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ON THE ČECH NUMBER OF $C_p(X)$

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Abstract. We discuss the Čech numbers of the spaces $C_p(X)$ and $C_p(X,I)$ (where the Čech number of a space Z is the pseudocharacter of Z in βZ). We establish the relation between the Čech numbers of $C_p(X)$ and of $C_p(X,I)$, find some upper and lower bounds for the Čech number of $C_p(X,I)$ in terms of the cardinal functions of X, and discuss the minimal possible infinite value that the Čech number of $C_p(X,I)$ can have.

All spaces considered in this paper are assumed Tychonoff. Given two spaces X and Y, we denote by $C_p(X,Y)$ the space of all continuous functions from X to Y equipped with the topology of pointwise convergence (that is, the topology of the subspace of the set of all functions from X to Y, Y^X, with the Tychonoff product topology). The space $C_p(X,\mathbb{R})$ is denoted as $C_p(X)$.

The symbols ω, $\mathbb{N}^+$, $\mathbb{R}$, I, $\mathbb{Q}$ and $\mathbb{P}$ stand for the set of all naturals, all positive naturals, the real line, segment $[0,1]$, the space of the rationals and the space ω^ω (homeomorphic to the space of irrationals in $\mathbb{R}$). We assume that all cardinals are equipped with the discrete topology (so the expression τ^λ means the Tychonoff product of λ copies of a discrete space of cardinality τ). The symbol $\mathfrak{c}$ denotes the cardinality of continuum.

A classical theorem of D.J. Lutzer and R.A. McCoy [LM] says that $C_p(X)$ is Čech complete if and only if X is countable and discrete. V.V. Tkachuk observed in [Tk, Theorem 1.13] that $C_p(X,I)$ is Čech complete if and only if X is discrete, thus, if and only if $C_p(X,I)=I^X$ (naturally, many arguments in this paper are modifications of the proofs in [LM] and [Tk]). It seems natural now to ask: *Given a space X, how many open sets are necessary to intersect in $\mathbb{R}^X$ (or I^X) to obtain $C_p(X)$ (or $C_p(X,I)$)?*. The above statements show that the answer is never ω.

1. The Čech number.

Recall that if $A\subset X$, then the *pseudocharacter of A in X* is defined as

$$\Psi(A,X)=\min\{|\mathcal{U}|:\mathcal{U}\text{ is a family of open sets in }X\text{ and }A=\bigcap\mathcal{U}\}.$$

Note that either $\Psi(A,X)=1$ or $\Psi(A,X)$ is infinite.

If τ is a cardinal, and A is a set in a space X, we say that A is *of type G_τ* (or *a G_τ-set*) in X if $\Psi(A,X)\le\tau$. Similarly, we say that A is *of type F_τ* (or *an F_τ-set*) if A is a union of at most τ closed sets in X (that is, $\Psi(X\setminus A,X)\le\tau$).

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The unions of at most λ G_τ sets are called $G_{\tau\lambda}$-sets, and the intersections of at most λ F_τ sets are called $F_{\tau\lambda}$-sets; following the tradition, we use the symbols σ for countable unions and δ for countable intersections.

1.1. Definition. The *Čech number* of a space X is

$$\check{C}(X) = \Psi(X, \beta X).$$

Obviously, $\check{C}(X) = 1$ if and only if X is locally compact, and $\check{C}(X) \leq \omega$ if and only if X is Čech complete.

Define the *k-covering number* of a space Z as

$$kcov(Z) = \min\{|\mathcal{K}| : \mathcal{K} \text{ is a compact cover of } Z\}.$$

Obviously, $kcov(X) \leq \tau$ if and only X is an F_τ set in any space that contains X.

The next statements are immediate:

1.2. Proposition. $\check{C}(X) = kcov(\beta X \setminus X)$.

1.3. Proposition. *If Y is a closed subspace of Z, then $kcov(Y) \leq kcov(Z)$.*

1.4. Proposition. *If $f\colon Z \to Y$ is a continuous mapping, and $f(Z) = Y$, then $kcov(Y) \leq kcov(Z)$.*

1.5. Proposition. *If $f\colon Z \to Y$ is a perfect mapping, and $f(Z) = Y$, then $kcov(Y) = kcov(Z)$.*

1.6. Proposition. *Always $kcov(X \times Y) = kcov(X) \cdot kcov(Y)$.*

The fact that for any perfect mapping $p\colon X \to Y$, $p^*(\beta X \setminus X) = \beta Y \setminus Y$ where $p^*\colon \beta X \to \beta Y$ is the continuous extension of p, together with Proposition 1.5 and Proposition 1.2 yields

1.7. Proposition. *If Y is a perfect image of X, then $\check{C}(X) = \check{C}(Y)$.*

We now can prove that (just like for the Čech completeness) we can use any compactification instead of βX to calculate the Čech number:

1.8. Proposition. *Let bX be a compactification of X. Then $\check{C}(X) = \Psi(X, bX) = kcov(bX \setminus X)$.*

Proof. The equality $\Psi(X, bX) = kcov(bX \setminus X)$ is trivial.

Let $i^*\colon \beta X \to bX$ be the continuous extension of the identity mapping $i\colon X \to X$. Then $i^*(\beta X \setminus X) = bX \setminus X$, and the restriction of i^* to $\beta X \setminus X$ is perfect, so $\check{C}(X) = kcov(\beta X \setminus X) = kcov(bX \setminus X) = \Psi(X, bX)$. $\square$

1.9. Proposition. *If Y is a closed subspace of X, then $\check{C}(Y) \leq \check{C}(X)$.*

Proof. Let bY be the closure of Y in βX, and let $\mathcal{U}$ be a family of open sets in βX such that $|\mathcal{U}| = \check{C}(X)$ and $X = \bigcap \mathcal{U}$. Then $\mathcal{V} = \{\, U \cap bY : U \in \mathcal{U}\,\}$ is a family of open sets in bY such that $Y = \bigcap \mathcal{V}$, and $|\mathcal{V}| \leq \check{C}(X)$. $\square$

1.10. Proposition. *If* $\{ X_\alpha : \alpha \in A \}$ *is a family of spaces and* $\prod\{ X_\alpha : \alpha \in A \}$ *is not locally compact, then*

$$\check{C}\left(\prod\{ X_\alpha : \alpha \in A \}\right) = |A| \cdot \sup\{ \check{C}(X_\alpha) : \alpha \in A \}\}.$$

Proof. Since $\check{C}(Y \times K) = \check{C}(Y)$ holds for every space Y and every compact space K, we can assume, without loss of generality, that X_α is not compact for every $\alpha \in A$.

Let $X = \prod\{ X_\alpha : \alpha \in A \}$. Since $bX = \prod\{ \beta X_\alpha : \alpha \in A \}$ is a compactification of X, it is sufficient to verify $\Psi(X, bX) = \tau$ where $\tau = |A| \cdot \sup\{ \check{C}(X_\alpha) : \alpha \in A \}$. First we prove that $\Psi(X, bX) \leq \tau$:

For every $\alpha \in A$, fix a family $\mathcal{U}_\alpha$ of open sets in βX_α so that $|\mathcal{U}_\alpha| = \check{C}(X_\alpha)$ and $X_\alpha = \bigcap \mathcal{U}_\alpha$. Put $\mathcal{V}_\alpha = \{ p_\alpha^{-1}(U) : U \in \mathcal{U}_\alpha \}$ where $p_\alpha \colon bX \to \beta X_\alpha$ is the projection, and $\mathcal{V} = \bigcup\{ \mathcal{V}_\alpha : \alpha \in A \}$. Then $\mathcal{V}$ is a family of open sets in bX, $|\mathcal{V}| \leq \tau$, and $X = \bigcap \mathcal{V}$.

Now, we will verify that $\Psi(X, bX) \geq \tau$. By Proposition 1.9, we have that $\Psi(X, bX) \geq sup\{\check{C}(X_\alpha) : \alpha \in A\}$.

Since X is not locally compact, $\Psi(X, bX) \geq \aleph_0$. So, to prove $\Psi(X, bX) \geq |A|$ it is enough to verify that $kcov(bX \setminus X) \geq |A|$ when A is infinite. Fix, for each $\alpha \in A$, $a_\alpha \in (\beta X_\alpha \setminus X_\alpha)$ and $b_\alpha \in X_\alpha$.

Let Z be the set of all points $(z_\alpha)_{\alpha \in A}$ of bX whose α-th coordinate is equal to either b_α or a_α for every $\alpha \in A$, and at most for one α, $z_\alpha = a_\alpha$. It is easy to see that every neighborhood in bX of the point $\tilde{b} = (b_\alpha)_{\alpha \in A}$ contains all but finitely many points of Z. It follows that Z is homeomorphic to the one-point compactification of the discrete space of cardinality $|A|$; hence, its intersection with $bX \setminus X$ (equal to $Z \setminus \{\tilde{b}\}$) is closed and discrete in $bX \setminus X$. Thus, $bX \setminus X$ has a closed discrete subspace of cardinality $|A|$. Since the k-covering number is hereditary with respect to closed sets, it follows that $kcov(bX \setminus X) \geq |A|$. □

1.11. Corollary. *If* $X_\alpha : \alpha \in A$ *is a family of subspaces of a space* X*, then*

$$\check{C}\left(\bigcap\{ X_\alpha : \alpha \in A \}\right) \leq |A| \cdot \sup\{ \check{C}(X_\alpha) : \alpha \in A \}.$$

This follows from Proposition 1.9, Proposition 1.10, and the fact that the intersection $\bigcap\{ X_\alpha : \alpha \in A \}$ is homeomorphic to a closed subspace of the product $\prod\{ X_\alpha : \alpha \in A \}$.

In particular,

1.12. Proposition. *If* Y *is a* G_τ*-set in* X*, then* $\check{C}(Y) \leq \tau \cdot \check{C}(X)$.

Proof. By Corollary 1.11, it is enough to verify that $\check{C}(G) \leq \check{C}(X)$ if G is open in X. Let $\mathcal{U}$ be a family of open sets in βX such that $|\mathcal{U}| = \check{C}(X)$ and $X = \bigcap \mathcal{U}$, and let G' be an open set in βX such that $G = G' \cap X$. Let bG be the closure of G in βX, and put $\mathcal{V} = \{G' \cap bG\} \cup \{ U \cap bG : U \in \mathcal{U} \}$. Then $\mathcal{V}$ is a family of open sets in the compactification bG of G, $G = \bigcap \mathcal{V}$, and $|\mathcal{V}| \leq \check{C}(X)$. □

2. The relation between the Čech numbers of $C_p(X)$ and $C_p(X,I)$.

We start from the comparison of the Čech numbers of $C_p(X)$ and $C_p(X,I)$; as we will see, the former is completely determined by the latter and the cardinality of X.

First note that $C_p(X,I)$ is a closed subspace of $C_p(X)$, so $\check{C}(C_p(X,I)) \le \check{C}(C_p(X))$.

The following lemma is a consequence of Proposition 1.10.

2.1. Lemma. *For every infinite cardinal τ, $\check{C}\,(\mathbb{R}^\tau) = \tau$.*

2.2. Proposition. *If $\check{C}\big(C_p(X,I)\big) \le \tau$, then there are disjoint clopen subspaces X_0 and X_1 of X such that $X = X_0 \cup X_1$, $|X_0| \le \tau$, and X_1 is discrete.*

Proof. For a finite set $A \subset X$ and a real number $\varepsilon > 0$ define the set

$$O(A,\varepsilon) = \{\, f \in I^X : |f(a)| < \varepsilon \text{ for all } a \in A \,\}.$$

Then the family

$$\{\, O(A,\varepsilon) : A \text{ is a finite set in } X,\ \varepsilon > 0 \,\}$$

is a base for I^X at the point 0 (the function equal to 0 at all points of X). Note that $O(A,\varepsilon)$ contains the set $Z(A)$ of all functions in I^X whose restrictions to A are zero.

Let $\mathcal{U}$ be a family of cardinality $\tau = \check{C}(C_p(X,I))$ of open sets in I^τ such that $C_p(X,I) = \bigcap \mathcal{U}$. For every $U \in \mathcal{U}$ there is a finite set $A_U \subset X$ and an $\varepsilon_U > 0$ such that $O(A_U,\varepsilon_U) \subset U$. In particular, $Z(A_U) \subset U$. Let $B = \bigcup\{\, A_U : U \in \mathcal{U} \,\}$. Then $|B| \le \tau$ if τ is infinite, and B is finite if $\tau = 1$. Furthermore, $Z(B) = \bigcap\{\, Z(A_U) : U \in \mathcal{U} \,\} \subset \bigcap \mathcal{U} = C_p(X,I)$; hence, every function in I^X whose restriction to B is equal to 0, is continuous. In particular, the characteristic function of $X \setminus B$ is continuous, whence B is clopen in X. Besides, every function from $X \setminus B$ to I is continuous, because it is the restriction to $X \setminus B$ of a function on X equal to 0 at every point of B. Thus, $X \setminus B$ is clopen and discrete.

To end the proof, put $X_0 = B$ and $X_1 = X \setminus B$ if τ is infinite, and $X_0 = \emptyset$ and $X_1 = X$ if $\tau = 1$. $\square$

Now all is ready to prove the main theorem of this section

2.3. Theorem. *For every infinite space X, $\check{C}(C_p(X)) = |X| \cdot \check{C}(C_p(X,I))$.*

Proof. Since always $\check{C}(C_p(X)) \ge \check{C}(C_p(X,I))$, we only need to prove $\check{C}(C_p(X)) \ge |X|$ and $\check{C}(C_p(X)) \le |X| \cdot \check{C}(C_p(X,I))$.

Suppose by contradiction that $\tau = \check{C}(C_p(X)) < |X|$. We have $\check{C}(C_p(X,I)) \le \tau$, so by Proposition 2.2, there is a clopen partition $\{X_0, X_1\}$ of X such that $|X_0| \le \tau$ and X_1 is discrete. Since $|X| > \tau$, it follows that $|X_1| = |X|$. We have

$$C_p(X) = C_p(X_0) \times C_p(X_1) = C_p(X_0) \times \mathbb{R}^{X_1},$$

so $\mathbb{R}^{X_1}$ is homeomorphic to a closed set in $C_p(X)$, and

$$|X| = |X_1| = \check{C}\big(\mathbb{R}^{X_1}\big) \le \check{C}(C_p(X)) = \tau,$$

a contradiction.

Let us now verify that $\check{C}(C_p(X)) \leq |X| \cdot \check{C}(C_p(X, I))$. Let $S = \mathbb{R} \cup \{-\infty, \infty\}$ be the compactification of $\mathbb{R}$ homeomorphic to I; then $C_p(X, S)$ is homeomorphic to $C_p(X, I)$, so it is enough to prove that $\check{C}(C_p(X)) \leq |X| \cdot \check{C}(C_p(X, S))$. But $C_p(X) = \mathbb{R}^X \cap C_p(X, S)$, and the required inequality follows from Lemma 2.1 and Corollary 1.11. □

3. $K(\tau, \lambda)$-analytic spaces.

As we have seen, the calculation of the Čech number of a space reduces to the calculation of the compact-covering number of its complement in a compactification; for the spaces of the form $C_p(X, I)$ we have I^X as a natural compactification. We will now introduce certain classes of spaces that arise naturally in the calculation of the compact-covering numbers.

3.1. Definition. Let τ and λ be cardinals such that $\tau \geq 1$. We say that a space X is $K(\tau, \lambda)$-*analytic* if X is a continuous image of a closed subspace of a product of τ^λ and a compact space.

Thus, for example, a space X is a $K(1, \lambda)$-space if and only if X is compact, X is a $K(\tau, 1)$-space if and only it is a union of $\leq \tau$ compact spaces, and the class of $K(\omega, \omega)$-spaces is exactly the class of all K-analytic spaces.

We denote the class of all $K(\tau, \lambda)$-analytic spaces as $\mathcal{K}(\tau, \lambda)$. Obviously, all compact sets are $K(\tau, \lambda)$-analytic for any $\tau \geq 1$, $\mathcal{K}(\tau, \lambda) \subset \mathcal{K}(\sigma, \lambda)$ if $\tau \leq \sigma$ and $\mathcal{K}(\tau, \lambda) \subset \mathcal{K}(\tau, \mu)$ if $\lambda \leq \mu$.

Since the k-covering number is not increased in continuous images, closed subspaces and products with compact spaces, we have

3.2. Proposition. *If X is $K(\tau, \lambda)$-analytic, then $kcov(X) \leq kcov(\tau^\lambda)$.*

3.3. Proposition. *Let τ and λ be infinite cardinals. The class $\mathcal{K}(\tau, \lambda)$ is invariant with respect to continuous images, closed subspaces, unions of families of cardinality $\leq \tau$, products of families of cardinality $\leq \lambda$, and intersections of families of cardinality $\leq \lambda$.*

Proof. The invariance with respect to continuous images and closed subspaces is immediate from the definition. The union of a family of cardinality $\leq \tau$ of $K(\tau, \lambda)$-analytic spaces can be represented as the continuous image of a closed subspace of the sum of τ copies of the product of τ^λ with a compact space, which is homeomorphic to the product of τ^λ with a compact space. The product of a family of cardinality $\leq \lambda$ of $K(\tau, \lambda)$-analytic spaces can be represented as a continuous image (under the product mapping) of a closed subspace (the product of closed subspaces) of the product of $(\tau^\lambda)^\lambda = \tau^\lambda$ with a compact space. Finally, an intersection of a family of cardinality $\leq \lambda$ of $K(\tau, \lambda)$-analytic spaces is homeomorphic to a closed subspace of the product of this family. □

Remark. Assume that τ is smaller than the first weakly inaccessible cardinal. Then ω^τ contains a closed discrete space of cardinality τ [Myc]; it follows that the classes $\mathcal{K}(\tau, \lambda)$ and $\mathcal{K}(\omega, \lambda)$ coincide whenever $\lambda \geq \tau$.

3.4. Corollary. *If X is an $F_{\tau\lambda}$-set in a compact space, then X is $K(\tau,\lambda)$-analytic, and hence $kcov(X) \le kcov(\tau^\lambda)$.*

It is easy to see that in fact, $\mathcal{K}(\tau,\lambda)$ is the minimal class of spaces that contains all compact spaces and is closed with respect to closed subspaces, continuous images, unions of families of cardinalities $\le \tau$ and products of families of cardinalities $\le \lambda$ (and also the minimal class of spaces that contains all compact spaces and is closed with respect to continuous images and $F_{\tau\lambda}$-subspaces).

3.5. Corollary. *If X is a $G_{\lambda\tau}$-set in a compact space, then $\check{C}(X) \le kcov(\tau^\lambda)$.*

Indeed, if X is a $G_{\lambda\tau}$-set in some compact space, then it is also $G_{\lambda\tau}$ in its closure in this space, hence, in some of its compactification, and the complement of a $G_{\lambda\tau}$-set is an $F_{\tau\lambda}$-set.

In some cases, we can calculate the numbers $kcov(\tau^\lambda)$. Of course, always $kcov(\tau^\lambda) \ge \tau$. Now, $kcov(\omega^\omega) = \mathfrak{d}$ (see, e.g., [vDou]; or we can consider this equality as the definition of $\mathfrak{d}$).

3.6. Proposition. *If $\omega \le \tau < \omega_\omega$, then $kcov(\tau^\omega) = \tau \cdot \mathfrak{d}$.*

Proof. We will prove the statement for $\tau = \omega_n$ by induction on n. If $\tau = \omega = \omega_0$, the statement is true. Assuming that the equality is already proved for some $\tau = \omega_n$, we have, using $\mathrm{cf}(\omega_{n+1}) > \omega$,

$$\omega_{n+1}^\omega = \bigcup\{\alpha^\omega : \alpha \in \omega_{n+1}\},$$

and since for every $\alpha \in \omega_{n+1}$ we have $|\alpha| \le \omega_n$, we have represented ω_{n+1}^ω as the union of ω_{n+1} subspaces whose k-covering numbers do not exceed $\omega_n \cdot \mathfrak{d}$. □

Similarly,

3.7. Proposition. *For every infinite cardinal $\tau \ge \lambda$, $kcov((\tau^+)^\lambda)) = \tau^+ \cdot kcov(\tau^\lambda)$.*

and, somewhat more generally,

3.8. Proposition. *If $\mathrm{cf}(\tau) > \lambda$, then*

$$kcov(\tau^\lambda) = \tau \cdot \sup\{\, kcov(\mu^\lambda) : \mu < \tau\}.$$

It is easy to see that $kcov((\omega_\omega)^\omega) > \omega_\omega$; the particular value of this number obviously is very dependent on additional set-theoretic assumptions.

Obviously, $kcov(\mathfrak{c}^\omega) = \mathfrak{c}$; we do not know the answer to the following:

3.9. Question. Is it true that $kcov(\mathfrak{d}^\omega) = \mathfrak{d}$?

4. The Čech numbers for some $C_p(X, I)$.

In this section we calculate the Čech numbers for some spaces $C_p(X, I)$. Since I^X is a compactification of $C_p(X, I)$, trivially, $\check{C}(C_p(X, I)) \le |I^X| = 2^{|X|}$.

4.1. Proposition. *Let X be a space such that $C_p(X,I)$ is an $F_{\tau\lambda}$-set in I^X. Then $\check{C}(C_p(X,I)) \le kcov((|X|\cdot\tau)^\lambda)$.*

Proof. If U is an open set in I^X, then, since I^X is compact and has the weight equal to $|X|$, U is the union of at most $|X|$ compact spaces, hence $K(|X|\cdot\tau,\lambda)$-analytic. By Proposition 3.3, it follows that every $G_{\lambda\tau}$-set in I^X is $K(|X|\cdot\tau,\lambda)$-analytic, and hence has the k-covering number less or equal to $kcov\big((|X|\cdot\tau)^\lambda\big)$. The statement now follows from Proposition 1.8 and the fact that the complement of an $F_{\tau\lambda}$-set is a $G_{\lambda\tau}$-set. □

For a space X, denote by X' the set of all nonisolated points of X.

4.2. Theorem. *Let X be a subspace of $C_p(Y)$. Then*

$$\check{C}(C_p(X,I)) \le kcov\big((|X|\cdot kcov(Y))^{kcov(X')+\omega}\big).$$

Proof. Let $\tau = kcov(Y)$ and $\lambda = kcov(X')+\omega$. By Proposition 4.1, it is enough to prove that under the conditions of the theorem, $C_p(X,I)$ is an $F_{\tau\lambda}$-set in I^X.

Let $X' = \bigcup\{\, K_\alpha : \alpha\in\lambda \,\}$ where each K_α is compact, and for every $\alpha\in\lambda$ let

$$C_\alpha = \{\, f\in I^X : f \text{ is continuous at every point of } K_\alpha \,\}.$$

Clearly,

$$C_p(X,I) = \bigcap\{\, C_\alpha : \alpha\in\lambda \,\},$$

so it suffices to prove that every C_α is an $F_{\tau\delta}$-set in I^X.

Since the k-covering number is not increased by finite products, countable unions, closed subspaces and continuous images, and $kcov(Z)\le\tau$ implies that Z is an F_τ-set in any larger space, the required statement follows from the next lemma. □

4.3. Lemma. *Let X be a subspace of $C_p(Y)$, K a compact set in X, and let C be the set of all functions in I^X that are continuous at every point of K. Then there is a family $\{\, B_{mn} : m\in\mathbb{N}^+, n\in\mathbb{N}^+ \,\}$ of subsets of I^X such that*

1. *$C = \bigcap_{m\in\mathbb{N}^+}\bigcup_{n\in\mathbb{N}^+} B_{mn}$, and*
2. *for any $m,n\in\mathbb{N}^+$, B_{mn} is a continuous image of a closed subspace of $Y^n\times I^X$.*

Proof of the lemma. For every $n,m\in\mathbb{N}^+$ let

$$B_{mn} = \{\, f\in I^X : \text{there is } (y_1,\dots,y_n)\in Y^n \text{ such that } |f(z)-f(x)|\le 1/m \text{ whenever } x\in K,\ z\in X, \text{ and } |z(y_i)-x(y_i)|<1/n,\ i=1,\dots,n \,\}$$

Claim 1. $C = \bigcap_{m\in\mathbb{N}^+}\bigcup_{n\in\mathbb{N}^+} B_{mn}$.

The inclusion $\bigcap_{m\in\mathbb{N}^+}\bigcup_{n\in\mathbb{N}^+} B_{mn}\subset C$ is trivial. To prove the inverse inclusion, let $f_0\in C$ and $m\in\mathbb{N}^+$. We will find an $n\in\mathbb{N}^+$ so that $f_0\in B_{mn}$.

For every $x \in K$ there are $n_x \in \mathbb{N}^+$ and points $y_{1x}, \dots, y_{n_x x} \in Y$ such that the inequalities $|z(y_{1x}) - x(y_{1x})| < 1/n_x$, ... , $|z(y_{n_x x}) - x(y_{n_x x})| < 1/n_x$ imply $|f_0(z) - f_0(x)| < 1/2m$. The sets of the form

$$U_x = \{\, z \in X : |z(y_{1x}) - x(y_{1x})| < 1/n_x, \dots, |z(y_{n_x x}) - x(y_{n_x x})| < 1/2n_x \,\}$$

$x \in X$, are open in I^X and cover K. Let $\{\, U_{x_1}, \dots, U_{x_k} \}$ be a finite subfamily of $\{\, U_x : x \in K \,\}$ that covers K. Put $n = 2(n_{x_1} + \dots + n_{x_k})$, and let $\bar{y} = \{y_1, \dots, y_n\}$ be a point in Y^n such that each point y_{jx_l}, $l \le k$, $j \le n_{x_l}$ is equal to at least one of y_i, $i \le n$.

Let us verify that $f_0 \in B_{mn}$. Suppose $x \in K$ and $z \in X$ are such that $|z(y_1) - x(y_1)| < 1/n$, ... , $|z(y_n) - x(y_n)| < 1/n$. Then there is an $l \le k$ such that $x \in U_{x_l}$, and since $n \ge 2n_{x_l}$ and the set $\{y_{1x_l}, \dots, y_{n_{x_l} x_l}\}$ is contained in the set $\{y_1, \dots, y_n\}$, we have

$$|x(y_{1x_l}) - x_l(y_{1x_l})| < 1/n_{x_l}, \dots, |x(y_{n_{x_l} x_l}) - x_l(y_{n_{x_l} x_l})| < 1/n_{x_l}$$

and

$$|z_(y_{1x_l}) - x_l(y_{1x_l})| < 1/n_{x_l}, \dots, |z_(y_{n_{x_l} x_l}) - x_l(y_{n_{x_l} x_l})| < 1/n_{x_l},$$

whence $|f_0(x) - f_0(x_l)| < 1/2m$, $|f_0(z) - f_0(x_l)| < 1/2m$, and $|f_0(x) - f_0(z)| < 1/m$.

Claim 2. *The set B_{mn} is a continuous image of a closed subspace of $Y^n \times I^X$.*

Let

$$F_{mn} = \{\, (y_1, \dots, y_n, f) \in Y^n \times I^X : |f(z) - f(x)| \le 1/m \text{ whenever } x \in K,\ z \in X, \text{ and } |z(y_i) - x(y_i)| < 1/n,\ i = 1, \dots, n \,\}.$$

Then B_{mn} is the image of F_{mn} under the projection to I^X, and it is sufficient to verify that F_{mn} is closed in $Y^n \times I^X$.

Suppose $(y_1^0, \dots, y_n^0, f_0) \in (Y^n \times I^X) \setminus F_{mn}$. Then there are $x_0 \in K$ and $z_0 \in X$ such that $|z_0(y_i^0) - x_0(y_i^0)| < 1/n$, $i = 1, \dots, n$, and $|f_0(z_0) - f_0(x_0)| > 1/m$. The set

$$U = \{\, (y_1, \dots, y_n, f) \in Y^n \times I^X : |z_0(y_i) - x_0(y_i)| < 1/n,\ i = 1, \dots, n, \text{ and } |f(z_0) - f(x_0)| > 1/m \,\}$$

is then a neighborhood of $(y_1^0, \dots, y_n^0, f_0)$ in $Y^n \times I^X$ disjoint from F_{mn}. □

Recall that a space X is an *Eberlein-Grothendieck space* (or an *EG-space*) [Arh1] if X is homeomorphic to a subspace of $C_p(K)$ for some compact space K. Note that, in particular, all metrizable spaces are EG-spaces [Arh1].

4.4. Corollary. *If X is an EG-space, then $\check{C}(C_p(X, I)) \le kcov\big((|X|)^{kcov(X')+\omega}\big)$.*

The compact EG-spaces are called *Eberlein compact spaces* [Arh1].

4.5. Corollary. *If X is an Eberlein compact space (in particular, if X is a metrizable compact space), then $\check{C}(C_p(X, I)) \le kcov(|X|^\omega)$.*

In particular,

4.6. Corollary. *If X is an Eberlein compact space, and $|X| < \omega_\omega$, then*

$$\check{C}(C_p(X,I)) \le |X| \cdot \mathfrak{d}.$$

Recall that if $Y \subset X$, then a family $\mathcal{B}$ of open sets in X is called *a external base for Y in X* if for every $y \in Y$ and a neighborhood U of y in X, there is a $B \in \mathcal{B}$ such that $y \in B \subset U$. The minimal cardinality of an external base for Y in X is called the *external weight of Y in X* and is denoted as $w(Y,X)$.

4.7. Corollary. *Always $\check{C}(C_p(X,I)) \le kcov((|X| \cdot w(X',X))^{kcov(X')+\omega})$.*

Call the *essential cardinality of a space X*, $ec(X)$, the minimal cardinality of a subspace of X whose complement is clopen and discrete. Thus, Proposition 2.2 says that

4.8. Corollary. $ec(X) \le \check{C}(C_p(X,I))$.

Every space X has a subspace X_0 such that $|X_0| = ec(X_0)$ and $X_1 = X \setminus X_0$ is clopen and discrete; we have $C_p(X,I) = C_p(X_0,I) \times C_p(X_1,I) = C_p(X_0,I) \times I^{X_1}$. Since I^{X_1} is compact, $\check{C}(C_p(X,I)) = \check{C}(C_p(X_0,I))$. It follows that in all the statements in this section we can replace $|X|$ by $ec(X)$. Corollary 4.8 also gives us the first lower bound for the Čech number of $C_p(X,I)$.

4.9. Corollary. *If X is a σ-compact EG-space of cardinality $\mathfrak{c}$, then $\check{C}(C_p(X,I)) = \check{C}(C_p(X)) = \mathfrak{c}$.*

In particular,

4.10. Corollary. *If X is an uncountable σ-compact metrizable space, then*

$$\check{C}(C_p(X,I)) = \check{C}(C_p(X)) = \mathfrak{c}.$$

We will now establish another lower bound.

4.11. Lemma. *Let $S = \{0\} \cup \{\, 1/n : n \in \mathbb{N}^+ \}$. Then $\check{C}(C_p(S,I)) = \mathfrak{d}$.*

Proof. Corollary 4.5, $\check{C}(C_p(S,I)) \le kcov(|S|^\omega) = kcov(\omega^\omega) = \mathfrak{d}$. On the other hand, $C_p(S,I)$ is an $F_{\sigma\delta}$ set, hence Borelian in I^S (a well-known fact, which also easily follows from Lemma 4.3), which is not a G_δ-set. By the Hurewicz theorem (see, e.g. Corollary 21.21 in [Kech]), $C_p(S,I)$ contains a closed homeomorphic copy of $\mathbb{Q}$, so $\check{C}(C_p(S,I)) \ge \check{C}(\mathbb{Q}) = kcov(\mathbb{P}) = \mathfrak{d}$. □

4.12. Corollary. *If X contains a convergent sequence, then $\check{C}(C_p(X,I)) \ge \mathfrak{d}$.*

Proof. Let $T \subset X$ be a convergent sequence. Fix a countable set $\{\, f_n : n \in \omega\}$ of continuous functions from X to I that separates points of T, and consider the diagonal product $F = \triangle\{\, f_n : n \in \omega \}\colon X \to I^\omega$. Obviously, the space $Y = F(X)$ is metrizable, and by the compactness of T, the restriction $F_T = F|T : T \to F(T)$ is a homeomorphism. By the Dugundji Extension Theorem [Dug], there is an extension operator $\psi\colon C_p(F(T),I) \to C_p(Y,I)$, that is, a mapping ψ such that $\psi(f)|F(T) = f$ for every $f \in C_p(F(T),I)$; it is easy to see from the construction of ψ in [Dug] that ψ is continuous with respect to the topologies of pointwise convergence. Define $\phi\colon C_p(T,I) \to C_p(X,I)$ by putting $\phi(g) = \psi((F_T^{-1} \circ g)) \circ F$ for all $g \in C_p(T,I)$.

Then ϕ is a continuous extension operator. The subspace $\phi(C_p(T,I))$ of $C_p(X,I)$ is homeomorphic to $C_p(T,I)$ under ϕ (the inverse mapping $g \mapsto g|T$ is continuous), and is a retract of $C_p(X,I)$ (with the retraction $g \mapsto \phi(g|T)$), hence is closed in $C_p(X,I)$. Therefore, $\check{C}(C_p(X,I)) \geq \check{C}(C_p(T,I)) = \check{C}(C_p(S,I)) = \mathfrak{d}$. □

4.13. Corollary. *If X is a non-discrete metrizable space, then $\check{C}(C_p(X,I)) \geq \mathfrak{d}$.*

4.14. Corollary. *If X is a σ-compact metrizable space, then either $\check{C}(C_p(X,I)) = 1$ (if X is discrete), or $\check{C}(C_p(X,I)) = \mathfrak{d}$ (if X is countable non-discrete), or $\check{C}(C_p(X,I)) = \mathfrak{c}$ (if X is uncountable).*

Proof. If X is countable and not discrete, then $\check{C}(C_p(X,I)) \leq \mathfrak{d}$ by Corollary 4.5 and $\check{C}(C_p(X,I)) \geq \mathfrak{d}$ by Corollary 4.13.

If X is uncountable, then $\check{C}(C_p(X,I)) = \mathfrak{c}$ by Corollary 4.10. □

Since every infinite Eberlein compact space contains a convergent sequence (see Theorem 3.3.6 in [Arh2]), and obviously, $ec(X) = |X|$ for every infinite compact space X, we get

4.15. Corollary. *If X is an infinite Eberlein compact space, then $\check{C}(C_p(X)) \geq |X| \cdot \mathfrak{d}$.*

Combining this with Corollary 4.6, we obtain

4.16. Proposition. *If X is an infinite Eberlein compact space, and $|X| < \omega_\omega$, then $\check{C}(C_p(X,I)) = |X| \cdot \mathfrak{d}$.*

The bounds we obtained for the Čech numbers of $C_p(X,I)$ for Eberlein compact spaces of cardinalities ω_ω and higher generally do not match, and we do not know now how to obtain the exact numbers, or even whether the cardinality of an Eberlein compact space X determines completely the Čech number of $C_p(X,I)$.

It seems worth to mention also the following consequence of Corollary 4.13.

4.17. Proposition. *If X is an infinite pseudocompact space, then $\check{C}(C_p(X,I)) \geq \mathfrak{d}$.*

Proof. If X is infinite and pseudocompact, then there is a continuous mapping F of X onto a non-discrete metrizable space M. By Theorem 7 in [Arh3], the mapping F is R-quotient, and by Proposition 0.4.10 in [Arh2], the dual mapping $F^*\colon C_p(M,I) \to C_p(X,I)$ is a closed embedding. Hence, $\check{C}(C_p(X,I)) \geq \check{C}(C_p(M,I)) \geq \mathfrak{d}$. □

In the end of the section we find the Čech number of the function space for the space $\mathbb{P} = \omega^\omega$.

Recall that a space X is called *analytic* if it is a continuous image of $\mathbb{P}$; a set A in a second-countable space Z is *coanalytic* if its complement in Z is analytic.

4.18. Theorem. $\check{C}(C_p(\mathbb{P},I)) = \mathfrak{c}$.

Proof. Since $ec(\mathbb{P}) = \mathfrak{c}$, we have $\check{C}(C_p(\mathbb{P},I)) \geq \mathfrak{c}$ by Corollary 4.8.

For every $A \subset \mathbb{P}$ let $r_A\colon C_p(\mathbb{P}) \to C_p(A)$ be the restriction mapping defined by $r_A(f) = f|A$. Denote $C_p(X|A) = r_A(C_p(X))$ and $C_p(X|A,I) = r_A(C_p(X,I))$.

It is proved in [Ok] that for every dense $A \subset \mathbb{P}$, the space $C_p(X|A)$ is coanalytic in $\mathbb{R}^A$. Since I^A is closed in $\mathbb{R}^A$, the analyticity is preserved by closed subsets, and $C_p(\mathbb{P}|A, I) = C_p(\mathbb{P}|A) \cap I^A$, the set $C_p(X|A, I)$ is coanalytic in I^A. Since $kcov(Z) \leq kcov(\mathbb{P}) = \mathfrak{d}$ for every analytic space Z, it follows that $C_p(X|A, I)$ is a $G_\mathfrak{d}$-set in I^A.

Obviously, a function $f\colon \mathbb{P} \to I$ is continuous if and only if its restrictions to every dense countable set in $\mathbb{P}$ admits a continuous extension to $\mathbb{P}$. In other words,

$$C_p(\mathbb{P}, I) = \bigcap \{ r_A^{-1}(C_p(\mathbb{P}|A, I)) : A \text{ is a countable dense subset of } \mathbb{P} \}.$$

For every dense countable A in $\mathbb{P}$, the preimage $r_A^{-1}(C_p(\mathbb{P}|A, I))$ of a $G_\mathfrak{d}$-set in I^A under the continuous mapping r_A is a $G_\mathfrak{d}$-set in $I^\mathbb{P}$. Since there are $\mathfrak{c}$ countable dense sets in $\mathbb{P}$, we obtain a representation of $C_p(\mathbb{P}, I)$ as the intersection of $\mathfrak{c}$ of $G_\mathfrak{d}$ sets in a compact space $I^\mathbb{P}$, and hence $\check{C}(C_p(\mathbb{P}, I)) \leq \mathfrak{c}$. □

4.19. Question. Let X be a metrizable analytic non-σ-compact space of cardinality $\mathfrak{c}$. Is it true that $\check{C}(C_p(X, I)) = \mathfrak{c}$?

5. What is the minimal infinite Čech number of $C_p(X, I)$?.

As we have already mentioned, it is an easy consequence of the results from [LM] and [Tk] that ω is never the Čech number of a space $C_p(X, I)$. It seems probable from the results of the previous section that the minimal possible infinite value of $\check{C}(C_p(X, I))$ might be $\mathfrak{d}$, but we could not prove or disprove this. Thus, the following question remains open:

5.1. Question. Is there a non-discrete space X such that $\check{C}(C_p(X, I)) < \mathfrak{d}$?

We have found a lower bound for infinite values of $\check{C}(C_p(X, I))$, which is consistently equal to $\mathfrak{c}$, and hence is greater than any "given" cardinal.

Recall that the *Novak number* of a space X is

$$nov(X) = \min\{ |\mathcal{C}| : X = \bigcup \mathcal{C} \text{ and every element of } \mathcal{C} \text{ is nowhere dense in } X \}.$$

Let $\mathfrak{N} = \min\{ \tau : I^\tau \text{ can be represented as a union of } \tau \text{ nowhere dense sets} \}$.

Of course, always $\omega_1 \leq \mathfrak{N} \leq \mathfrak{c}$, and Martin's Axiom implies $\mathfrak{N} = \mathfrak{c}$. Trivially, $\mathfrak{N} \leq nov(\mathbb{R}) \leq \mathfrak{d}$.

5.2. Question. Is it true that $\mathfrak{N} = nov(\mathbb{R})$?

5.3. Theorem. *If X is a non-discrete space, then $\check{C}(C_p(X, I)) \geq \mathfrak{N}$.*

Proof. Suppose there is a non-discrete space X with $\check{C}(C_p(X, I)) < \mathfrak{N}$. By Proposition 2.2, there is a clopen set X_0 in X such that $|X_0| < \mathfrak{N}$ and $X_1 = X \setminus X_0$ is discrete. Then X_0 is not discrete, and $C_p(X_0, I)$ is homeomorphic to a closed subspace of $C_p(X, I)$, whence $\check{C}(C_p(X_0, I)) \leq \check{C}(C_p(X, I)) < \mathfrak{N}$. Thus, we can assume without loss of generality that $|X| < \mathfrak{N}$. By Theorem 2.3, we get $\check{C}(C_p(X)) < \mathfrak{N}$.

Since X is not discrete, there is a discontinuous function $f_0 \in \mathbb{R}^X$. We have then $(f_0 + C_p(X)) \cap C_p(X) = \emptyset$, so $(\mathbb{R}^X \setminus C_p(X)) \cup (\mathbb{R}^X \setminus (f_0 + C_p(X))) = \mathbb{R}^X$.

Since $C_p(X)$ is a dense set in $\mathbb{R}^X$ that is the intersection of $< \mathfrak{N}$ open sets, so is $f_0 + C_p(X)$, and hence both $\mathbb{R}^X \setminus C_p(X)$ and $\mathbb{R}^X \setminus (f + C_p(X))$ are unions of $< \mathfrak{N}$ nowhere dense sets in $\mathbb{R}^X$. Thus, we have obtained a representation of $\mathbb{R}^X$ with $|X| < \mathfrak{N} \leq nov(\mathbb{R})$ as a union of $< \mathfrak{N}$ nowhere dense sets, a contradiction with the definition of $\mathfrak{N}$. □

We have already mentioned several open problems in the text. The next question also seems very interesting, and not easily resolvable:

5.4. Question. What is the Čech number of ΣI^τ? (Here ΣI^τ is the Σ-product of τ copies of the segment I. Note that ΣI^τ is homeomorphic to $C_p(X, I)$ where X is the one-point Lindelöfication of the discrete space of cardinality τ).

In particular, what is the Čech number of ΣI^{ω_1}? Is it equal to $\mathfrak{d}$?

Note that Corollary 4.8 and Corollary 4.7 yield $\tau \leq \check{C}(\Sigma I^\tau) \leq kcov(\tau^\omega)$.

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