$ *is* $p$*-real* (*respectively, relative to* $Y$).

(5) *If* $\{U_n\colon n<\omega\}$ *is a locally* $p$*-finite family of open sets in* $X$ (*respectively, such that* $Y\cap U_n\neq\emptyset$), *then* $|\{n<\omega\colon U_n\neq\emptyset\}|<\aleph_0$.

**Proof.** We give the proof for the relative case.

(1) $\Rightarrow$ (2) Let $x\in X$ be a $p$-limit of $(U_n)_{n<\omega}$ with $Y\cap U_n$ for every $n<\omega$, and let $F\in p$. If $V$ is a neighborhood of $x$ then $G_V=\{n<\omega\colon V\cap U_n\neq\emptyset\}\in p$. Let $m\in F\cap G_V$. We have that $U_m\cap V\neq\emptyset$; so, $x\in\mathrm{cl}_X(\bigcup_{n\in F}U_n)$.

(2) $\Rightarrow$ (3) Let $(U_n)_{n<\omega}$ be a family of nonempty open sets that $p$-generates $\{A_j\colon j\in J\}$ relative to $Y$. Thus, for each $j\in J$, there exists $F_j\in p$ such that $\bigcup_{n\in F_j}U_n\subset A_j$. Therefore

$$\bigcap_{j\in J}\mathrm{cl}_X(A_j)\supset\bigcap_{j\in J}\mathrm{cl}_X\left(\bigcup_{n\in F_j}U_n\right)\supset\bigcap_{F\in p}\mathrm{cl}_X\left(\bigcup_{n\in F}U_n\right)\neq\emptyset.$$

(3) $\Rightarrow$ (4) This implication is trivial.

(4) $\Rightarrow$ (5) Assume that $\mathcal{U}=(U_n)_{n<\omega}$ is a locally $p$-finite family of nonempty open sets in $X$ such that $Y\cap U_n\neq\emptyset$. For each $x\in X$ let $V_x\in\mathcal{N}(x)$ be a cozero neighborhood of $x$ such that $\{n<\omega\colon V_x\cap U_n\neq\emptyset\}\notin p$. We have then that $\mathcal{V}=\{V_x\colon x\in X\}$ is a cover of $X$. It happens now that $\mathcal{W}=\{X\setminus V_x\colon x\in X\}$ is a family of zero sets $p$-generated by $(U_n)_{n<\omega}$ relative to $Y$, so it has the finite intersection property. Let $\mathcal{Z}$ be a $z$-ultrafilter on $X$ containing $\mathcal{W}$. By hypothesis, $\mathcal{Z}$ is $p$-real relative to $Y$, hence there is $x_0\in\bigcap\mathcal{W}$; but this means that $\mathcal{V}$ does not cover $X$, which is a contradiction.

(5) $\Rightarrow$ (1) It is easy to prove this implication. $\square$

**Definition 2.3.** Let $X$ be a space, $Y\subset X$ and $p\in\omega^*$.

(1) A collection $\{f_n\colon n<\omega\}$ of real-valued functions defined on $X$ is *locally* $p$*-zero relative to* $Y$ if for each $y\in Y$ we can find a neighborhood $V_y\in\mathcal{N}(y)$ which satisfies

$$\left\{n<\omega\colon V_y\subset f_n^{-1}(\{0\})\right\}\in p.$$

(2) A collection $\{f_n\colon n<\omega\}$ of real-valued functions defined on $X$ is *locally* $p$*-bounded relative to* $Y$ if there exists $r>0$ such that, for each $y\in Y$ we can find a neighborhood $V_y\in\mathcal{N}(y)$ which satisfies

$$\left\{n<\omega\colon V_y\subset f_n^{-1}([-r,r])\right\}\in p.$$

(3) A collection $\{f_n\colon n<\omega\}$ of real-valued functions defined on $X$ is *strongly locally* $p$*-bounded* (or $p$*-equicontinuous*) *relative to* $Y$ if for each $r>0$ and each $y\in Y$ there exists $V_{(y,r)}\in\mathcal{N}(y)$ such that

$$\left\{n<\omega\colon V_{(y,r)}\subset f_n^{-1}([-r,r])\right\}\in p.$$

(4) A collection of functions $\{f_n\colon\ n<\omega\}$ is *$p$-bounded relative to $Y$* if there exists $F\in p$ such that $\{f_n(y)\colon\ n\in F,\ y\in Y\}$ is bounded in $\mathbb{R}$.

We will say that $\{f_n\colon\ n<\omega\}$ is *locally $p$-zero* (respectively, *locally $p$-bounded, strongly locally $p$-bounded, $p$-bounded*) if the subset $Y$ coincides to the whole space $X$.

**Theorem 2.4.** *Let $X$ be a space, $Y\subset X$ and $p\in\omega^*$. Then, the following are equivalent*:

(1) *$X$ is $p$-pseudocompact* (*respectively, $Y$ is $p$-bounded in $X$*).
(2) *Each locally $p$-zero collection $\{f_n\in C(X)\colon\ n<\omega\}$ is $p$-bounded* (*respectively, relative to $Y$*).
(3) *Each strongly locally $p$-bounded collection $\{f_n\in C(X)\colon\ n<\omega\}$ is $p$-bounded* (*respectively, relative to $Y$*).
(4) *Each locally $p$-bounded collection $\{f_n\in C(X)\colon\ n<\omega\}$ is $p$-bounded* (*respectively, relative to $Y$*).

**Proof.** Observe that every locally $p$-zero family $\mathcal{F}=\{f_n\colon\ n<\omega\}$ is strongly locally $p$-bounded, and this implies that $\mathcal{F}$ is locally $p$-bounded. Then we have that the implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious. We are going to prove (2) $\Rightarrow$ (1) $\Rightarrow$ (4). We give the proof for the relative case.

(2) $\Rightarrow$ (1) Assume that $Y$ is not $p$-bounded in $X$. Then there is a sequence $(U_n)_{n<\omega}$ of nonempty open sets in $X$ such that $Y\cap U_n\neq\emptyset$ for each $n<\omega$, which is locally $p$-finite. For each $n<\omega$, we take $y_n\in Y\cap U_n$ and a continuous function $f_n: X\to[0,n]$ defined by $f_n(y_n)=n$ and $f_n(y)=0$ if $y\notin U_n$.

**Claim.** *The collection $\mathcal{F}=\{f_n\colon\ n<\omega\}$ is locally $p$-zero.*

Indeed, since $(U_n)_{n<\omega}$ is locally $p$-finite, for each $x\in X$ there exists $V_x\in\mathcal{N}(x)$ such that $\{n<\omega\colon\ V_x\cap U_n\neq\emptyset\}\notin p$. Thus $F=\{n<\omega\colon\ V_x\cap U_n=\emptyset\}\in p$. So, $\{n<\omega\colon\ V_x\subset f_n^{-1}(0)\}\supset F\in p$.

Now, for each $F\in p$ and each $n<\omega$ there is $n_F\in F$ with $n_F>n$. By definition, $f_{n_F}(y_{n_F})=n_F>n$. Hence, $\{f_n(x)\colon\ n\in F,\ x\in Y\}$ is not bounded. This means that $\mathcal{F}$ is not $p$-bounded relative to $Y$.

(1) $\Rightarrow$ (4) Assume that $Y$ is $p$-bounded in $X$ and let $\mathcal{F}=\{f_n\in C(X)\colon\ n<\omega\}$ be a locally $p$-bounded collection. So there exists $r>0$ such that for each $x\in X$ we can find $V_x\in\mathcal{N}(x)$ satisfying $\{n<\omega\colon\ V_x\subset f_n^{-1}([-r,r])\}\in p$. For each $n<\omega$, let $U_n=f_n^{-1}(\mathbb{R}\setminus[-r,r])$ and for each $F\in p$ consider the set $A_F=\bigcup_{n\in F}U_n$. Suppose that $Y\cap A_F\neq\emptyset$ for every $F\in p$. Observe that $\mathcal{A}$ is $p$-generated relative to $Y$. Since $Y$ is $p$-bounded in $X$, then $\bigcap_{F\in p}\mathrm{cl}_X(A_F)\neq\emptyset$ (Theorem 2.2). Let $x_0\in\bigcap_{F\in p}\mathrm{cl}_X(A_F)$. Thus, for each $F\in p$ and each $V\in\mathcal{N}(x_0)$ there exists $n(F,V)\in F$ and $x(F,V)\in V$ such that $|f_{n(F,V)}(x(F,V))|>r$. We define the set $G_V=\{n(F,V)\colon\ F\in p\}\subset\omega$ for each $V\in\mathcal{N}(x_0)$. If $G_V\notin p$, then $H_V=\omega\setminus G_V\in p$; so, because of the definition of $G_V$ and $n(H_V,V)$, it happens that $n(H_V,V)\in H_V\cap G_V$, which is not possible. Therefore $G_V\in p$ for every $V\in\mathcal{N}(x_0)$. We also define the set

$$T_V=\bigl\{n<\omega\colon\ |f_n(V)|\leqslant r\bigr\}\quad\text{for } V\in\mathcal{N}(x_0).$$

**Claim.** *For each $V \in \mathcal{N}(x_0)$, $T_V \cap G_V = \emptyset$.*

In fact, if $n \in G_V$ then $n = n(F, V)$ for some $F \in p$. But $|f_n(x(F, V))| > r$ with $x(F, V) \in V$, so $n \notin T_V$.

Since $G_V \in p$, $T_V \notin p$; though this contradicts our hypothesis about $\mathcal{F}$ in the point $x_0$. This contradiction was obtained by assuming that $Y \cap A_F \neq \emptyset$ for every $F \in p$; so there must be $E \in p$ such that $Y \cap A_E = \emptyset$. That is, for each $n \in E$, $Y \cap U_n = \emptyset$. This implies that for every $y \in Y$ and every $n \in E$, $|f_n(y)| \leqslant r$. Therefore $\mathcal{F}$ is $p$-bounded with respect to $Y$. $\square$

## 3. $p$-pseudocompactness and $p$-compactness

In this section we are going to realize how different $p$-pseudocompactness and $p$-compactness can be, even in classes of spaces with strong properties as that of sequentially compact $\alpha$-pseudocompact spaces with arbitrary $\alpha$. We construct our examples of $\alpha$-pseudocompact spaces satisfying an additional property $\mathcal{P}$ that is hereditary with respect to subspaces satisfying property $\mathcal{Q}$. The basic construction consists of: First we take a space $X$ that satisfies $\mathcal{P}$, then convenient compact spaces $bX$ and $K$, where $bX$ is a compactification of $X$ and $K \subset bX$. Finally we choose a cardinal number $\kappa$ such that:

(i) $\mathrm{cf}(\kappa) > \max\{\omega, \alpha\}$,
(ii) $(K \times [0, \kappa)) \cup (X \times \{\kappa\})$ is $C^*$-embedded in $(K \times [0, \kappa)) \cup (bX \times \{\kappa\})$, and
(iii) $(X \times \{\kappa\})$ satisfies $Q$ in $(K \times [0, \kappa)) \cup (X \times \{\kappa\})$.

**Definition 3.1.** We will say that a space $X$ is *ultrapseudocompact* if $X$ is $p$-pseudocompact for every $p \in \omega^*$.

Given an infinite cardinal number $\alpha$ and $p \in \omega^*$, there exist spaces which are $p$-pseudocompact and $\alpha$-pseudocompact. In fact, the space of ordinal numbers $[0, \alpha^+)$ is ultrapseudocompact and $\alpha$-pseudocompact; and, of course, every compact space $X$ is ultrapseudocompact and $\alpha$-pseudocompact for every cardinal number $\alpha$. On the other hand, there are $\alpha$-pseudocompact spaces which are not $p$-pseudocompact for any $p \in \omega^*$.

**Example 3.2.** Let $\alpha$ be a cardinal number. There exists a space $Y$ which is $\alpha$-pseudocompact and is not $p$-pseudocompact for any $p \in \omega^*$.

**Proof.** In fact, in [15] it was shown that if $X$ is a pseudocompact subspace of $\beta(\omega)$ with $\omega \subset X$, and $\kappa$ is a cardinal number with $\mathrm{cf}(\kappa) > 2^{\mathfrak{c}}$, then $X$ is homeomorphic to the regular closed subspace $X \times \{\kappa\}$ of the space $Y = Y(X, \kappa) = (\omega^* \times [0, \kappa)) \cup (X \times \{\kappa\})$, where $Y$ has the topology inherited by that of the product in $\beta(\omega) \times [0, \kappa]$. The space $Y$ is $C^*$-embedded in $(\omega^* \times [0, \kappa)) \cup (\beta(\omega) \times \{\kappa\}) \subset \beta(\omega) \times [0, \kappa]$, so $\beta(Y) = (\omega^* \times [0, \kappa)) \cup (\beta(\omega) \times \{\kappa\})$. Moreover, $Y$ is $\alpha$-pseudocompact if $\mathrm{cf}(\kappa) > \alpha$, because, in this case, $Y$ is $G_\alpha$-dense in $\beta(Y)$ (see Theorem 1.2 in [14]). It is shown by Comfort [3] and Frolík [9] that all powers of $\Sigma(p) = \omega \cup T(p)$ are pseudocompact for every $p \in \omega^*$, and García-Ferreira proved in [11] that if $p \in \omega^*$ is not *RK*-minimal, then $\Sigma(p)$ is not $q$-pseudocompact for every $q \in \omega^*$. Therefore, the space $Y(\Sigma(p), \kappa)$, where $p \in \omega^*$ is not *RK*-minimal and $\mathrm{cf}(\kappa) > \max\{2^{\mathfrak{c}}, \alpha\}$, is $\alpha$-pseudocompact and is not $p$-pseudocompact for any $p \in \omega^*$ because this property is inherited by regular closed subsets. $\square$

It is clear that every $p$-compact space is $p$-pseudocompact for $p \in \omega^*$, but these two properties are not equivalent. Indeed, the space $Y = (A(\omega) \times [0, \kappa]) \setminus \{(x_0, \kappa)\}$ where $A(\omega) = \omega \cup \{x_0\}$ is the one-point compactification of the natural numbers, and $\mathrm{cf}(\kappa) > \omega$, is an ultrapseudocompact locally compact and $\alpha$-pseudocompact space for every $\alpha < \mathrm{cf}(\kappa)$, and it is not $p$-compact for any $p \in \omega^*$, because it is not countably compact. Even more, we can give an example of a totally countably compact space with these properties, answering Question 6.5 in [12] affirmatively:

**Example 3.3.** Let $\alpha$ be a cardinal number. There exists a space $Y$ which is an ultrapseudocompact, totally countably compact and $\alpha$-pseudocompact space; besides, $Y$ is not $q$-compact for any $q \in \omega^*$.

**Proof.** We obtain this example by modifying Example 2.14 in [27]. For each $q \in \omega^*$ let $K_q$ be the subspace $\beta(\omega) \setminus \{q\}$ of $\beta(\omega)$. Every infinite set $E$ in $K_q$ has a cluster point $r$ in $\beta(\omega)$ with $r \neq q$. Thus any closed neighborhood of $r$ which does not contain $q$ will be a compact subset of $K_q$ containing an infinite subset of $E$. Then $K_q$ is totally countably compact, but is not $q$-compact since the only $q$-limit point in $\beta(\omega)$ of the countable set $\omega \subset K_q$ is $q$. Take a cardinal number $\kappa$ such that $\mathrm{cf}(\kappa) > \max\{\omega, \alpha\}$. For each $q \in \omega^*$, let $Z_q = \beta(\omega) \times \{q\}$ be a copy of $\beta(\omega)$. Let $x_0$ be a point not belonging to any $Z_q$, and let

$$Z = \left( \bigoplus_{q \in \omega^*} Z_q \right) \cup \{x_0\}$$

be the one-point compactification of the free topological sum of the spaces $Z_q$. Let $X$ be equal to $(\bigoplus_{q \in \omega^*} (K_q \times \{q\})) \cup \{x_0\}$ with the topology inherited from $Z$. Space $X$ is totally countably compact and is not even $q$-pseudocompact for any $q \in \omega^*$ (see [27]). Now, consider the subspace $Y = (Z \times [0, \kappa)) \cup (X \times \{\kappa\})$ of the compact space $Z \times [0, \kappa]$. The space $Y$ is totally countably compact. Besides, $X \times \{\kappa\}$ is closed in $Y$, so $Y$ is not $q$-compact for any $q \in \omega^*$ because this property is hereditary with respect to closed subsets. If $U$ is an open set in $Y$, then $U \cap (Z \times [0, \kappa)) \neq \emptyset$, so if $(U_n)_{n<\omega}$ is a sequence of open sets in $Y$ and for each $n < \omega$, $(x_n, y_n) \in U_n \cap (Z \times [0, \kappa))$, then $\{y_n\}_{n<\omega} \subset [0, \lambda]$ for some $\lambda < \kappa$. Since $Z \times [0, \lambda]$ is a compact space, there is a $p$-limit point $a \in Y$ of the sequence $(x_n, y_n)_{n<\omega}$. The point $a$ is a $p$-limit point of $(U_n)_{n<\omega}$. Therefore, $Y$ is ultrapseudocompact. Finally, $Y$ is $\alpha$-pseudocompact because, $Y$ is $C^*$-embedded in $Z \times [0, \kappa]$, so $\beta(Y) = Z \times [0, \kappa]$, and $Y$ is $G_\alpha$-dense in $\beta(Y)$ because of $\mathrm{cf}(\kappa) > \alpha$. $\square$

Now we are going to discuss the relationship between $p$-pseudocompactness and $p$-compactness in the class of sequentially compact spaces. First some definitions.

A space $X$ is *finally $\alpha$-compact*, where $\alpha$ is a cardinal number, if for every open cover of $X$ there exists a subcover of cardinality less than $\alpha$.

Define the quasi-order $\leqslant^*$ on ${}^\omega\omega$ by $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many $n < \omega$. A subset of ${}^\omega\omega$ is called *unbounded* if it is unbounded in $({}^\omega\omega, \leqslant^*)$.

Define the quasi-order $\subset^*$ on $\mathcal{P}(\omega)$ by

$$F \subset^* G \quad \text{if } F \setminus G \text{ is finite.}$$

We say that $A$ is a *pseudo-intersection* of a family $\mathcal{F}$ if $A \subset^* F$ for each $F \in \mathcal{F}$. We call $\mathcal{T} \subset [\omega]^\omega$ a *tower* if $\mathcal{T}$ is well ordered by $\supset^*$ and has no infinite pseudointersection. We say that a family of countable sets has the *strong finite intersection property* (*sfip*) if every nonempty finite subfamily has an infinite intersection.

Consider the following cardinal numbers introduced by van Douwen in [6]:

$$\mathfrak{b} = \min\bigl\{|B|\colon\ B \text{ is an unbounded subset of } {}^\omega\omega\bigr\};$$

$$\mathfrak{p} = \min\bigl\{|\mathcal{F}|\colon\ \mathcal{F} \text{ is a subfamily of } [\omega]^\omega \text{ with the } \textit{sfip} \text{ which has no infinite pseudointersection}\bigr\};$$

$$\mathfrak{t} = \min\bigl\{|\mathcal{T}|\colon\ \mathcal{T} \text{ is a tower}\bigr\}.$$

It was proved in [6], Theorem 3.7, that

$$\mathfrak{t} = \min\bigl\{\mathcal{T}\colon\ \mathcal{T} \subset [\omega]^\omega \text{ is well ordered by } \subset^* \text{ and } \forall T \in \mathcal{T},\ (\omega \setminus T) \text{ is infinite, and } \forall H \in [\omega]^\omega\ \exists T \in \mathcal{T} \text{ such that } (H \cap T) \text{ is infinite}\bigr\}.$$

**Example 3.4.** Let $\alpha$ be a cardinal number. There is a sequentially compact ultrapseudocompact $\alpha$-pseudocompact locally compact and zero-dimensional space $Y$, which is not ultracompact.

**Proof.** Let $X = [\omega, \mathfrak{t}) \cup \omega$. There exists

$$\mathcal{T} = \{H_\lambda\colon\ \omega \leqslant \lambda < \mathfrak{t}\} \subset [\omega]^{\leqslant\omega}$$

such that

(1) $H_\omega = \emptyset$ and $|H_\lambda| = \aleph_0$ for every $\omega < \lambda < \mathfrak{t}$;
(2) $\omega \leqslant \gamma < \lambda$ implies $H_\gamma \subset^* H_\lambda$;
(3) for every $\omega < \lambda < \mathfrak{t}$, $|\omega \setminus H_\lambda| = \aleph_0$; and
(4) for every $H \in [\omega]^\omega$ there exists $\omega \leqslant \lambda < \mathfrak{t}$ such that $|H \cap H_\lambda| = \aleph_0$.

We topologize $X$ as follows: $\omega$ and points of $\omega$ are isolated, and a basic neighborhood of $\omega < \lambda < \mathfrak{t}$ has the form

$$N(\gamma, \lambda; F) = (\gamma, \lambda] \cup \bigl((H_\lambda \setminus H_\gamma) \setminus F\bigr),$$

where $\omega \leqslant \gamma < \lambda$ and $F \in [\omega]^{<\omega}$. It was proved by van Douwen [6, Example 7.1], that $X$ is a non-compact zero-dimensional separable sequentially compact locally compact normal space. Even more, he proved that $X$ is an almost compact space (that is $|\beta(X) \setminus X| = 1$). Thus, $X$ is $\alpha$-pseudocompact for every $\alpha < \mathfrak{t}$. Besides, $\mathrm{cl}_X\, \omega = X$ is not compact, so $X$ is not $\omega$-bounded, that is, $X$ is not ultracompact.

Let $A(X)$ be the one-point compactification of $X$ (in this case $A(X) = \beta(X)$), and let $\kappa$ be a cardinal number with cofinality bigger than $\max(\omega, \alpha, \mathfrak{t})$. The space $A(X)$ is still sequentially compact because of Theorem 6.3 in [6]. Consider the product $A(X) \times [0, \kappa]$ and the subspace $Y = (A(X) \times [0, \kappa)) \cup (X \times \{\kappa\})$. Since $X$ and $A(X) \times [0, \lambda]$ are sequentially compact for every ordinal number $\lambda$ (see [6, Theorem 6.9]), then $Y$ is sequentially compact; and because of the fact that $X$ is closed in $Y$, this space is not ultracompact. Using similar arguments as in Example 3.4, it is possible to prove that $Y$ is ultrapseudocompact and $\alpha$-pseudocompact; moreover, $Y$ is locally compact and zero-dimensional. $\square$

**Example 3.5** $[\mathfrak{b} = \mathfrak{c}]$. Let $\alpha$ be a cardinal number. There is a sequentially compact zero-dimensional ultrapseudocompact $\alpha$-pseudocompact space which is not $p$-compact for any $p \in \omega^*$.

**Proof.** In [6, Example 13.1], van Douwen constructed, for each $p \in \omega^*$ a first countable countably compact (hence sequentially compact) locally compact zero-dimensional (and separable) space $X_p$ such that $\prod_{p\in\omega^*} X_p$ is not countably compact. We have that the space $X_p$ is not $p$-compact (it is not even $p$-pseudocompact). For each $p \in \omega^*$, let $K_p = \beta(X_p)$ be the Stone–Čech compactification of $X_p$, and let $K = \bigoplus_{p\in\omega^*} K_p$ be the free topological sum of the family $\{K_p\colon\ p \in \omega^*\}$. Take the one point compactification of the space $K$, $\widehat{K} = K \cup \{x_0\}$, where $x_0$ is a point not belonging to $K$. Choose a cardinal number $\kappa$ such that $\mathrm{cf}(\kappa) > \max\{\omega, \alpha, \mathfrak{c}\}$, and consider the product $\widehat{Y} = \widehat{K} \times [0, \kappa]$. Finally, consider the subspace $Y = (\widehat{K} \times [0, \kappa)) \cup (X \times \{\kappa\})$ of $\widehat{Y}$, where $X$ is the subspace $(\bigoplus_{p\in\omega^*} X_p) \cup \{x_0\}$ of $\widehat{K}$. Because of the same arguments given in the previous examples, we have that $Y$ is ultrapseudocompact ($\mathrm{cf}(\kappa) > \omega$). It is not $p$-compact for any $p \in \omega^*$ because there is a sequence in $X_p \times \{\kappa\}$ without a $p$-limit in $Y$. Using similar reasoning to that in Example 3.3, we conclude that $Y$ is sequentially compact. Besides, $Y$ is $\alpha$-pseudocompact because it is $C^*$-embedded and $G_\alpha$-dense in $\widehat{Y}$ (here we are using the facts that $\mathrm{cf}(\kappa) > \max\{\alpha, \mathfrak{c}\}$ and $|X_p| = \mathfrak{c}$ for every $p \in \omega^*$). $\square$

One of the common characteristics of all the examples we have already constructed is that none of them is neither normal nor first countable. Using Corollary 6.6 in [27], we deduce that every normal finally $\mathfrak{p}$-compact space which is $p$-pseudocompact for at least a $p \in \omega^*$ is ultra-compact; and applying Theorem 6.8 in [27] we conclude that assuming $\mathfrak{p} > \omega_1$, every perfectly normal $p$-pseudocompact space is compact. Furthermore, if $p \in \omega^*$ and $(U_n)_{n<\omega}$ is a sequence of open sets in $\omega^* \setminus \{p\}$, then, since $p$ does not have a countable local base, there is an open set $U$ in $\omega^*$ that contains $p$ and such that $U_n \setminus U \neq \emptyset$. If $q_n \in U_n \setminus U$, then $K = \mathrm{cl}_{\omega^*}\{q_n\colon\ n < \omega\}$ is a compact subset in $\omega^* \setminus \{p\}$. Thus, the $r$-limit point of $(q_n)_{n<\omega}$ in $K$ is an $r$-limit point of $(U_n)_{n<\omega}$ where $r \in \omega^*$. Therefore, $\omega^* \setminus \{p\}$ is an ultrapseudocompact space. On the other hand, if $p \in \omega^*$ is not a weakly $P$-point, then $\omega^* \setminus \{p\}$ is not ultracompact because if $p$ belongs to the closure of $\{q_n\colon\ n < \omega\}$, then there is $q \in \omega^*$ such that $p$ is the $q$-limit of the sequence $(q_n)_{n<\omega}$. It is known that if $p \in \omega^*$ is an accumulation point of some countable discrete subset of $\omega^*$, then $\omega^* \setminus \{p\}$ is not normal. Besides, under CH, every $\omega^* \setminus \{p\}$ is not normal (see, for example, [22]) and it is not known yet if it is consistent with ZFC that there exists a non $P$-point $p \in \omega^*$ for which $\omega^* \setminus \{p\}$ is normal. (Also note that in the class of paracompact spaces, $p$-pseudocompactness implies compactness, because in this class pseudocompactness and compactness coincide.) These facts suggest the following questions:

**Question 3.6.** Is it consistent with ZFC that, for every $p \in \omega^*$, each $p$-pseudocompact space satisfying $P \in \mathcal{P}$ must be $p$-compact, where $\mathcal{P} = \{$normality, perfect normality, collectionwise normality, normality + countable paracompactness$\}$?

**Question 3.7.** Is every $p$-pseudocompact first countable Tychonoff (respectively, normal) space, a $p$-compact space?

## 4. $p$-boundedness and products

It was proved in [11, Theorem 1.3], that an arbitrary product of $p$-bounded subsets is also a $p$-bounded subset. In this section we improve this result by showing a version of the classical Glisckberg Theorem on pseudocompactness which characterizes when the arbitrary product of pseudocompact spaces is pseudocompact by using the distribution of the functor of the Stone–Čech compactification. As a consequence we obtain that, for each cardinal $\alpha$, $C_\alpha$-compactness is preserving under products of $C_\alpha$-compact $p$-bounded subsets. First, we need several notations. Let $f \in C(X \times Y)$. For each $x \in X$ we will denote by $f_x$ the function from $Y$ into $\mathbb{R}$ defined by the requirement $f_x(y) = f(x, y)$ whenever $y \in Y$. For $y \in Y$, the function $f^y$ is defined in a similar way. We will denote by $\gamma X$ the *Dieudonné completion* or *universal completion* of $X$. For each $f \in C(X)$, $f^\gamma$ indicates the continuous extension of $f$ to $\gamma X$. Let $\mathcal{H}$ be a family of real-valued continuous function on $X \times Y$ and let $A, B$ be two subsets of $X$ and $Y$, respectively. We shall define the families $\mathcal{H}_B, \mathcal{H}_A$ as follows:

$$\mathcal{H}_B = \{f^y\colon\ y \in B,\ f \in \mathcal{H}\}, \qquad \mathcal{H}_A = \{f_x\colon\ x \in A,\ f \in \mathcal{H}\}.$$

The symbols $\mathcal{H}^*_B|A, \mathcal{H}^*_A|B$ denote the families:

$$\mathcal{H}^*_B|A = \{F^y\colon\ f^y \in \mathcal{H}_B\}, \qquad \mathcal{H}^*_A|B = \{F_x\colon\ f_x \in \mathcal{H}_A\},$$

where $F^y$, $F_x$ are defined as

$$F^y = (f^y)^\gamma|_{\mathrm{cl}_{\gamma X} A}, \qquad F_x = (f_x)^\gamma|_{\mathrm{cl}_{\gamma Y} B}$$

for $(x, y) \in X \times Y$.

We recall that a family $\mathcal{H}$ of real-valued continuous functions on $X$ is said to be *equicontinuous* if for each $x \in X$ and every $\varepsilon > 0$ there exists a neighborhood $V$ of $x$ such that $|f(y) - f(x)| < \varepsilon$ whenever $y \in V$ and $f \in \mathcal{H}$. $\mathcal{H}$ is called *pointwise bounded* if $\{f(x)\colon\ f \in \mathcal{H}\}$ is bounded in $\mathbb{R}$ for every $x \in X$. Let

$$\operatorname{osc}(f, V) = \sup\bigl\{\bigl|f(x) - f(y)\bigr|\colon\ x, y \in V\bigr\}.$$

**Theorem 4.1.** *If $\mathcal{H}$ is an equicontinuous pointwise bounded family of real-valued functions on $X \times Y$, then for each $(x_0, y_0) \in X \times \gamma Y$ and each $\varepsilon > 0$ there exists a regular closed neighborhood $V_{y_0}$ of $y_0$ in $\gamma Y$ such that*

$$\operatorname{osc}(F_{x_0}, V_{y_0}) < \varepsilon$$

*whenever $F_{x_0} \in \mathcal{H}^*_Y|X$.*

**Proof.** Let $(x_0, y_0) \in X \times \gamma Y$ and let $\varepsilon > 0$. Define

$$\mathcal{H}_{x_0} = \{f_{x_0}\colon\ f \in \mathcal{H}\}.$$

It is clear that $\mathcal{H}_{x_0}$ is equicontinuous and pointwise bounded on $Y$. By [24, Theorem 7], the family $\{f^\gamma_{x_0}\colon\ f \in \mathcal{H}\}$ is equicontinuous (and pointwise bounded) in $\gamma Y$. Therefore, we can find a regular closed neighborhood $V_{y_0}$ of $y_0$ in $\gamma Y$ such that $|f^\gamma_{x_0}(y) - f^\gamma_{x_0}(y_0)| < \frac{1}{2}\varepsilon$ whenever $y \in V_{y_0}$. The result now follows from the triangle inequality. □

**Remark 4.2.** Notice that, in the previous theorem, we can replace $X \times \gamma Y$ and $F_{x_0}$ by $\gamma X \times Y$ and $F^{y_0}$, respectively.

The following result on extensions will be used in the sequel. In the proof we follow the patterns given in [19, Lemma 4.4].

**Theorem 4.3.** *Let $A$, $B$ be two $p$-bounded subsets of $X$ and $Y$, respectively. Let $\mathcal{H}$ be an equicontinuous pointwise bounded family of real-valued functions on $X \times Y$. Then $\mathcal{H}^*_A|B$ is equicontinuous on $\operatorname{cl}_{\gamma Y} B$.*

**Proof.** Suppose that there exists $y_0 \in \operatorname{cl}_{\gamma Y} B$ such that $\mathcal{H}^*_A|B$ is not equicontinuous at $y_0$. We will define by induction a sequence $(f^n)_{n<\omega} \subset \mathcal{H}$, a sequence $((a_n, y_n))_{n<\omega}$ of points in $X \times Y$, and two sequences $(V^*_n)_{n<\omega}$, $(U_n \times V_n)_{n<\omega}$ of regular closed subsets of $\gamma Y$ and $X \times Y$ respectively, satisfying:

(1) $|F^n_{a_n}(y_n) - F^n_{a_n}(y_0)| > \varepsilon$ for each $n < \omega$,

(2) for each $n < \omega$, $V^*_n$ is a neighborhood of $y_0$ (in $\gamma Y$) and $\operatorname{osc}(F^n_{a_n}, V^*_n) < \frac{1}{5}\varepsilon$,

(3) for each $n < \omega$, $(U_n \times V_n)$ is a neighborhood of $(a_n, y_n)$ (in $X \times Y$) with $\operatorname{osc}(f^n, U_n \times V_n) < \frac{1}{5}\varepsilon$,

(4) for each $n < \omega$, $\operatorname{int}_Y V_n \subset \operatorname{int}_{\gamma Y} V^*_{n-1}$ and $\operatorname{int}_{\gamma Y} V^*_n \subset \operatorname{int}_{\gamma Y} V^*_{n-1}$.

Since $\mathcal{H}^*_A|B$ is not equicontinuous at $y_0$, then there exist $\varepsilon > 0$, $(x, y) \in A \times B$ and $f \in \mathcal{H}$ such that

$$\bigl|F_x(y) - F_x(y_0)\bigr| > \varepsilon.$$

For $n = 1$, we define $a_1 = x$, $y_1 = y$ and $f^1 = f$. Since $f^1$ is continuous on $X \times Y$, we can find a regular closed neighborhood (in $X \times Y$), $U_1 \times V_1$, of $(a_1, y_1)$ such that

$$\operatorname{osc}(f^1, U_1 \times V_1) < \tfrac{1}{5}\varepsilon.$$

By Theorem 4.1, there exists a regular closed neighborhood (in $\gamma Y$) $V^*_1$ of $y_0$ such that

$$\operatorname{osc}(F_{a_1}, V^*_1) < \tfrac{1}{5}\varepsilon$$

for each $f \in \mathcal{H}$. This completes the step $n = 1$. For $n > 1$, since $\mathcal{H}^*_A|B$ is not equicontinuous at $y_0$, there exist $y_n \in \operatorname{int}_{\gamma Y} V^*_{n-1} \cap B$, $a_n \in A$ and $f^n \in \mathcal{H}$ such that

$$\left|F^n_{a_n}(y_n) - F^n_{a_n}(y_0)\right| > \varepsilon.$$

Let $U_n \times V_n$ be a regular closed neighborhood (in $X \times Y$) of $(a_n, y_n)$ with $\operatorname{int}_Y V_n \subset \operatorname{int}_{\gamma Y} V^*_{n-1}$ such that

$$\operatorname{osc}(f^n, U_n \times V_n) < \tfrac{1}{5}\varepsilon.$$

By Theorem 4.1, we can find a neighborhood (in $\gamma Y$) $V^*_n$ of $y_n$ with $V^*_n \subset V^*_{n-1}$ such that

$$\operatorname{osc}(F^n_{a_n}, V^*_n) < \tfrac{1}{5}\varepsilon.$$

This completes the induction. Now, since $V_n \cap B \neq \emptyset$ for each $n < \omega$, the sequence $(V_n)_{n<\omega}$ has a $p$-limit $\widehat{y} \in Y$. Applying condition (4) it is easy to check that $\widehat{y}$ is also a cluster point of the sequence $(V^*_n)_{n<\omega}$ and, consequently, $\widehat{y} \in V^*_n$ for each $n < \omega$. Since each $(f^n)^{(\widehat{y})}$ is continuous on $X$, there exists a sequence $(T_n)_{n<\omega}$ of regular closed sets in $X$ such that $a_n \in \operatorname{int}_X T_n \subset U_n$ and $\operatorname{osc}((f^n)^{(\widehat{y})}, T_n) < \frac{1}{5}\varepsilon$ for each $n < \omega$. Let $\widehat{x}$ be a $p$-limit of $(T_n)_{n<\omega}$. Then $(\widehat{x}, \widehat{y})$ is both a $p$-limit of $(T_n \times V_n)_{n<\omega}$ and of $(T_n \times V^*_n)_{n<\omega}$. Since $\mathcal{H}$ is equicontinuous on $X \times Y$, we can find an open subset $U \times V$ with $(\widehat{x}, \widehat{y}) \in U \times V$ such that

$$\left|f(x, y) - f(\widehat{x}, \widehat{y})\right| < \tfrac{1}{5}\varepsilon$$

whenever $f \in \mathcal{H}$ and $(x, y) \in U \times V$.

Let $H = \{n < \omega\colon (U \times V) \cap (T_n \times V_n) \neq \emptyset\}$. By condition (4), $H$ is contained in $\{n < \omega\colon (U \times V) \cap (T_n \times V^*_n) \neq \emptyset\}$. It follows from condition (3) that

$$\left|f^n(\widehat{x}, \widehat{y}) - f^n(a_n, y_n)\right| < \tfrac{2}{5}\varepsilon$$

whenever $n \in H$.

On the other hand, since $\widehat{y} \in V^*_n$, for each $n < \omega$, $|f^n(a_n, \widehat{y}) - F^n_{a_n}(y_0)| < \frac{1}{5}\varepsilon$ for each $n < \omega$. Moreover, because $\operatorname{osc}((f^n)^{(\widehat{y})}, T_n) < \frac{1}{5}\varepsilon$ for each $n < \omega$, if $a \in T_n$ then

$$\left|f^n(a, \widehat{y}) - f^n(a_n, \widehat{y})\right| < \tfrac{1}{5}\varepsilon.$$

As a consequence, $|f^n(a_n, y_n) - F^n_{a_n}(y_0)| < \varepsilon$ whenever $n \in H$. This contradicts that

$$\left|F^n_{a_n}(y_n) - F^n_{a_n}(y_0)\right| > \varepsilon$$

for $n = 1, 2, \ldots.$ □

We recall that a subset $A$ of a space $X$ is said to be *bounded* (in $X$) if every real-valued continuous function on $X$ is bounded on $A$. It is not hard to see that a subset $A$ is bounded in $X$ if every sequence of open sets (in $X$) meeting $A$ has a cluster point in $X$. So, if $p \in \omega^*$, every $p$-bounded subset is bounded and a topological space is pseudocompact if and only if it is bounded in itself. However there exist pseudocompact spaces which are not $p$-pseudocompact for any $p \in \omega^*$ (see Example 3.2 and [17]). We make mention of a lemma proved by Pupier in [23, Lemma 3.3]. For each $A \subset X$, we will denote by $A_\gamma$ the uniform space defined as $A$ endowed with the restriction of the finest uniformity on $X$.

**Lemma 4.4.** *Let $A$, $B$ be two bounded subsets of $X$ and $Y$, respectively. Then the following assertions are equivalent*:

(1) *For each equicontinuous and pointwise bounded family $\mathcal{H}$ in $C(X \times Y)$, $\mathcal{H}_A$ (respectively, $\mathcal{H}_B$) is uniformly equicontinuous on $B_\gamma$ (respectively, on $A_\gamma$).*
(2) $(A \times B)_\gamma = A_\gamma \times B_\gamma$.

We recall that a compactification of a space $X$ is a compact space $K$ such that $X$ is dense in $K$. Two compactifications $K_1, K_2$ of $X$ are called equivalent if there exists a homeomorphism $\Phi$ from $K_1$ onto $K_2$ which leaves $X$ pointwise fixed. We will write $K_1 = K_2$ whenever $K_1$ and $K_2$ are two equivalent compactifications of $X$. If $A \subset X$, it is well known that $\mathrm{cl}_{\gamma X} A$ is the completion of $A_\gamma$ (see, for example, [7, Theorems 8.3.6 and 8.3.12] for details). Let $p \in \omega^*$. Since every $p$-bounded subset $A$ is bounded, $\mathrm{cl}_{\gamma X} A$ is a compactification of $A$. In fact, it is well known that $\mathrm{cl}_{\gamma X} A = \mathrm{cl}_{\beta X} A$ for each bounded subset $A$ of $X$. We can apply this fact and the previous results in order to obtain:

**Theorem 4.5.** *If $A_i$ is a $p$-bounded subset of $X_i$ for $i = 1, 2, \ldots, n$, then*

$$\mathrm{cl}_{\gamma X} \prod_{i=1}^{n} A_i = \prod_{i=1}^{n} \mathrm{cl}_{\gamma X_i} A_i,$$

*where $X = \prod_{i=1}^{n} X_i$.*

**Proof.** First, we shall study the case $n = 2$. Applying Theorem 4.3, the subsets $A_1$ and $A_2$ satisfy condition (1) in Lemma 4.4. So, $\mathrm{cl}_{\gamma(X_1 \times X_2)}(A_1 \times A_2)$ and $\mathrm{cl}_{\gamma X_1} A_1 \times \mathrm{cl}_{\gamma X_2} A_2$ are completions of the uniform space $(A_1 \times A_2)_\gamma$. So, because the identity mapping on $A_1 \times A_2$ is a uniform isomorphism on $(A_1 \times A_2)_\gamma$, it is extendable to a uniform isomorphism from $\mathrm{cl}_{\gamma(X_1 \times X_2)}(A_1 \times A_2)$ onto $\mathrm{cl}_{\gamma X_1} A_1 \times \mathrm{cl}_{\gamma X_2} A_2$ (Theorem 8.3.11 in [7]).

Now, since $p$-boundedness is preserving under arbitrary products, the general case follows from a straightforward induction argument. □

**Corollary 4.6.** *Let $\{p_i\}_{i=1}^{n} \subset \omega^*$ be such that there exists $q \in \omega^*$ with $q \leqslant_{RK} p_i$ for each $i = 1, 2, \ldots, n$. If $A_i$ is a $p_i$-bounded subset of $X_i$ for $i = 1, 2, \ldots, n$, then the restriction to $\prod_{i=1}^{n} A_i$ of each real continuous function on $\prod_{i=1}^{n} X_i$ admits a continuous extension to $\prod_{i=1}^{n} \mathrm{cl}_{\gamma X_i} A_i$.*

**Corollary 4.7.** *Let $X = \prod_{i=1}^{n} X_i$ and let $\{p_i\}_{i=1}^{n} \subset \omega^*$ be such that there exists $q \in \omega^*$ with $q \leqslant_{RK} p_i$ for each $i = 1, 2, \ldots, n$. If $A_i$ is a $p_i$-bounded subset of $X_i$ for $i = 1, 2, \ldots,$ then*

$$\mathrm{cl}_{\gamma X} \prod_{i=1}^{n} A_i = \prod_{i=1}^{n} \mathrm{cl}_{\gamma X_i} A_i.$$

As for each bounded subset $A$ of $X$, $\mathrm{cl}_{\gamma X} A = \mathrm{cl}_{\beta X} A$, Theorem 4.5 is a version of the classical Glisckberg Theorem on pseudocompactness in the realm of $p$-bounded subsets. Our goal in the sequel is to obtain a version of the Glisckberg Theorem for arbitrary products. Lemma 4.9 below is a straightforward version of Lemma 2.5 in [15]. We first need the following result; its proof is a routine adaptation of the proof of Lemma 2.1 in [8] and it is left to the reader.

**Lemma 4.8.** *Let $X, Y$ be two topological spaces and let $A, B$ be two infinite subsets of $X$ and $Y$, respectively. If $A \times B$ is not bounded in $X \times Y$, then there exists a locally finite family $\{U_n \times V_n\}_{n<\omega}$ of nonempty canonical open sets of $X \times Y$ meeting $A \times B$ such that the families $\{U_n\}$, $\{V_n\}$ are pairwise disjoint.*

**Lemma 4.9.** *Let $X, Y$ be two topological spaces and let $A, B$ be two infinite subsets of $X$ and $Y$, respectively. If $\mathrm{cl}_{\beta(X\times Y)}(A \times B) = \mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$, then $A \times B$ is bounded in $X \times Y$.*

**Proof.** Suppose that $A \times B$ is not bounded in $X \times Y$. According to Lemma 4.8 we can find a locally finite family $\{U_n \times V_n\}_{n<\omega}$ of open canonical sets (in $X \times Y$) meeting $A \times B$ such that $\{U_n\}_{n<\omega}$ and $\{V_n\}_{n<\omega}$ are pairwise disjoint. Choose, for each $n < \omega$, a point $t_n = (x_n, y_n) \in (U_n \times V_n) \cap (A \times B)$. Now, for each $n < \omega$, consider a real-valued continuous function $f_n$ on $X \times Y$ satisfying:

$$0 \leqslant f_n \leqslant 1, \quad f_n(t_n) = 1, \qquad f_n\big((X \times Y) \setminus (U_n \times V_n)\big) = 0.$$

Since $\{U_n \times V_n\}_{n<\omega}$ is locally finite, the real-valued function $f$ on $X \times Y$ defined as

$$f(x, y) = \sup_{n<\omega} f_n(x, y), \quad (x, y) \in X \times Y$$

is a bounded real-valued continuous function. Let $g = f^\beta|_{\mathrm{cl}_{\beta(X\times Y)}(A\times B)}$. By hypothesis, $\mathrm{cl}_{\beta(X\times Y)}(A \times B)$ is a compactification of $A \times B$ equivalent to $\mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$. So, we can consider that $g$ is defined on $\mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$. Let $(x, y) \in \mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$ a cluster point of the sequence $\{(x_n, y_n)\}_{n<\omega}$. Then, for each canonical open set (in $\mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$) $U \times V$ containing $(x, y)$, we can choose $n, m \in \omega$ with $n \neq m$ such that $(x_n, y_n)$ and $(x_m, y_m)$ belong to $U \times V$. So, $(x_n, y_m) \in U \times V$ and, consequently, $g(x, y) = 0$. But, since $(x, y)$ is a cluster point of the sequence $\{(x_n, y_n)\}_{n<\omega}$, $g(x, y) = 1$, a contradiction. $\square$

**Theorem 4.10.** *Let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces, $X = \prod_{\alpha\in I} X_\alpha$ and $\{A_\alpha\}_{\alpha\in I}$ be a family of sets such that $A_\alpha$ is bounded in $X_\alpha$ for each $\alpha \in I$. If*

$$\mathrm{cl}_{\beta X} \prod_{\alpha\in I} A_\alpha = \prod_{\alpha\in I} \mathrm{cl}_{\beta X_\alpha} A_\alpha,$$

*then $\prod_{\alpha\in I} A_\alpha$ is bounded in $X$.*

**Proof.** First notice that, as the product of a compact subset and a bounded subset is bounded (see, for example, Proposition 1 in [2]), we can suppose, without loss of generality, that the family $\{A_\alpha\}_{\alpha\in I}$ contains at least two infinite sets $A_\sigma$ and $A_\eta$. Let $J = I \setminus \{\sigma\}$. We shall prove that $\mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha$ and $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$ are equivalent compactifications of $\prod_{\alpha\in J} A_\alpha$ where $Y=\prod_{\alpha\in J} X_\alpha$. To see this, according to [16, 10E] it suffices to prove that the functions in $C^*(\prod_{\alpha\in J} A_\alpha)$ that are extendable to $\mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha$ are the same as those extendable to $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$.

Let $f\in C(\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha)$. Since $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$ is compact, there exists a continuous extension $g$ of $f$ to $\prod_{\alpha\in J}\beta X_\alpha$. Consider $h = g|_{\prod_{\alpha\in J} X_\alpha}$. It is clear that the restriction of $h^\beta$ to $\mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha$ is a continuous extension of $f|_{\prod_{\alpha\in J} A_\alpha}$. Conversely, if $f\in C(\mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha)$, the restriction of $f$ to $\prod_{\alpha\in J} A_\alpha$ admits a continuous extension to $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$. In fact, since $\mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha$ is compact, there is a real-valued continuous function $g$ on $\beta Y$ such that

$$g|_{\prod_{\alpha\in J} A_\alpha} = f|_{\prod_{\alpha\in J} A_\alpha}.$$

Consider a real-valued continuous function $h$ on $\prod_{\alpha\in I} X_\alpha$ defined as

$$h(x_\alpha) = g\big(\pi_J(x_\alpha)\big),\quad x_\alpha\in\prod_{\alpha\in I} X_\alpha,$$

where $\pi_J$ is the projection map from $\prod_{\alpha\in I} X_\alpha$ onto $\prod_{\alpha\in J} X_\alpha$. Then $h\in C^*(\prod_{\alpha\in I} X_\alpha)$ and, consequently, it admits a continuous extension $h^\beta$ to $C(\beta(\prod_{\alpha\in I} X_\alpha))$. As

$$\mathrm{cl}_{\beta X}\prod_{\alpha\in I} A_\alpha \quad\text{and}\quad \prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha} A_\alpha$$

are equivalent compactifications of $\prod_{\alpha\in I} A_\alpha$, there exists $m\in C^*(\prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha} A_\alpha)$ such that

$$m|_{\prod_{\alpha\in I} A_\alpha} = h|_{\prod_{\alpha\in I} A_\alpha}.$$

Now fix $x_\sigma\in A_\sigma$. It is clear that the function $m^*$ defined as

$$m^*(x_\alpha) = m(x_\sigma, x_\alpha)\quad\text{whenever } (x_\alpha)\in\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$$

is a continuous function on $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$ satisfying

$$f|_{\prod_{\alpha\in J} A_\alpha} = m^*|_{\prod_{\alpha\in J} A_\alpha}.$$

Thus, $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha} A_\alpha$ and $\mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha$ are equivalent compactifications of $\prod_{\alpha\in J} A_\alpha$. Therefore we have

$$\mathrm{cl}_{\beta X}\prod_{\alpha\in I} A_\alpha = \prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha} A_\alpha = \mathrm{cl}_{\beta X_\sigma} A_\sigma \times \mathrm{cl}_{\beta Y}\prod_{\alpha\in J} A_\alpha.$$

Since $A_\sigma$ and $\prod_{\alpha\in J} A_\alpha$ are infinite (because $\prod_{\alpha\in J} A_\alpha$ contains $A_\eta$), the desired result follows by Lemma 4.9. $\square$

Let $X=\prod_{\alpha\in I} X_\alpha$. Consider a family of sets $\{A_\alpha\}_{\alpha\in I}$ with $A_\alpha\subset X_\alpha$ for each $\alpha\in I$ and let $f\in C(X)$. For each finite subset $J$ of $I$ and each $b\in\prod_{\alpha\in I\setminus J} A_\alpha$, we will denote by $f_J(-,b)$ the function from $\prod_{\alpha\in J} A_\alpha$ into $\mathbb{R}$ defined by the requirement that

$f_J(a,b) = f(a,b)$ whenever $a \in \prod_{\alpha\in J} A_\alpha$. We will denote by $\mathcal{H}(f,J)$ the family defined as

$$\mathcal{H}(f,J) = \left\{ f_J(-,b):\ b \in \prod_{\alpha\in I\setminus J} A_\alpha \right\}.$$

The following results about extensions of maps and projection maps are needed. We recall that a map $f$ from $X$ into $Y$ is said to be $z$-closed if $f(Z)$ is closed in $Y$ whenever $Z$ is a zero set in $X$.

**Theorem 4.11** (Taĭmanov [25]). *Let $S$ be a dense subspace of a topological space $Y$ and let $\phi$ be a continuous map from $S$ into a compact Hausdorff space $T$. Suppose that $\mathcal{B}$ is a base for the closed sets of $T$ which is closed under finite intersections. Then $\phi$ can be continuously extended over $Y$ if and only if for every pair $B_1$, $B_2$ of disjoint elements of $\mathcal{B}$ the inverse images $\phi^{-1}(B_1)$ and $\phi^{-1}(B_2)$ have disjoint closures in $Y$.*

**Theorem 4.12** (Comfort and Hager [5]). *The following conditions on the product space $X \times Y$ are equivalent*:

(1) *The projection map $p_X$ from $X \times Y$ onto $X$ is $z$-closed.*
(2) *If $f$ is a bounded real-valued continuous function on $X \times Y$, then $\{f^y:\ y \in Y\}$ is an equicontinuous family on $X$.*

**Theorem 4.13.** *Let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces, $X = \prod_{\alpha\in I} X_\alpha$ and $\{A_\alpha\}_{\alpha\in I}$ be a family of sets such that $A_\alpha$ is bounded in $X_\alpha$ for each $\alpha \in I$. Then, the following assertions are equivalent*:

(1) *$\prod_{\alpha\in I} A_\alpha$ is bounded in $\prod_{\alpha\in I} X_\alpha$ and, for each finite subset $J$ of $I$ and each $f \in C(\prod_{\alpha\in I} X_\alpha)$, the family $\mathcal{H}(f,J)$ admits an equicontinuous extension to $\prod_{\alpha\in J} \mathrm{cl}_{\beta X_\alpha} A_\alpha$.*
(2) *For each $f \in C(\prod_{\alpha\in I} X_\alpha)$, $f|_{\prod_{\alpha\in I} A_\alpha}$ admits a continuous extension to $\prod_{\alpha\in I} \mathrm{cl}_{\beta X_\alpha} A_\alpha$.*
(3) $\mathrm{cl}_{\beta X} \prod_{\alpha\in I} A_\alpha = \prod_{\alpha\in I} \mathrm{cl}_{\beta X_\alpha} A_\alpha$.

**Proof.** (1) $\Rightarrow$ (2) Let $f \in C(\prod_{\alpha\in I} X_\alpha)$. First, following the patterns given in [7, 3.12.20(a)], we shall prove that, for every $\varepsilon > 0$, there exists a finite set $S_0 \subset I$ with the property that if for $x, y \in \prod_{\alpha\in I} A_\alpha$ we have that if $p_\alpha(x) = p_\alpha(y)$ whenever $\alpha \in S_0$, then $|f(x) - f(y)| < \varepsilon$.

Assume that there is no such $S_0$. Fix $\alpha_1 \in I$. Then, there exist $x_1, y_1 \in \prod_{\alpha\in I} A_\alpha$ such that $p_{\alpha_1}(x_1) = p_{\alpha_1}(y_1)$ and $|f(x_1) - f(y_1)| \geqslant \frac{1}{2}\varepsilon$.

Let $U^1 = \prod_{\alpha\in I} U^1_\alpha$, $V^1 = \prod_{\alpha\in I} V^1_\alpha$ be open neighborhoods of $x_1, y_1$, respectively, such that

$$\mathrm{osc}(f, U^1) < \tfrac{1}{8}\varepsilon, \qquad \mathrm{osc}(f, V^1) < \tfrac{1}{8}\varepsilon,$$

and $U^1_{\alpha_1} = V^1_{\alpha_1}$. Let $S_1 = \{\alpha_1\}$ and define $S_2$ in the following way:

$$\{\alpha \in I:\ U^1_\alpha \neq X_\alpha \text{ or } V^1_\alpha \neq X_\alpha\}.$$

By assumption (since $S_2$ is finite), we can find $x_2, y_2$ such that $p_\alpha(x_2) = p_\alpha(y_2)$ whenever $\alpha \in S_2$ and $|f(x_2) - f(y_2)| \geqslant \frac{1}{2}\varepsilon$.

In this way, by induction, we can find two sequences of points $(x_n)_{n<\omega}$, $(y_n)_{n<\omega}$ in $\prod_{\alpha\in I} A_\alpha$, a sequence $(S_n)_{n<\omega}$ of finite subsets of $I$ and two sequences of open subsets in $\prod_{\alpha\in I} X_\alpha$, $(U^n)_{n<\omega}$, $(V^n)_{n<\omega}$, satisfying:

(1) $|f(x_n) - f(y_n)| \geqslant \frac{1}{2}\varepsilon$ for all $n < \omega$,

(2) $\operatorname{osc}(f, U^n) < \frac{1}{8}\varepsilon$, $\operatorname{osc}(f, V^n) < \frac{1}{8}\varepsilon$ for all $n < \omega$,

(3) If $S_n = \{\alpha \in I\colon U^{n-1}_\alpha \neq X_\alpha \text{ or } V^{n-1}_\alpha \neq X_\alpha\}$, then $S_n \subseteq S_{n+1}$ for all $n < \omega$,

(4) $U^n_{\alpha_i} = V^n_{\alpha_i}$ whenever $\alpha_i \in S_n$.

Since $U^n \cap \prod_{\alpha\in I} A_\alpha \neq \emptyset$ for all $n < \omega$ and $\prod_{\alpha\in I} A_\alpha$ is bounded in $\prod_{\alpha\in I} X_\alpha$, $\{U^n\}_{n<\omega}$ has a cluster point $z \in \prod_{\alpha\in I} X_\alpha$. Let $W$ be an open neighborhood of $z$ with $\operatorname{osc}(f, W) < \frac{1}{8}\varepsilon$. The fact that $z$ is a cluster point of $(U^n)_{n<\omega}$ joint conditions (3) and (4) imply that we can find $n < \omega$ such that

$$W \cap U^n \neq \emptyset, \qquad W \cap V^n \neq \emptyset.$$

Applying condition (2) we have that $|f(z) - f(x_n)| < \varepsilon/4$ and $|f(z) - f(y_n)| < \frac{1}{4}\varepsilon$ which contradicts condition (1).

Now, consider the function $g = f|_{\prod_{\alpha\in I} A_\alpha}$. Since $\prod_{\alpha\in I} A_\alpha$ is bounded in $\prod_{\alpha\in I} X_\alpha$, we can find a compact interval $[\alpha, \beta]$ such that $g(\prod_{\alpha\in I} A_\alpha)$ is contained in $[\alpha, \beta]$. Set $Y = \prod_{\alpha\in I} \operatorname{cl}_{\beta X_\alpha} A_\alpha$. Let $K_1, K_2$ be two pairwise disjoint closed subsets of $[\alpha, \beta]$. By Taĭmanov's theorem we only need to prove that

$$\operatorname{cl}_Y g^{-1}K_1 \cap \operatorname{cl}_Y g^{-1}K_2 = \emptyset.$$

Because $[\alpha, \beta]$ is compact, there exists $\varepsilon > 0$ such that $|k_1 - k_2| > \varepsilon$ whenever $(k_1, k_2) \in K_1 \times K_2$. Let $J$ be a finite subset of $I$ such that $|g(x) - g(y)| < \frac{1}{3}\varepsilon$ whenever $p_\alpha(x) = p_\alpha(y)$, $\alpha \in J$ and $x, y \in \prod_{\alpha\in I} A_\alpha$. For convenience, in the sequel we will denote $\prod_{\alpha\in J} \operatorname{cl}_{\beta X_\alpha} A_\alpha$ by $M$. We shall first prove that

$$\operatorname{cl}_M p_J\big(g^{-1}(K_1)\big) \cap \operatorname{cl}_M p_J\big(g^{-1}(K_2)\big) = \emptyset.$$

Suppose that, contrary to what we claim, there exists

$$p \in \operatorname{cl}_M p_J\big(g^{-1}(K_1)\big) \cap \operatorname{cl}_M p_J\big(g^{-1}(K_2)\big).$$

Since $\{f_J(-, b)\colon b \in \prod_{\alpha\in I\setminus J} A_\alpha\}$ has an equicontinuous extension $\{f_J^*(-, b)\colon b \in \prod_{\alpha\in I\setminus J} A_\alpha\}$ to $\prod_{\alpha\in J} \operatorname{cl}_{\beta X_\alpha} A_\alpha$ we can find two points $x = (x_\alpha)_{\alpha\in J} \in p_J(g^{-1}(K_1))$ and $y = (y_\alpha)_{\alpha\in J} \in p_J(g^{-1}(K_2))$ such that

$$\big|f_J^*(p, b) - f_J^*(x, b)\big| < \tfrac{1}{3}\varepsilon, \qquad \big|f_J^*(p, b) - f_J^*(y, b)\big| < \tfrac{1}{3}\varepsilon$$

whenever $b \in \prod_{\alpha\in I\setminus J} A_\alpha$. Let $b_1, b_2$ be in $\prod_{\alpha\in I\setminus J} A_\alpha$ with $(x, b_1) \in g^{-1}(K_1)$ and $(y, b_2) \in g^{-1}(K_2)$. Then,

$$\begin{aligned}\big|f(x, b_1) - f(y, b_2)\big| &\leqslant \big|f(x, b_1) - f_J^*(p, b_1)\big| \\ &\quad + \big|f_J^*(p, b_1) - f(y, b_1)\big| + \big|f(y, b_1) - f(y, b_2)\big| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \big|f(y, b_1) - f(y, b_2)\big|.\end{aligned}$$

Since $p_J(y,b_1)=p_J(y,b_2)$, we have that $|f(y,b_1)-f(y,b_2)|<\frac{1}{3}\varepsilon$, and consequently, $|f(x,b_1)-f(y,b_2)|<\varepsilon$. But this leads us to a contradiction, because $f(x,b_1)\in K_1$ and $f(y,b_2)\in K_2$. Thus,

$$\mathrm{cl}_M\, p_J\big(g^{-1}(K_1)\big)\cap \mathrm{cl}_M\, p_J\big(g^{-1}(K_2)\big)=\emptyset.$$

Now, suppose that there is $q\in \mathrm{cl}_Y\, g^{-1}(K_1)\cap \mathrm{cl}_Y\, g^{-1}(K_2)$. Then every open neighborhood $W$ of $q$ in $Y$ such that

$$W=\prod_{\alpha\in J}V_\alpha\times\prod_{\alpha\in I\setminus J}\mathrm{cl}_{\beta X_\alpha}A_\alpha$$

where $\prod_{\alpha\in J}V_\alpha$ is a basic open set in $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha}A_\alpha$, meets $g^{-1}(K_1)\cap g^{-1}(K_2)$. Therefore

$$p_J(q)\in \mathrm{cl}_M\, p_J\big(g^{-1}(K_1)\big)\cap \mathrm{cl}_M\, p_J\big(g^{-1}(K_2)\big),$$

a contradiction.

(2) $\Rightarrow$ (3) Both $\mathrm{cl}_{\beta(\prod_{\alpha\in I}X_\alpha)}A_\alpha$ and $\prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha}A_\alpha$ are compactifications of $\prod_{\alpha\in I}A_\alpha$. By (2), the functions on $\prod_{\alpha\in I}A_\alpha$ continuously extendable to $\mathrm{cl}_{\beta(\prod_{\alpha\in I}X_\alpha)}\prod_{\alpha\in I}A_\alpha$ are the same than the functions continuously extendable to $\prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha}A_\alpha$. So, by [16, 10E(2)], $\mathrm{cl}_{\beta(\prod_{\alpha\in I}X_\alpha)}A_\alpha$ and $\prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha}A_\alpha$ are equivalent compactifications of $\prod_{\alpha\in I}A_\alpha$.

(3) $\Rightarrow$ (1) By Theorem 4.10, $\prod_{\alpha\in I}A_\alpha$ is bounded in $\prod_{\alpha\in I}X_\alpha$. Let $f\in C(\prod_{\alpha\in I}X_\alpha)$ and let $J$ be a finite subset of $I$. Consider the continuous extension $g$ of $f|_{\prod_{\alpha\in I}A_\alpha}$ to $\prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha}A_\alpha$. Since

$$\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha}A_\alpha\times\prod_{\alpha\in I\setminus J}\mathrm{cl}_{\beta X_\alpha}A_\alpha$$

is compact, the projection map $p_{\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha}A_\alpha}$ is $z$-closed. So, according to Theorem 4.12, $\mathcal{H}(g,J)$ is equicontinuous on $\prod_{\alpha\in J}\mathrm{cl}_{\beta X_\alpha}A_\alpha$ and the proof is complete. □

We can apply Theorem 4.13 in order to obtain

**Corollary 4.14.** *Let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces and let $X=\prod_{\alpha\in I}X_\alpha$. Consider, for each $\alpha\in I$, a subset $A_\alpha$ of $X_\alpha$ such that $A_\alpha$ is $p_\alpha$-bounded in $X_\alpha$ with $p_\alpha\in\omega^*$. If there exists $p\in\omega^*$ such that $p\leqslant_{RK}p_\alpha$ for all $\alpha\in I$, then*

$$\mathrm{cl}_{\beta X}\prod_{\alpha\in I}A_\alpha=\prod_{\alpha\in I}\mathrm{cl}_{\beta X_\alpha}A_\alpha.$$

Let $\alpha$ be a cardinal number. A subset $A$ of $X$ is said to be $C_\alpha$-compact if $f(A)$ is a compact subset of $\mathbb{R}^\alpha$ for each continuous function from $X$ into $\mathbb{R}^\alpha$. $C_\omega$-subsets are called $C$-compact subsets. The cardinal number

$$\rho(A,X)=\sup\{\alpha:\ A \text{ is } C_\alpha\text{-compact in } X\}$$

is called *the degree of pseudocompactness* of $A$ in $X$. The reader might consult [14] for basic results on degree of pseudocompactness. Since $A$ is $C_\alpha$-compact in $X$ if and only if $A$ is $G_\alpha$-dense in $\mathrm{cl}_{\beta X} A$ (see Theorem 1.2 in [14]), we can apply Theorem 4.23 and Lemma 3.3 in [14] in order to obtain

**Corollary 4.15.** *Let $p \in \omega^*$. If, for each $\alpha \in I$, $A_\alpha$ is a $p$-bounded, $C$-compact subset of $X_\alpha$, then*

$$\rho\left(\prod_{\alpha\in I} A_\alpha, \prod_{\alpha\in I} X_\alpha\right) = \min\left\{\rho(A_\alpha, X_\alpha):\ \alpha \in I\right\}.$$

*In particular, the product of a family of $p$-bounded, $C_\alpha$-compact subsets is a $C_\alpha$-compact subset.*

The interval $(0, 1)$ is a trivial example of a $p$-bounded subset (in $\mathbb{R}$) for all $p \in \omega^*$ which is not $C$-compact. The following example improves this result. For each topological space $X$, we denote by $F(X)$ the free topological group generated by $X$ (see Sections 2.3 and 9.20 in [4] for definition). It is well known that $X$ is a closed $C$-embedded subset of $F(X)$.

**Example 4.16.** For every cardinal number $\alpha > \omega$ there exists a bounded subset of a topological group $G$ which is not $C_\alpha$-compact in $G$.

**Proof.** Let $\alpha > \omega$ be a cardinal number. Let $X$ be a pseudocompact space which is not $C_\alpha$-compact in itself [14, Corollary 2.8]. Consider the free topological group $F(X)$ over $X$. Since $X$ is $C$-embedded in $F(X)$, $X$ is not $C_\alpha$-compact in $F(X)$. Moreover, because every bounded subset of a topological group is $p$-bounded for all $p \in \omega^*$ (see Remark 4.17 below), $X$ is $p$-bounded in $F(X)$ for all $p \in \omega^*$. $\square$

**Remark 4.17.**

(1) As a consequence of Theorem 2.6 and Theorem 2.8 in [15], condition (1) in Theorem 4.13 is satisfied when considering a family of pseudocompact subsets whose finite products are also pseudocompact and whose product is bounded in the whole space. So, if $\{P_\alpha\}_{\alpha\in I}$ is a family of pseudocompact spaces such that $\prod_{\alpha\in J} P_\alpha$ is pseudocompact for each finite subset $J$ of $I$ and $\prod_{\alpha\in I} P_\alpha$ is bounded in $\prod_{\alpha\in I} X_\alpha$, then

$$\mathrm{cl}_{\beta(\prod_{\alpha\in I} X_\alpha)} P_\alpha = \prod_{\alpha\in I} \mathrm{cl}_{\beta X_\alpha} P_\alpha.$$

In addition, Comfort's example [3] of a non-pseudocompact product space whose finite subproducts are pseudocompact spaces, points out that we can not omit that $\prod_{\alpha\in I} P_\alpha$ be bounded.

(2) A slight modification of the proof of Theorem 1 of [26] shows that every bounded subset of a topological group $G$ is $p$-bounded for each $p \in \omega^*$ (see Theorem 4.3 of [13] for details). So, Corollaries 4.14 and 4.15 give alternative proofs of Corollary 3 in [18], Corollary 3.8 in [14] and Corollary 4.1 in [19].

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