Topology and its Applications 183 (2015) 110–126

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# The Menger property on $C_p(X,2)$

D. Bernal-Santos [*,1], Á. Tamariz-Mascarúa [2]

*Departamento de Matemáticas, Facultad de Ciencias, UNAM, Circuito exterior s/n, Ciudad Universitaria, CP 04510, México D.F., México*

ARTICLE INFO

*Article history:*
Received 1 October 2014
Received in revised form 23 December 2014
Accepted 25 December 2014
Available online xxxx

*MSC:*
primary 54C35, 54D35, 03G10
secondary 54D45, 54C45

*Keywords:*
Space of continuous functions
Menger space
Hurewicz space
Lindelöf space
Generalized ordered topological space
Subspace of ordinals
Strong $P$-point
$\Psi$-space

ABSTRACT

A space $X$ is said to have the Menger property (or simply $X$ is Menger) if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers of $X$, there exists a sequence of finite sets $\langle \mathcal{F}_n : n \in \omega\rangle$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a cover of $X$ and $\mathcal{F}_n \subset \mathcal{U}_n$ for every $n \in \omega$. We prove: (1) If $X$ is a subspace of $C_p(Y)$, where $Y^n$ is Menger for every $n \in \omega$, and $X'$ (the set of non-isolated points of $X$) is compact, then $C_p(X,2)$ is Menger; (2) If $C_p(X,2)$ is Menger and $X$ is normal, then $X'$ is countably compact; (3) For a first countable $GO$-space without isolated points $L$, $C_p(L,2)$ is Menger if and only if $C_p(L,2)$ is Lindelöf and $L$ is countably compact; and for a subspace $L$ of an ordinal, $C_p(L,2)$ is Menger if and only if $C_p(L,2)$ is Lindelöf and $L'$ is countably compact; (4) For every $\mathcal{F} \in \omega^*$, $C_p(\omega \cup \{\mathcal{F}\},2)$ is Menger if and only if $\mathcal{F}$ is a strong $P$-point; (5) Assuming the Continuum Hypothesis, there is a maximal almost disjoint family $\mathcal{A}$ for which the space $C_p(\Psi(\mathcal{A}),2)$ is Menger.



## 1. Introduction

M. Scheepers started the identification and classification of common prototypes for selective properties appearing in classical and modern works. In [12,21,23,24] we can find good surveys of this field of *Selective Principles in Mathematics*. Two of the main prototypes in the field are defined as follows [21]. Fix a topological space $X$, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of subsets of $X$. The following are properties which $X$ may or may not have [21]:

* Corresponding author.
*E-mail addresses:* shen_woo@me.com (D. Bernal-Santos), atamariz@unam.mx (Á. Tamariz-Mascarúa).

[1] The first author was supported by CONACyT grant No. 322416/233751.
[2] The research of the second author was supported by PAPIIT IN115312.

http://dx.doi.org/10.1016/j.topol.2014.12.022
0166-8641/

$\mathrm{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of members of $\mathcal{A}$, there exists, for each $n \in \omega$, $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\} \in \mathcal{B}$.

$\mathrm{S}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of members of $\mathcal{A}$, there exists, for each $n \in \omega$, a finite subset $\mathcal{F}_n \subset \mathcal{U}_n$, such that $\bigcup\{\mathcal{F}_n : n \in \omega\} \in \mathcal{B}$.

When $\mathcal{A}$ and $\mathcal{B}$ coincide with the collection $\mathcal{O}$ of all open covers of $X$, then, in the case of metric spaces, $\mathrm{S}_{fin}(\mathcal{O},\mathcal{O})$ is the property shown by W. Hurewicz [11] to be equivalent to K. Menger's property E [15]. And $\mathrm{S}_1(\mathcal{O},\mathcal{O})$ is F. Rothberger's property traditionally known as $C''$ [17].

So we will say that a space $X$ has the *Menger property* (or simply $X$ is *Menger*) if $X$ has property $\mathrm{S}_{fin}(\mathcal{O},\mathcal{O})$, and a space $X$ is *Rothberger* if $X$ has property $\mathrm{S}_1(\mathcal{O},\mathcal{O})$.

Every $\sigma$-compact space and every Rothberger space has the Menger property and every space with the Menger property is Lindelöf.

Naturally, M. Scheepers's prototypes of selective principles have been analyzed in the class of spaces of real-valued continuous functions defined on a space $X$ with its pointwise convergence topology, $C_p(X)$; see for example [9,18,19,25]. With respect to the Menger property, in [1] the following theorem is proved:

**Theorem.** *$C_p(X)$ is Menger if and only if $X$ is finite.*

For spaces of the form $C_p(X,2)$, of course, there is not an equivalent result to the previous theorem. If the space $X$ is discrete, $C_p(X,2)$ is compact and so it is Menger. And the space $C_p(2^\omega,2)$ is countable and thus it is Menger as well. Moreover, Á. Tamariz-Mascarúa and A. Contreras-Carreto in [7] show that if $X$ is a $EG$-space and $X'$ is Eberlein compact, then $C_p(X,2)$ is $\sigma$-compact (hence, Menger). Therefore, the class of spaces for which $C_p(X,2)$ is Menger is not trivial.

In this article we are going to analyze when $C_p(X,2)$ is Menger for several classes of spaces $X$. In Section 3, we list some general remarks on Menger spaces. Section 4 is devoted to obtain some general results about Menger property on spaces of the form $C_p(X,2)$. In Sections 5, 6, 7 and 8 we analyze sufficient and necessary conditions in order to have $C_p(X,2)$ Menger when $X$ is a $GO$-space without isolated points, a subspace of ordinals, a countable space with exactly one non-isolated point and a $\Psi$-space, respectively.

With respect to the Rothberger property in $C_p(X,2)$, the first author of this paper made an analysis in [2].

## 2. Notation and terminology

All spaces under consideration are assumed to be Tychonoff, i.e., $T_{3\frac{1}{2}}$. Given a space $X$, $X'$ denotes the set of non-isolated points of $X$. For spaces $X$ and $Y$, $C_p(X,Y)$ is the subspace of $Y^X$ consisting of the continuous functions from $X$ to $Y$ (i.e., $C(X,Y)$ with the topology of the pointwise convergence). As usual, $C_p(X)$ will mean $C_p(X,\mathbb{R})$. For a space $X$, $n \in \omega$, points $x_0,\dots,x_n \in X$, $f \in C_p(X)$ and a positive real number $\delta$, we will denote by $[f;x_0,\dots,x_n;\delta]$ the set $\{g \in C_p(X) : \forall i(0 \le i \le n \to |f(x_i) - g(x_i)| < \delta)\}$. Recall that for every space $X$ and every discrete space $Y$, there exists a zero-dimensional space $Z$ such that $C_p(X,Y) \cong C_p(Z,Y)$. So, where reference is made to $C_p(X,Y)$ where $Y$ is discrete, we will assume that $X$ is a zero-dimensional space. Let $iw(X)$ be the minimal cardinal $\kappa$ such that $X$ has a weaker Tychonoff topology of weight $\kappa$; evidently, the statement $iw(X) = \omega$ is equivalent to saying that $X$ has a weaker separable metrizable topology. A space $X$ has *countable fan tightness* if for any $x \in X$ and any sequence $\langle A_n : n \in \omega\rangle$ of subsets of $X$ such that $x \in \bigcap_{n\in\omega} \mathrm{cl}\,(A_n)$, we can choose a finite set $B_n \subset A_n$ for each $n \in \omega$ in such a way that $x \in \mathrm{cl}\,(\bigcup\{B_n : n \in \omega\})$. A space $X$ has a *countable tightness* (which is denoted by $t(X) \le \omega$) if any $x \in X$ and $A \subset X$, if $x \in \mathrm{cl}\,(A)$, then there is a countable set $B \subset A$ such that $x \in \mathrm{cl}\,(B)$. A family $\mathcal{P}$ of non-empty subsets of a space $X$ is said to be $\pi$-*network* at $x \in X$ if every neighborhood of $x$ contains some member of $\mathcal{P}$. For any set $X$, $[X]^{<\omega}$ will denote the set of all finite subsets of $X$. The set of ordinals strictly less than an ordinal $\alpha$ equipped with its order topology will be denoted simply by $\alpha$.

## 3. Properties of Menger spaces

As we have already mentioned:

**Proposition 3.1.** *Every $\sigma$-compact space is Menger and every Menger space is a Lindelöf space.*

Some other properties of Menger spaces are as follows:

**Proposition 3.2.** *Any closed subspace of a Menger space is a Menger space and the continuous image of a Menger space is a Menger space.*

For the purposes of this paper, the following equivalent formulation of Menger space will be useful.

**Lemma 3.3.** *A space $X$ is Menger if and only if for any sequence of open covers $\langle \mathcal{U}_n : n \in \omega \rangle$ such that, for every $n \in \omega$, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, there exists a sequence of finite sets $\langle \mathcal{F}_n : n \in \omega \rangle$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a cover of $X$ and $\mathcal{F}_n \subset \mathcal{U}_n$ for each $n \in \omega$.*

**Proof.** We only have to show the sufficiency. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of $X$. By recursion we define a new sequence $\langle \mathcal{U}'_n : n \in \omega \rangle$ such that, for each $n \in \omega$, $\mathcal{U}'_{n+1}$ refines $\mathcal{U}'_n$ and $\mathcal{U}_{n+1}$. Let $\mathcal{U}'_0 = \mathcal{U}_0$. Suppose that $\mathcal{U}'_0, \ldots, \mathcal{U}'_n$ have been defined. We define $\mathcal{U}'_{n+1} = \{U \cap V : U \in \mathcal{U}_n \wedge V \in \mathcal{U}'_n\}$. Then the sequence $\langle \mathcal{U}'_n : n \in \omega \rangle$ satisfies the required properties. □

We shall need the following results.

**Proposition 3.4 ([22]).** *The countable union of Menger spaces is Menger.*

**Proposition 3.5 ([22]).** *If $X$ is a Menger space and $Y$ is $\sigma$-compact, then $X \times Y$ is Menger.*

A space $X$ is a *$P$-space* if all $G_\delta$-sets in $X$ are open.

**Proposition 3.6 ([26]).** *A $P$-space is Menger if and only if it is Lindelöf.*

The typical example of a Lindelöf space which is not Menger is the space of irrationals $\omega^\omega$. As a consequence of this we have the following:

**Proposition 3.7.** *For any space $X$, $X^\omega$ is Menger if and only if $X$ is compact.*

**Proof.** Suppose that $X^\omega$ is Menger and $X$ is not compact. Since $X$ is Lindelöf, $X$ is not countably compact. Then $X$ contains a closed countable discrete subspace $D$. In this manner $D^\omega$ is a closed subspace $X^\omega$ homeomorphic to $\omega^\omega$. But this is a contradiction to Proposition 3.2. □

As the Lindelöf property, the Menger property is not productive. In [14] A. Lelek gives an example (assuming the continuum hypothesis) of a Menger space $X$ such that $X^2$ is not Menger.

## 4. The Menger property on $C_p(X, 2)$

Recall that where reference is made to $C_p(X, Y)$ where $Y$ is discrete, we will assume that $X$ is a zero-dimensional space.

The following is shown in [1]:

($\star$) *If $C_p(X)$ is a Lindelöf space, then each finite power of $X$ has countable tightness.*

A space $X$ has *countable supertightness* at $x \in X$ if any $\pi$-network at $x$ consisting of finite subsets of $X$ contains a countable $\pi$-network at $x$. If $X$ has this property in each of its points we say that $X$ has *countable supertightness*, and we denote this fact by $st(X) \leq \omega$. Clearly, countable supertightness implies countable tightness. With this new notion of tightness, we obtain a more general result than ($\star$) for $C_p(X,2)$:

**Proposition 4.1.** *If $C_p(X,2)$ is Lindelöf, then $st(X^n) \leq \omega$ for any $n \in \omega$.*

**Proof.** Fix $k \in \omega$, a point $x = (x_1, \dots, x_k) \in X^k$ and a $\pi$-network $\mathcal{P}$ at $x$ consisting of finite subsets of $X^k$. We take open neighborhoods $U_1, \dots, U_k$ such that, for each $i, j \in \{1, \dots, k\}$, $x_i \in U_i$, $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $U = U_1 \times \cdots \times U_k$. We can suppose that each member of $\mathcal{P}$ is contained in $U$. Since the space $C_p(X,2)$ is Lindelöf, the closed subspace

$$\varPhi = \big\{f \in C_p(X,2) : \forall i\big(1 \leq i \leq k \to f(x_i) = 1\big)\big\}$$

of $C_p(X,2)$ is Lindelöf. For each $F \in \mathcal{P}$, we define $H_F = \bigcup\{\pi_i[F] : i \in \{1, \dots, k\}\}$, where $\pi_i$ is the projection of $X^k$ over the i-th coordinate, and $V_F = \{f \in C_p(X,2) : \forall x(x \in H_F \to f(x) = 1)\}$. Given $f \in \varPhi$, for each $i \in \{1, \dots, k\}$, there is an open subset $V_i \subset U_i$ such that $x_i \in V_i$ and $f[V_i] \subset \{1\}$. Since $\mathcal{P}$ is a $\pi$-network, there is $F \in \mathcal{P}$ such that $F \subset V_1 \times \cdots \times V_k$. So, $f[\pi_i[F]] \subset \{1\}$ for each $i \in \{1, \dots, k\}$ and consequently $f \in V_F$. This shows that $\{V_F : F \in \mathcal{P}\}$ is an open cover of $\varPhi$. Therefore, there is a countable subset $\mathcal{P}'$ of $\mathcal{P}$ such that $\{V_F : F \in \mathcal{P}'\}$ forms an open cover of $\varPhi$. Let us prove that $\mathcal{P}'$ is a $\pi$-network at $x$.

Let $W = W_1 \times \cdots \times W_k$ be an open subset which contains $x$. We can assume that $W_i = W_j$ if $x_i = x_j$ and $W_i \subset U_i$ for each $i, j \in \{1, \dots, k\}$. We choose $f \in C_p(X,2)$ such that

$$f\left[X \setminus \bigcup_{i=1}^{k} W_i\right] \subset \{0\}$$

and $f(x_i) = 1$ for each $i \in \{1, \dots, k\}$. Thus $f \in \varPhi$, and consequently, there is $F \in \mathcal{P}'$ such that $f \in V_F$. Now, if $(y_1, \dots, y_k) \in F$, since $F \subset U$, $y_i \in U_i$ for each $i \in \{1, \dots, k\}$. Moreover, due to the fact that $f \in V_F$, $y_1, \dots, y_k \in \bigcup_{i=1}^k W_i$. However, $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$, then $y_i \in W_i$ for each $i \in \{1, \dots, k\}$. This shows that $F \subset W$. $\square$

M. Sakai introduces the following notion.

**Definition 4.2 ([20]).** A space $X$ has *countable fan tightness for finite sets* if for each point $x \in X$ and each sequence $\langle \mathcal{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ of $\pi$-networks at $x$ consisting of finite subsets of $X$, there is, for each $n \in \omega$, a finite subfamily $\mathcal{G}_n \subset \mathcal{P}_n$ such that $\bigcup\{\mathcal{G}_n : n \in \omega\}$ is a $\pi$-network at $x$.

The following equivalent formulation for countable fan tightness for finite sets will be useful.

**Lemma 4.3.** *A space $X$ has a countable fan tightness for finite sets if and only if for each point $x \in X$ and any decreasing sequence $\langle \mathcal{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ of $\pi$-networks at $x$, there are, for each $n \in \omega$, finite subfamilies $\mathcal{G}_n \subset \mathcal{P}_n$ such that $\bigcup\{\mathcal{G}_n : n \in \omega\}$ is a $\pi$-network at $x$.*

**Proof.** The necessity is clear; we show the sufficiency. Let $\langle \mathcal{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ be a sequence of $\pi$-networks at $x \in X$. For each $n \in \omega$, we define $\mathcal{P}'_n = \bigcup_{n \leq k} \mathcal{P}_k$. Then, by hypothesis, there is a sequence of finite sets $\langle \mathcal{F}'_n : n \in \omega \rangle$ where $\mathcal{F}'_n \subset \mathcal{P}'_n$, such that $\bigcup_{n \in \omega} \mathcal{F}'_n$ is a $\pi$-network at $x$. Hence, if we define $\mathcal{F}_n = (\bigcup_{k \leq n} \mathcal{F}'_k) \cap \mathcal{P}_n$, $\langle \mathcal{F}_n : n \in \omega \rangle$ is the required sequence. $\square$

Making a modification of the proof of Proposition 4.1 we have the following.

**Proposition 4.4.** *If the space $C_p(X,2)$ is Menger, then $X^n$ has countable fan tightness for finite sets for any $n \in \omega$.*

**Proof.** We fix a $k \in \omega$, a point $x = (x_1, \ldots, x_k) \in X^k$ and a sequence $\langle \mathcal{P}_n : n \in \omega \rangle$ of $\pi$-networks at $x$ consisting of finite subsets of $X$. We take open subsets $U_1, \ldots, U_k$ of $X$ such that, for each $i, j \in \{1, \ldots, k\}$, $x_i \in U_i$, $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $U = U_1 \times \cdots \times U_k$. We can suppose that, for every $n \in \omega$, each member of $\mathcal{P}_n$ is contained in $U$. Since the space $C_p(X,2)$ is Menger, the closed subspace

$$\Phi = \big\{ f \in C_p(X,2) : \forall i \big( 1 \leq i \leq k \to f(x_i) = 1 \big) \big\}$$

of $C_p(X,2)$ is Menger. For each $F \in [X^k]^{<\omega}$, we define $H_F = \bigcup \{ \pi_i[F] : i \in \{1, \ldots, k\}\}$, where $\pi_i$ is the projection of $X^k$ over the $i$-th coordinate, and we set $V_F = \{ f \in C_p(X,2) : \forall x (x \in H_F \to f(x) = 1)\}$. For each $n \in \omega$, let

$$\mathcal{U}_n = \{ V_F : F \in \mathcal{P}_n \}.$$

Given $f \in \Phi$, for each $i \in \{1, \ldots, k\}$, there is an open subset $V_i \subset U_i$ such that $x_i \in V_i$ and $f[V_i] \subset \{1\}$. Since $\mathcal{P}_n$ is a $\pi$-network, there is $F \in \mathcal{P}_n$ such that $F \subset V_1 \times \cdots \times V_k$. So $f[\pi_i[F]] \subset \{1\}$ for each $i \in \{1, \ldots, k\}$. Thus $f \in V_F \in \mathcal{U}_n$. This implies that $\mathcal{U}_n$ is an open cover of $\Phi$. Therefore, since $\Phi$ is Menger, there is a sequence of finite sets $\langle \mathcal{F}_n : n \in \omega \rangle$ such that $\bigcup_{n \in \omega} \mathcal{F}_n$ forms a cover of $\Phi$ and $\mathcal{F}_n \subset \mathcal{U}_n$ for every $n \in \omega$. Choosing a finite subset $\mathcal{P}'_n \subset \mathcal{P}_n$ such that $\mathcal{F}_n$ is equal to $\{ V_F : F \in \mathcal{P}'_n \}$ and using similar argumentation to that developed in the proof of Theorem 4.1, we can prove that $\bigcup_{n \in \omega} \mathcal{P}'_n$ is a $\pi$-network at $x$. $\square$

The converse of Propositions 4.1 and 4.4 are false. The following example can be found in [1, II.1.7].

**Example 4.5.** Let $X$ be the well-known “double arrow” compact space; that is, $X$ is the set $[0,1] \times 2$ endowed with the topology generated by the lexicographic order. For each $a \in (0,1)$, we define $f_a : X \to 2$ as follows:

$$f_a(x) = \begin{cases} 0, & \text{if } x \leq (a,0); \\ 1, & \text{if } x \geq (a,1). \end{cases}$$

Then, the subspace $A = \{ f_a : a \in (0,1)\}$ is closed and discrete in $C_p(X,2)$. Hence, $C_p(X,2)$ is not a Menger space. However, $X^n$ has countable fan tightness for finite sets (because $X^n$ satisfies the first axiom of countability) for each $n \in \omega$.

Proposition 3.7 leaves out the possibility that $C_p(X,2)^\omega \cong C_p(X,2^\omega)$ have the Menger property when $X$ is not a discrete space. We will only analyze the finite power of the spaces $C_p(X,2)$, and for this we have the following:

**Proposition 4.6.** *For any space $X$, $C_p(X,2)^n$ is Menger for any $n \in \omega$ if and only if $C_p(X,k)$ is Menger for any $k \in \omega$.*

**Proof.** This is immediate of the fact that $C_p(X,2)^n$ is homeomorphic to $C_p(X,2^n)$ for any $n \in \omega$ and the fact that a closed subspace of a Menger space is Menger. $\square$

**Definition 4.7.** A space $X$ is an *Eberlein–Grothendieck-space*, or an *EG-space*, if it is homeomorphic to a subspace of $C_p(Y)$ for some compact space $Y$. We say that $X$ is *Eberlein compact* if $X$ is a compact *EG*-space.

M. Sakai [20] shows that $C_p(X)$ has countable fan tightness for finite sets if and only if $X^n$ is Menger for each $n \in \omega$. Making use of this fact, it is clear that, indeed, all $EG$-spaces have countable fan tightness for finite sets. With everything we have already said, it is natural to conjecture that the spaces $X$ with $C_p(X,2)$ Menger are subspaces of $C_p(Y)$ where $Y^n$ is Menger for each $n \in \omega$. The following result shows that this is true as long as $X'$ is compact.

**Theorem 4.8.** *Let $X$ be a subspace of $C_p(Y)$ where $Y^k$ is Menger for every $k \in \omega$. If $X'$ is compact, then $C_p(X,2)^n$ is Menger for each $n \in \omega$.*

**Proof.** We adapt, for our purposes, the respective part of the proof of Theorem 4.15 from [7]. We only show that $C_p(X,2)$ is Menger; the $(n \geq 2)$-cases are shown similarly. For each $n \in \omega$, we define

$$F_n = \{\varphi \in 2^X : \exists (y_1, \dots, y_n) \in Y^n \forall f \in X' \big(\varphi\big[[f; y_1, \dots, y_n; 1/n]\big] = \{\varphi(f)\}\big)\},$$

where $[f; y_1, \dots, y_n; 1/n] = \{g \in C_p(Y) : \forall i(1 \leq i \leq n \to |f(y_i) - g(y_i)| < 1/n)\}$. Each $F_n$ is Menger. Indeed, for each $n \in \omega$, $F_n$ coincides with $\pi_2[S_n]$, where $\pi_2$ is the projection of $Y^n \times 2^X$ over $2^X$ and

$$S_n = \{(y_1, \dots, y_n, \varphi) \in Y^n \times 2^X : \forall f \in X' \big(g \in [f; y_1, \dots, y_n; 1/n] \to \varphi(f) = \varphi(g)\big)\}.$$

Now, Proposition 3.5 ensures that $Y^n \times 2^X$ is Menger. Then, to prove that $F_n$ is Menger, by Proposition 3.2, it is sufficient to show that $S_n$ is Menger. And to do this we proceed as follows: Let $(y_1^0, \dots, y_n^0, \varphi_0) \in Y^n \times 2^X \setminus S_n$. This means that there are $f_0 \in X'$ and $g_0 \in X$ such that $g_0 \in [f_0; y_1^0, \dots, y_n^0; 1/n]$ and $\varphi_0(f_0) \neq \varphi_0(g_0)$. Then, the open set

$$\left(\prod_{i=1}^{n} |f_0 - g_0|^{-1}\big[[0, 1/n)\big]\right) \times \{\varphi \in 2^X : \varphi(f_0) = \varphi_0(f_0) \wedge \varphi(g_0) = \varphi_0(g_0)\}$$

of $Y^n \times 2^X$ contains the point $(y_1^0, \dots, y_n^0, \varphi_0)$ and does not intersect $S_n$. Therefore, $S_n$ is a closed subset of the Menger space $Y^n \times 2^X$, and so it is Menger.

Now we will show that $C_p(X,2)$ is equal to $\bigcup_{n\in\omega} F_n$ and, by Proposition 3.4, our theorem will be proved. First observe that $C_p(X,2)$ is contained in $\bigcup_{n\in\omega} F_n$. In fact, fix a function $\varphi \in C_p(X,2)$. Since $\varphi$ is continuous, for each $f \in X'$ we can take a neighborhood $U_f$ of $f$ in $C_p(Y)$ such that $\varphi(g) = \varphi(f)$ if $g \in U_f \cap X$. Now, for each $f \in X'$ there are $n_f \in \omega$ and $y_1^f, \dots, y_{n_f}^f \in Y$ for which

$$f \in \big[f; y_1^f, \dots, y_{n_f}^f; 1/n_f\big] \cap X \subset U_f \cap X.$$

Since $X'$ is a compact space, there are points $f_0, \dots, f_k \in X'$ such that

$$X' \subset \big[f_0; y_1^{f_0}, \dots, y_{n_{f_0}}^{f_0}; 1/(2n_{f_0})\big] \cup \dots \cup \big[f_k; y_1^{f_k}, \dots, y_{n_{f_k}}^{f_k}; 1/(2n_{f_k})\big].$$

For each $f \in X'$ we take

$$V_f = \big[f; y_1^{f_0}, \dots, y_{n_{f_0}}^{f_0}, y_1^{f_1}, \dots, y_{n_{f_1}}^{f_1}, \dots, y_1^{f_k}, \dots, y_{n_{f_k}}^{f_k}; 1/l\big] \cap X$$

with $l = 2(n_{f_0} + \dots + n_{f_k})$. It is evident that the collection $\mathcal{V} = \{V_f : f \in X'\}$ covers $X'$. We have that $\mathcal{V}$ refines $\{U_f \cap X : f \in X'\}$. Indeed, if $f \in X'$, $f$ must belong to

$$\big[f_j; y_1^{f_j}, \dots, y_{n_{f_j}}^{f_j}; 1/(2n_{f_j})\big]$$

for some $j \in \{1, \dots, k\}$. Then, if $g \in V_f$ we have

$$\left|g\left(y_i^{f_j}\right) - f_j\left(y_i^{f_j}\right)\right| \leq \left|g\left(y_i^{f_j}\right) - f\left(y_i^{f_j}\right)\right| + \left|f\left(y_i^{f_j}\right) - f_j\left(y_i^{f_j}\right)\right| < \frac{1}{l} + \frac{1}{2n_{f_j}} \leq \frac{1}{n_{f_j}}.$$

Therefore, $g \in [f_j; y_1^{f_j}, \dots, y_{n_{f_j}}^{f_j}; 1/(n_{f_j})] \cap X$, and the latter set is contained in $U_{f_j} \cap X$.

Now we prove that $\varphi$ belongs to $F_l$. First note the following: if $f \in X'$, $g \in X$ and they satisfy $|f(x) - g(x)| < 1/l$ for all $x \in \{y_1^{f_0}, \dots, y_{n_{f_0}}^{f_0}, \dots, y_1^{f_k}, \dots, y_{n_{f_k}}^{f_k}\}$, then $g \in V_f$ and, consequently, $f, g \in U_h \cap X$ for some $h \in X'$. Because of the choice of $U_h$, $\varphi(g) = \varphi(h) = \varphi(f)$. Therefore, for each $f \in X'$, if $g \in [f; y_1^{f_0}, \dots, y_{n_{f_0}}^{f_0}, \dots, y_1^{f_k}, \dots, y_{n_{f_k}}^{f_k}; 1/l]$, $\varphi(f) = \varphi(g)$. This shows that $\varphi \in F_l$.

Finally we prove that, for each $n \in \omega$, $F_n \subset C_p(X, 2)$; that is, each element of $F_n$ is a continuous function. Let $\varphi \in F_n$ and $f \in X$. If $f$ is an isolated point of $X$ then $\varphi$ is continuous at $f$. Suppose $f \in X'$. By definition of $F_n$, there is $(y_1, \dots, y_n) \in Y^n$ such that $\varphi[[f; y_1, \dots, y_n; 1/n] \cap X] = \{\varphi(f)\}$. Since $[f; y_1, \dots, y_n; 1/n] \cap X$ is an open subset of $X$ containing $f$, $\varphi$ is continuous at $f$. $\square$

Given a space $X$, $C_p^*(X, \omega)$ denotes the subspace of $C_p(X)$ consisting of all bounded functions with values in $\omega$.

**Corollary 4.9.** *Let $X$ be a space and suppose that $X'$ is compact. Then the following statements are equivalent:*

(a) $C_p(X, 2)^n$ *is Menger for each* $n \in \omega$*;*
(b) $X \subset C_p(Y)$ *for some space* $Y$ *such that* $Y^n$ *is Menger for each* $n \in \omega$*;*
(c) $C_p^*(X, \omega)$ *is Menger.*

**Proof.** The equivalence of (a) and (c) is immediate from the fact that $C_p^*(X, \omega) = \bigcup_{n \in \omega} C_p(X, n)$ (see Propositions 3.2 and 3.4). (a) implies (b) follows from Theorem 4.8 and the fact that $C_p(X, 2)^n$ is homeomorphic to $C_p(X, 2^n)$ for each $n \in \omega$. And the proof of (b) implies (a) is as follows: For each $x \in X$, we define $\widetilde{x} : C_p(X, 2) \to 2$ as $\widetilde{x}(f) = f(x)$. It is not difficult to show that the function $x \mapsto \widetilde{x}$ is an embedding of $X$ into $C_p(C_p(X, 2))$. Then $Y = C_p(X, 2)$ is the required space. $\square$

A subspace $Y$ of a space $X$ is *bounded* in $X$ if for every continuous function $f : X \to \mathbb{R}$, $f \restriction Y$ is a bounded function, or equivalently, if every sequence of open sets in $X$, which meets $Y$, has an accumulation point in $X$.

Since $\mathbb{Q}$ and $\omega^\omega$ are second countable, $C_p(\mathbb{Q}, 2)$ and $C_p(\omega^\omega, 2)$ are Lindelöf. The following result rules out the possibility that $C_p(\mathbb{Q}, 2)$ and $C_p(\omega^\omega, 2)$ satisfy the Menger property.

**Theorem 4.10.** *If $C_p(X, 2)$ is Menger, then $X'$ is bounded in $X$.*

**Proof.** We proceed by contradiction. Suppose that there exists a sequence $\langle O_n : n \in \omega \rangle$ of open subsets of $X$ without accumulation points in $X$ such that $O_n \cap X' \neq \emptyset$. We can suppose without loss of generality that each element of the sequence is open and closed, and that any two different elements of this sequence are disjoint. Let $Y = X \setminus \bigcup_{n \in \omega} O_n$. Since the sequence $\langle O_n : n \in \omega \rangle$ does not have accumulation points, $Y$ is open and closed. Moreover, the family $\{O_n : n \in \omega\} \cup \{Y\}$ forms a partition of $X$ in clopen subsets of $X$. Then $C_p(X, 2)$ is homeomorphic to

$$\left(\prod_{n \in \omega} C_p(O_n, 2)\right) \times C_p(Y, 2).$$

For each $n \in \omega$, since $C_p(X, 2)$ is Menger, $C_p(O_n, 2)$ is Menger, and hence, Lindelöf. On the other hand, since each $O_n$ contains a non-isolated point of $X$, $C_p(O_n, 2)$ is a proper dense subspace of $2^{O_n}$. So, since

$C_p(O_n, 2)$ is Lindelöf, then $C_p(O_n, 2)$ is not countably compact; in particular, it contains a countable discrete closed subspace $D_n$. In this manner, $\prod_{n\in\omega} D_n$ is a closed subspace of $C_p(X, 2)$, and given that $C_p(X, 2)$ is Menger, $\prod_{n\in\omega} D_n$ is Menger, which is impossible since it is homeomorphic to $\omega^\omega$. □

**Corollary 4.11.** *If $X$ is a normal space and $C_p(X, 2)$ is Menger, then $X'$ is countably compact.*

**Proof.** By Theorem 4.10, $X'$ is bounded in $X$. Since $X$ is a normal space and $X'$ is a closed subset of $X$, $X'$ is pseudocompact. Again, by the normality of $X$, $X'$ is countably compact. □

**Corollary 4.12.** *Let $X$ be a Lindelöf space. Then $C_p(X, 2)^n$ is Menger for any $n \in \omega$ if and only if $X'$ is compact and $X \subset C_p(Y)$ for some space $Y$ such that $Y^n$ is Menger for each $n \in \omega$.*

**Proof.** If $C_p(X, 2)^n$ is Menger for any $n \in \omega$, then $C_p(X, n)$ is Menger for each $n \in \omega$ and, by Corollary 4.9, $X \subset C_p(Y)$ for some space $Y$ such that $Y^n$ is Menger for each $n \in \omega$. Furthermore, applying Corollary 4.11, $X'$ is countably compact and hence compact since $X$ is a Lindelöf space. The proof of the converse is a consequence of Theorem 4.8. □

A space is *σ-pseudocompact* if it is the countable union of pseudocompact subspaces. The following theorem appears in [1, III.4.23].

**Theorem 4.13 ([1]).** *If $X$ contains a dense σ-pseudocompact subspace, then every countably compact subspace of $C_p(X)$ is compact.*

**Corollary 4.14.** *Let $X$ be a space with $iw(X) = \omega$. Then the following statements are equivalent.*

(a) *$X'$ is compact and $X \subset C_p(Y)$ for some space $Y$ such that $Y^n$ is Menger for any $n \in \omega$;*
(b) *$C_p(X, 2)^n$ is Menger for any $n \in \omega$ and $X$ is a normal space.*

**Proof.** By Theorem 4.8, (a) implies that $C_p(X, 2)^n$ is Menger for any $n \in \omega$. Moreover, since $X$ is regular and $X'$ is compact, then $X$ is normal. Now suppose (b), since $X \subset C_p(C_p(X, 2))$, to prove (a) it is sufficient to show that $X'$ is compact. The normality of $X$ and Corollary 4.11 imply that $X'$ is a countably compact space. Given that $iw(X) = \omega$, $C_p(X, 2)$ is separable. By Theorem 4.13, $X'$ is compact. □

On metric spaces we have the following.

**Theorem 4.15.** *Let $X$ be a metrizable space. Then the following statements are equivalent.*

(a) *$C_p(X, 2)$ is Menger;*
(b) *$C_p(X, 2)^n$ is Menger for each $n \in \omega$;*
(c) *$C_p(X, 2)$ is σ-compact;*
(d) *$X'$ is compact.*

**Proof.** Since every metrizable space is an $EG$-space (see [1, IV.1.25]), $X$ is an $EG$-space. Then, if $X'$ is countably compact, it is compact being $X$ metrizable; so, $X'$ is Eberlein compact, and by Corollary 4.12 in [7], $C_p(X, 2)$ is σ-compact. This proves that (d) implies (c). Clearly (c) implies (b) and (b) implies (a). Finally, by Corollary 4.11, (a) implies (d). □

## 5. The Menger property on $C_p(L,2)$ when $L$ is a $GO$-space

A space $L$ is a *GO-space* (Generalized Ordered space) if it is a subspace of a linearly ordered topological space. Observe that if $L$ is a countable $GO$-space, then $L$ is zero-dimensional, separable and metrizable (and $C_p(L,2)$ is Lindelöf). Then, by Theorem 4.15 we obtain:

**Proposition 5.1.** *Let $L$ be a countable GO-space. Then the following statements are equivalents.*

(a) $C_p(L,2)$ *is Menger;*
(b) $C_p(L,2)$ *is $\sigma$-compact;*
(c) $L'$ *is compact.*

Now we are going to characterize the Menger property on $C_p(L,2)$ when $L$ is an uncountable $GO$-space. We will follow some notations, terminology and constructions due to R.Z. Buzyakova in [6]. First we will review a construction of the Dedekind completion of a given $GO$-space $L$.

**Definition 5.2.** An ordered pair $\langle A,B\rangle$ of disjoint closed subsets of a $GO$-space $L$ is called a *Dedekind section* if $A\cup B=L$, $\max A$ or $\min B$ do not exist, and $A$ is to the left of $B$; that is, for every $a\in A$ and $b\in B$, $a<b$ holds. A pair $\langle L,\emptyset\rangle$ ($\langle\emptyset,L\rangle$) is also a Dedekind section if $\max L$ ($\min L$) does not exist.

**Definition 5.3.** The *Dedekind completion* of $L$, denoted by $cL$, is constructed as follows. The set $cL$ is the union of $L$ and the set of all Dedekind sections of $L$. The order on $cL$ is natural: the order on $cL$ among elements of $L$ coincides with the order on $L$ of these elements. If $x\in L$ and $y=\langle A,B\rangle\in cL\setminus L$ then $x$ is less (greater) than $y$ if $x\in A$ ($x\in B$). If $x=\langle A_1,B_1\rangle$ and $y=\langle A_2,B_2\rangle$ are elements of $cL\setminus L$, then $x$ is less than $y$ if $A_1$ is a proper subset of $A_2$. Consider now $cL$ with the order topology generated by the order just defined. We will denote by $\infty$ and $-\infty$ the supremum and infimum, respectively, of $cL$.

Observe that for every $GO$-space $L$, $cL$ is a compact linearly ordered space.

For a given $GO$-space $L$ we consider the space $T(L)$:

**Definition 5.4.** An element $x\in cL$ is in $T(L)$ if and only if $x\in cL\setminus L$, or $x\in L$ and either $x$ is the smallest or the greatest element in $L$ or $x$ has an immediate successor in $L$. Points of $T(L)$ that are in $L$ are declared isolated. The other points inherit base neighborhoods from the Dedekind completion $cL$.

Observe that $T(L)$ is a $GO$-space. Indeed, $T(L)$ can be obtained from $cL$ as follows. For each $x\in L$ that has an immediate successor $x^+$ in $L$, insert a new point $p_x$ between $x$ and $x^+$. If $x\in L$ is the smallest element of $L$, we add a point $p_{-\infty}$ to the left of $x$; and if $x\in L$ is the greatest element of $L$, we add a point $p_\infty$ to the right of $x$. The resulting space is a compact linearly ordered topological space containing $cL$ as a closed subspace. The subspace of this structure that consists of all $p_x$'s and $cL\setminus L$ is a copy of $T(L)$. Thus we can think of $T(L)$ as a $GO$-space with the order inherited from $cL$. R.Z. Buzyakova presents in [6] some examples of $T(L)$ for some particular $GO$-spaces.

If $x_1,\dots,x_n\in cL$ and $-\infty\le x_1\le\dots\le x_n\le\infty$ then by $f=f^0_{x_1,\dots,x_n}$ we denote the function from $L$ to 2 defined by

$$f\bigl[(x_i,x_{i+1}]\cap L\bigr]=\{i \mod 2\},$$

for each $i\in\{1,\dots,n\}$. The rightmost formula is simply $\{1\}$ if $n$ is odd and $\{0\}$ otherwise. The functions $f^1_{x_1,\dots,x_n}$ are defined similarly by changing $\{0\}$ with $\{1\}$ in the above formulas.

**Definition 5.5.** A function $f$ from $X$ to 2 belongs to $S_p(L,2)$ if and only if there exists $-\infty \leq x_1 \leq \cdots \leq x_n \leq \infty$ in $T(L)$ such that $f = f^0_{x_1,\ldots,x_n}$ or $f = f^1_{x_1,\ldots,x_n}$

Observe that $[-\infty, x_1] \cap L$, $(x_1, x_2] \cap L, \ldots, (x_n, \infty] \cap L$ are clopen because $x_1, \ldots, x_n \in T(L)$. Therefore $f^0_{x_1,\ldots,x_n}$ is continuous and $S_p(L,2) \subset C_p(L,2)$. The topology of $S_p(L,2)$ is the topology of subspace of $C_p(L,2)$. For each $n \geq 1$, we define

$$S_n = \{f \in S_p(L,2) : \exists x_1, \ldots, x_n \in T(L)(f = f^0_{x_1,\ldots,x_n} \vee f = f^1_{x_1,\ldots,x_n})\}.$$

Observe that $S_p(L,2) = \bigcup_{1 \leq n} S_n$, and $S_p(L,2) = C_p(L,2)$ if $L$ is countably compact.

We are going to denote by $S^*$ the subspace $\{f^1_x : x \in T(L)\}$ of $S_1$.

**Lemma 5.6 ([6]).** *Let $L$ be a GO-space. Then for any $f \in S_p(L,2)$, there exist $f_1, \ldots, f_k \in S^*$ such that $f = f_1 + \cdots + f_k$.*

More properties on $S^*$ are given in [6]. One of these is the following:

**Theorem 5.7 ([6]).** *The subspace $S^*$ of $S_p(L,2)$ is homeomorphic to $T(L)$.*

For a countably compact $GO$-space $L$, R.Z. Buzyakova proves in [6] that $C_p(L,2)$ is Lindelöf if and only if $T(L)$ is Lindelöf. We show that in fact this is a sufficient condition in order to have $C_p(L,2)$ Menger.

**Theorem 5.8.** *Let $L$ be a first countable GO-space without isolated points. The following statements are equivalent.*

(a) *$C_p(L,2)$ is Lindelöf and $L$ is countably compact,*
(b) *$T(L)$ is Lindelöf and $L$ is countably compact,*
(c) *$T(L)^n$ is Menger for each $n \in \omega$ and $L$ is countably compact,*
(d) *$C_p(L,2)^n$ is Menger for each $n \in \omega$,*
(e) *$C^*_p(L,\omega)$ is Menger,*
(f) *$C_p(L,2)$ is Menger.*

**Proof.** The equivalence $(a) \Leftrightarrow (b)$ is Theorem 4.1 in [6]. We suppose (b) and we are going to prove (c). Let us show that $T(L)$ is a $P$-space. For each $n \in \omega$, let $U_n$ be an open subset of $T(L)$. We are going to prove that $F = \bigcap_{n\in\omega} U_n$ is open. Take any $x$ in this intersection. If $x \in L$ then $x$ is isolated in $T(L)$. If $x \notin L$ then, due to the countably compactness of $L$, $x$ is unreachable by nontrivial countable sequence in $cL$, and therefore, in $T(L)$. In both cases, we conclude that $x$ is in the interior of $F$. This shows that $T(L)$ is a $P$-space. Then $T(L)^n$ is a $P$-space for $n \in \omega$. Applying Noble's theorem [16], a countable power of a Lindelöf $P$-space is Lindelöf, $T(L)^\omega$ is Lindelöf, and hence, $T(L)^n$ is Lindelöf for any $n \in \omega$. But Lindelöf property agrees with Menger property in $P$-spaces (see Proposition 3.6). Then $T(L)^n$ is Menger for any $n \in \omega$.

Now suppose (c). Given that $T(L)$ is homeomorphic to $S^*$ (see Theorem 5.7) and the countable union of Menger spaces is Menger, the topological sum $\bigoplus_{k\in\omega}(S^*)^k$ is Menger. Moreover, every finite power of this space is Menger. Besides, if we define the continuous function $\mathcal{F} : \bigoplus_{k\in\omega}(S^*)^k \to S_p(L,2)$ as $\mathcal{F}(F) = f_1 + \cdots + f_k$ where $F = (f_1, \ldots, f_k) \in (S^*)^k$. Then by Lemma 5.6, $\mathcal{F}$ is surjective. Then, for each $n \in \omega$, the function $\mathcal{F}^n : (\bigoplus_{k\in\omega}(S^*)^k)^n \to S_p(L,2)^n$ defined by $\mathcal{F}^n(F_1, \ldots, F_n) = (\mathcal{F}(F_1), \ldots, \mathcal{F}(F_n))$ is a surjective continuous function. Thus, $S_p(L,2)^n = C_p(L,2)^n$ is Menger. This shows that (c) implies (d).

(d) implies (e) is trivial. Since $C_p(L,2)$ is a close subspace of $C^*_p(L,\omega)$, by Proposition 3.2, (e) implies (f). Finally, if we suppose (f), then, by Corollary 4.11, $L = L'$ is countably compact and clearly $C_p(L,2)$ is Lindelöf. This proves that (f) implies (a). □

**Corollary 5.9.** *Let $L$ be a first countable countably compact GO-space without isolated points. Then, $C_p(L,2)$ is Lindelöf if and only if $C_p(L,2)$ is Menger.*

**Problem 5.10.** Determine when $C_p(L,2)$ is Menger, when $L$ is a first countable $GO$-space (without any restriction about the isolated points in $L$).

## 6. The Menger property on $C_p(X,2)$ when $X$ is a subspace of ordinals

By a *subspace of ordinals* we are referring to a subspace of an ordinal $\alpha$. As we have already said, the set of ordinals lower than an ordinal $\alpha$ endowed with its order topology is denoted by $\alpha$. As a corollary of Proposition 5.1 we have the following.

**Corollary 6.1.** *Let $\alpha \in \omega_1$. Then the following statements are equivalent.*

(a) *$C_p(\alpha,2)$ is Menger,*
(b) *$C_p(\alpha,2)$ is $\sigma$-compact,*
(c) *$\alpha$ is a successor ordinal.*

If $X$ is normal and $C_p(X,2)$ is Menger, then $X'$ is countably compact and $X$ has countable fan tightness (see Corollary 4.11 and Proposition 4.1). Then, when $X$ is a subspace of ordinals and $C_p(X,2)$ is Menger, $X'$ must be countably compact and $X$ is first countable. In the following statements we see that these properties are enough.

The proof of the following theorem was suggested to the referee by Professor Piotr Szewczak. His proof is simpler than one we gave in a previous version of this paper.

**Theorem 6.2.** *Let $X$ be a subspace of ordinals and $n \in \omega \setminus \{0\}$. Then $C_p(X,2)^n$ is Menger if and only if $C_p(X,2)^n$ is Lindelöf and $X'$ is countably compact.*

**Proof.** Remember that $C_p(X,2)^n$ is homeomorphic to $C_p(X,2^n)$. We are going to show our theorem when $n=1$ (the proof for $2^n$ instead of 2 is similar). It is obvious that if $C_p(X,2)$ is Menger, then $C_p(X,2)$ is Lindelöf. Moreover, by Corollary 4.11, $X'$ is countably compact. Reciprocally, we suppose that $C_p(X,2)$ is Lindelöf and $X'$ is countably compact. We will show that $C_p(X,2)$ is Menger. We will prove this fact by induction over $\alpha = \sup X$. Let us assume that the statement is true for every $\beta < \alpha$. Let $\delta = \sup X'$.

**Case I.** If $\delta < \alpha$, then $X' \subset Z = X \cap (\delta+1)$, Z is clopen in $X$ and $X \setminus Z$ is clopen and discrete in X. Therefore $X = Z \oplus (X \setminus Z)$ and $C_p(X,2) \cong C_p(Z,2) \times 2^{X \setminus Z}$. It is easy to see that $C_p(Z,2)$ is Lindelöf as a closed subspace of $C_p(X,2)$ and $Z' = X'$ so it is countably compact. From the assumption we have that $C_p(Z,2)$ is Menger and by compactness of $2^{X\setminus Z}$ the space $C_p(X,2)$ is also Menger.

**Case II.** If $\delta = \alpha$ and there is in $X$ an increasing countable sequence $\langle \alpha_n : n \in \omega\rangle$ which converges to $\alpha$. Then by $\delta = \alpha$ and countably compactness of $X'$ we infer that $\alpha \in X'$. Let us observe that family $\{X \cap (\alpha_n, \alpha] : n \in \omega\}$ forms a base at $\alpha$. Since every $f \in C_p(X,2)$ is continuous at $\alpha$ and $\alpha \in X'$ so there is $k \in \omega$ such that $f[(\alpha_n,\alpha]] = \{f(\alpha)\}$. Let $A_n^j = \{f \in C_p(X,2) : \forall x(x \in (\alpha_n,\alpha] \to f(x) = j)\}$ for every $n \in \omega$ and $j \in 2$. We have that

$$C_p(X,2) = \bigcup \{A_n^j : n \in \omega \wedge j \in 2\}$$

Every $A_n^j$ is homeomorphic to $C_p(Z_n,2)$ where $Z_n = X \cap (\alpha+1)$. Since $Z_n$ is clopen in $X$ we can easily verify that $C_p(Z_n,2)$ is Lindelöf and $Z_n'$ is countably compact. Now it follows from inductive assumption that $A_n^j \cong C_p(Z_n, 2)$ is Menger. Because $C_p(X, 2) = \bigcup\{A_n^j : n \in \omega \wedge j \in 2\}$ we conclude that $C_p(X, 2)$ is Menger.

**Case III.** If $\delta = \alpha$ and the cofinality of $\alpha$ in $X$ is not countable. Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of countable open covers of $C_p(X, 2)$ consisting of open basis sets. Let us observe that each element $U \in \mathcal{U}_n$ is in the form $U = \prod_{x\in X} U(x) \cap C_p(X, 2)$ where $U(x) \neq 2$ only if $x \in X_U \subset X$, where $X_U$ is finite. Let us observe that there is some $\beta \in X$ such that $\bigcup\{X_U : \exists n(n \in \omega \wedge U \in \mathcal{U}_n)\} \subset X \cap (\beta + 1) = Z$. Clearly $\beta = \sup Z$. Then

$$\forall U \forall x\big(\exists n(n \in \omega \wedge U \in \mathcal{U}_n) \wedge x \in X \setminus Z \to U(x) = 2\big). \qquad (*)$$

It is easy to see that $Z$ is a clopen subset of $X$ and $Z' = X \cap (\beta + 1)$. Hence $Z'$ is countably compact as a closed subset of countably compact space $X'$. Since $C_p(X, 2)$ is homeomorphic to $C_p(Z, 2) \times C_p(X \setminus Z, 2)$ we have that $C_p(Z, 2)$ as a closed subset of $C_p(X, 2)$ is Lindelöf. By inductive assumption $C_p(Z, 2)$ is Menger. Now let $\mathcal{U}'_n = \{U \cap C_p(Z, 2) : U \in \mathcal{U}_n\}$ for every $n \in \omega$. Then $\langle \mathcal{U}'_n : n \in \omega\rangle$ is a sequence of open cover of $C_p(Z, 2)$. Therefore, there are $\mathcal{V}'_n \in [\mathcal{U}'_n]^{<\omega}$ such that $\bigcup\{\mathcal{V}'_n : n \in \omega\}$ covers $C_p(Z, 2)$. For every $n \in \omega$ pick $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ such that $\mathcal{V}_n = \{U \cap C_p(Z, 2) : U \in \mathcal{V}'_n\}$. By $(*)$ we have that $\bigcup\{\mathcal{V}_n : n \in \omega\}$ covers $C_p(X, 2)$. □

It is shown in [4] that for every countably compact first countable subspace $X$ of ordinals, $C_p(X, 2)^n$ is Lindelöf for each $n \in \omega$. Therefore:

**Corollary 6.3.** *For any countably compact first countable subspace $X$ of ordinals, $C_p(X, 2)^n$ is Menger for each $n \in \omega$.*

It is shown in [5] that every first countable subspace $X$ of ordinals with countable extent has $C_p(X, 2)^n$ is Lindelöf for each $n \in \omega$. So, we obtain:

**Corollary 6.4.** *Let $X$ be a first countable subspace of ordinals with countable extent. Then $C_p(X, 2)^n$ is Menger for each $n \in \omega$ if and only if $X'$ is countably compact.*

Corollary 4.11 shows that in the class of normal spaces $X$, if $C_p(X, 2)$ is Menger, then $X'$ is countably compact. With the same hypotheses we cannot imply the compactness of $X'$. Indeed, by Corollary 6.3, $C_p(\omega_1, 2)$ is Menger.

Observe, on the one hand, that the ordinal number $X = \omega \cdot \omega$ is countable, metrizable ordinal subspace such that $C_p(X, 2)$ is Lindelöf but $C_p(X, 2)$ is not Menger (see Theorem 6.2). So, it is not possible to add the statement "$C_p(X, 2)$ is Lindelöf" in the list of equivalent claims neither in Theorem 4.15 nor in Proposition 5.1, nor in Corollary 6.1 (compare with Corollary 5.9). On the other hand, the converse of Corollary 6.3 is not true. Indeed, $C_p(\omega, 2)$ is Menger and $\omega$ is not countably compact. A nondiscrete example of the same fact is the countable metrizable ordinal subspace $Y = (\omega \cdot \omega + 1) \setminus \{\omega\}$.

Moreover, it is natural to conjecture that the class of subspaces of ordinals $X$ for which $C_p(X, 2)$ is Menger is equal to the class of ordinal subspaces which are the topological sum of two subspaces, one of them a discrete subspace and the other a first countable countably compact ordinal subspace. This is not true. In fact, consider $X = \{\omega_1 \cdot n : n \leq \omega\} \cup \bigcup_{n\in\omega}\{(\omega_1 \cdot n) + m : m \in \omega\}$. We have that $C_p(X, 2)$ is Menger but $X$ cannot be expressed as the sum of a discrete subspace plus a countably compact first countable ordinal subspace. Also, it is natural to conjecture that Corollary 6.3 is valid for any *GO*-space (or *LOTS*) not only for subspaces of ordinals, but Example 4.5 shows that this is false.

In [5] R.Z. Buzyakova asks if $C_p(X, 2)$ is Lindelöf when $X$ is a first countable subspace of ordinals and $X'$ is countably compact. We ask the same question in the following form (see Theorem 6.2):

**Problem 6.5.** Is there a first countable ordinal subspace $X$ with $X'$ countably compact such that $C_p(X,2)$ is not Menger?

## 7. The Menger property on $C_p(X,2)$ when $X$ is a countable simple space

A space $X$ is called *simple* if $X$ has exactly one non-isolated point. For any filter $\mathcal{F}$ on $\omega$, we define the space $\omega \cup \{\mathcal{F}\}$ as follows: any $n \in \omega$ is declared isolated and the sets $A \cup \{\mathcal{F}\}$, where $A \in \mathcal{F}$, form a base of neighborhoods of $\mathcal{F}$. Any countable simple space is homeomorphic to $\omega \cup \{\mathcal{F}\}$ for a filter $\mathcal{F}$ on $\omega$.

It is proved in [1, III.3.3] that if $A_\tau$ denotes the one-point compactification of the discrete space of cardinality $\tau$, then $A_\tau$ is Eberlein compact. And therefore $C_p(A_\tau,2)$ is Menger. It is also shown in [1, III.1.7] that if $\mathcal{F} \in \omega^*$, $\omega \cup \{\mathcal{F}\}$ is not an *EG*-space; that is, $\omega \cup \{\mathcal{F}\}$ cannot be embedded in a space $C_p(Y)$ where $Y$ is compact. The following results shows that, under certain conditions, $\omega \cup \{\mathcal{F}\}$ can be embedded in a space $C_p(Y)$ for some space $Y$ for which $Y^n$ is Menger for every $n \in \omega$. These conditions are set in the following definitions.

**Definition 7.1 ([13]).** An ultrafilter $\mathcal{F} \in \omega^*$ is a *strong P-point* if for any sequence $\langle \mathcal{C}_n : n \in \omega\rangle$ of compact subspaces of $\mathcal{F}$ (considering $\mathcal{F}$ as a subset of $2^\omega$ with the product topology) there is an interval partition $\langle I_n : n \in \omega\rangle$ of $2^\omega$ such that for each choice of $X_n \in \mathcal{C}_n$ we have

$$\bigcup_{n\in\omega}(I_n \cap X_n) \in \mathcal{F}.$$

Given a filter $\mathcal{F}$ on $\omega$ we define $\mathcal{F}^{<\omega}$ to be the filter on $[\omega]^{<\omega} \setminus \{\emptyset\}$ generated by $\{[F]^{<\omega} \setminus \{\emptyset\} : F \in \mathcal{F}\}$. Note that the filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega} \setminus \{\emptyset\}$ is not an ultrafilter even if $\mathcal{F}$ is.

**Definition 7.2 ([3]).** A filter $\mathcal{F}$ on a countable set $S$ is a $P^+$-filter if for any $\subset$-descending sequence $\langle X_n : n \in \omega\rangle \subset \mathcal{F}^+$, there is an $X \in \mathcal{F}^+$ such that $X \subset^* X_n$ for all $n \in \omega$, where $\mathcal{F}^+ = \{X \subset S : S\setminus X \notin \mathcal{F}\}$.

The elements of $\mathcal{F}^+$ are called *positive sets (with respect to $\mathcal{F}$)*. Then, a filter $\mathcal{F}$ is a $P^+$-filter if every decreasing sequence of positive sets has a positive pseudointersection. The definition of a strong $P$-point that we will use is the following.

**Theorem 7.3 ([3]).** *An ultrafilter $\mathcal{F} \in \omega^*$ is a strong $P$-point if and only if $\mathcal{F}^{<\omega}$ is a $P^+$-filter.*

The following result was conjectured by M. Hrušák and is the key to characterize the Menger property on $C_p(\omega \cup \{\mathcal{F}\},2)$.

**Proposition 7.4.** *Let $\mathcal{F}$ be a filter on $\omega$. The space $X = \omega \cup \{\mathcal{F}\}$ has countable fan tightness for finite sets if and only if $\mathcal{F}^{<\omega}$ is a $P^+$-filter.*

**Proof.** First note that $\mathcal{P} \in (\mathcal{F}^{<\omega})^+$ if and only if $\mathcal{P}$ is a $\pi$-network at $\mathcal{F}$ in $X$. Suppose that $\mathcal{F}^{<\omega}$ is a $P^+$-filter. Let $\langle \mathcal{P}_n \subset [X]^{<\omega} : n \in \omega\rangle$ be a decreasing sequence of $\pi$-networks at $\mathcal{F}$ (see Lemma 4.3). Given that $\mathcal{F}^{<\omega}$ is a $P^+$-filter, and $\langle \mathcal{P}_n \subset [X]^{<\omega} : n \in \omega\rangle$ is a decreasing sequence of positive sets with respect to $\mathcal{F}^{<\omega}$, $\langle \mathcal{P}_n : n \in \omega\rangle$ has a positive pseudointersection $\mathcal{P} \in (\mathcal{F}^{<\omega})^+$. Since $\mathcal{P} \setminus \mathcal{P}_0$ is finite, $\mathcal{P} \cap \mathcal{P}_0$ is a $\pi$-network at $\mathcal{F}$. Then, if we suppose that $\bigcap_{n\in\omega}\mathcal{P}_n = \{p_n : n \in \omega\}$, and define $\mathcal{K}_n = (\mathcal{P} \cap \mathcal{P}_n \setminus \mathcal{P}_{n+1}) \cup \{p_n\}$ for each $n \in \omega$, $\bigcup_{n\in\omega}\mathcal{K}_n = \mathcal{P} \cap \mathcal{P}_0$ is a $\pi$-network at $\mathcal{F}$. Observe that $\mathcal{K}_n$ is finite because $(\mathcal{P}\cap\mathcal{P}_n\setminus\mathcal{P}_{n+1})\cup\{p_n\} \subseteq (\mathcal{P}\setminus\mathcal{P}_{n+1})\cup\{p_n\}$ and $\mathcal{P}$ is a pseudointersection of the family $\{\mathcal{P}_n : n < \omega\}$.

Reciprocally, suppose that $\omega \cup \{\mathcal{F}\}$ has countable fan tightness for finite sets. Let $\langle \mathcal{P}_n : n \in \omega\rangle$ be a decreasing sequence of positive sets. Since $\langle \mathcal{P}_n : n \in \omega\rangle$ is a sequence of $\pi$-networks at $\mathcal{F}$, there is a sequence of finite sets $\langle \mathcal{F}_n : n \in \omega\rangle$ such that $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{F}_n$ is a $\pi$-network at $\mathcal{F}$ and $\mathcal{F}_n \subset \mathcal{P}_n$ for each $n \in \omega$. Then $\mathcal{P}$ is a positive set and, since $\mathcal{P} \setminus \mathcal{P}_{n+1} \subset \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$, $\mathcal{P}$ is a pseudointersection of $\langle \mathcal{P}_n : n \in \omega\rangle$. □

**Corollary 7.5.** *Let $\mathcal{F} \in \omega^*$. The subspace $\omega \cup \{\mathcal{F}\}$ of $\beta\omega$ has countable fan tightness for finite sets if and only if $\mathcal{F}$ is a strong $P$-point.*

**Theorem 7.6.** *Let $\mathcal{F}$ be a filter on $\omega$ and $X = \omega \cup \{\mathcal{F}\}$. Then, the following statements are equivalent:*

(a) *$C_p(X, 2)$ is Menger;*
(b) *$C_p(X, 2)^n$ is Menger for any $n \in \omega$;*
(c) *$C_p^*(X, \omega)$ is Menger;*
(d) *$\mathcal{F}^{<\omega}$ is a $P^+$-filter.*

**Proof.** If $C_p(X, 2)$ is Menger, then, by Propositions 4.4 and 7.4, $\mathcal{F}^{<\omega}$ is a $P^+$-filter. Now assume that $\mathcal{F}^{<\omega}$ is a $P^+$-filter in $\omega$. We are going to show that $C_p(X, 2)$ is Menger (the proof for $n$ instead of 2 is similar). For each $k \in 2$ and $F \subset \omega$, we define $A^k_F = \{f \in 2^\omega : \forall x(x \in F \to f(x) = k)\}$. Note that $C_p(X, 2)$ is homeomorphic to the subspace $\bigcup\{A^k_F : F \in \mathcal{F} \wedge k \in 2\}$ of $2^\omega$. Then, to see that $C_p(X, 2)$ is Menger, by Proposition 3.4, it is enough to show that $\bigcup\{A^k_F : F \in \mathcal{F}\}$ is Menger for each $k \in 2$. However, $\bigcup\{A^k_F : F \in \mathcal{F}\}$ is homeomorphic to $\bigcup\{A^m_F : F \in \mathcal{F}\}$ for $k, m \in 2$. So, it is enough to prove that $A = \bigcup\{A^0_F : F \in \mathcal{F}\}$ is Menger. To simplify the notations, we write $A_F$ to mean $A^0_F$ and, if $F$ is a single point $x$, we write $A_x$ instead of $A_F$.

Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of countable covers of $A$ such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n \in \omega$ (see Lemma 3.3). We can suppose that each $\mathcal{U}_n$ is closed under finite unions. For each open subset of $A$, let $Y_U = \{H \in [\omega]^{<\omega} : A_H \subset U\}$. And we define $Z_n = \bigcup_{U\in\mathcal{U}_n} Y_U$ for each $n \in \omega$. In view of the fact that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, $Z_{n+1} \subset Z_n$. Moreover, $Z_n$ is a positive set. Indeed, if $F \in \mathcal{F}$, since $A_F$ is compact, there is an element $U \in \mathcal{U}_n$ containing $A_F$. Given that $A_F = \bigcap_{x\in F} A_x$ and $A_x$ is compact, there is $H \in [F]^{<\omega}$ such that $A_H = \bigcap_{x\in H} A_x \subset U$. Then $[F]^{<\omega} \cap Z_n \neq \emptyset$. Now, since $\mathcal{F}^{<\omega}$ is a $P^+$-filter, the sequence of positive sets $\langle Z_n : n \in \omega\rangle$ has a positive pseudointersection $\widetilde{Z} \in (\mathcal{F}^{<\omega})^+$. Suppose that $\bigcap_{n\in\omega} Z_n = \{b_n : n \in \omega\}$, then we define, for each $n \in \omega$, $\mathcal{P}_n = (\widetilde{Z} \cap Z_n \setminus Z_{n+1}) \cup \{b_n\}$. In the same way as in Proposition 7.4 we infer that $\mathcal{P}_n$ is finite for every $n \in \omega$. In this manner $Z = \bigcup_{n\in\omega} \mathcal{P}_n = \widetilde{Z} \cap Z_0 \in (\mathcal{F}^{<\omega})^+$. For each $n \in \omega$ and $H \in \mathcal{P}_n$, we choose $U_H \in \mathcal{U}_n$ such that $A_H \subset U_H$. We define $\mathcal{F}_n = \{U_H : h \in \mathcal{P}_n\}$ for each $n \in \omega$. Given $f \in A$, there is $F \in \mathcal{F}$ such that $f \in A_F$. Since $Z$ is a positive set, $[F]^{<\omega} \setminus \{\emptyset\}$ intersects $Z$ and, hence, intersects some $\mathcal{P}_n$. Consequently, if $H \in ([F]^{<\omega} \setminus \{\emptyset\}) \cap \mathcal{P}_n$, then $A_F \subset A_H \subset U_H$. This proves that $f \in U_H$. That is, $\bigcup_{n\in\omega} \mathcal{F}_n$ is a cover of $A$.

The implication $(b) \Rightarrow (c)$ is a consequence of Proposition 3.4 and the equality $C_p^*(X, \omega) = \bigcup_{n\in\omega} C_p(X, n)$. The implication $(c) \Rightarrow (b)$ is a consequence of Proposition 3.2 and the fact that each $C_p(X, n)$ is a closed subset of $C_p^*(X, \omega)$. □

As a consequence of Theorems 7.3 and 7.6 we conclude:

**Corollary 7.7.** *Let $\mathcal{F} \in \omega^*$ and $X$ the subspace $\omega \cup \{\mathcal{F}\}$ of $\beta\omega$. Then the following statements are equivalent:*

(a) *$C_p(X, 2)$ is Menger;*
(b) *$C_p(X, 2)^n$ is Menger for any $n \in \omega$;*
(c) *$C_p^*(X, \omega)$ is Menger;*
(d) *$\mathcal{F}$ is a strong $P$-point.*

As previously mentioned, Example III.1.7 in [1] shows that $\omega \cup \{\mathcal{F}\}$ is not an $EG$-space and, by Theorem 4.16 in [7], $C_p(\omega \cup \{\mathcal{F}\}, 2)$ is not $\sigma$-compact. Then, by Corollary 7.7, if $\mathcal{F}$ is a strong $P$-point, $C_p(\omega \cup \{\mathcal{F}\}, 2)$ is a Menger space which is not $\sigma$-compact.

## 8. The Menger property on $C_p(\Psi(\mathcal{A}), 2)$

An *almost disjoint family* of subsets of $\omega$ is a collection $\mathcal{A}$ of subsets of $\omega$ such that each element in $\mathcal{A}$ is infinite, and if $A, B \in \mathcal{A}$, $|A \cap B| < \aleph_0$. An almost disjoint family $\mathcal{A}$ is maximal if it is not proper subfamily of an another almost disjoint family. For an infinite maximal almost disjoint family (mad) $\mathcal{A}$ on $\omega$, a $\Psi$-space is a space $\Psi(\mathcal{A})$ whose underlying set is $\omega \cup \mathcal{A}$ and the topology is given by: All points of $\omega$ are isolated, and the neighborhood base at $A \in \mathcal{A}$ consists of all sets $\{A\} \cup A \setminus F$ where $F$ is a finite subset of $\omega$. A. Dow shows in [8] that if $\mathfrak{b} > \omega_1$, for each mad family $\mathcal{A}$, $C_p(\Psi(\mathcal{A}), 2)$ is not Lindelöf and, hence, in this case, $C_p(\Psi(\mathcal{A}), 2)$ is not Menger. M. Hrušák, P.J. Szeptycki and Á. Tamariz-Mascarúa show in [10], assuming CH, the existence of a *Mrówka mad family* $\mathcal{A}$ (that is, the one-point compactification of $\Psi(\mathcal{A})$ coincides with its Stone–Čech compactification) such that $C_p(\Psi(\mathcal{A}), 2)$ is Lindelöf.

For a mad family $\mathcal{A}$ and $j \in 2$, we define the closed subspace $\sigma_n^j(\mathcal{A}) = \{f \in C_p(\Psi(\mathcal{A}), 2) : |f^{-1}(j) \cap \mathcal{A}| \leq n\}$ of $C_p(\Psi(\mathcal{A}), 2)$. If $\mathcal{A}$ is a Mrówka family, then $C_p(\Psi(\mathcal{A}), 2) = \bigcup_{n\in\omega, j\in 2} \sigma_n^j(\mathcal{A})$. For every $n \in \omega$, $\sigma_n^0(\mathcal{A})$ is homeomorphic to $\sigma_n^1(\mathcal{A})$. We are going to write $\sigma_n(\mathcal{A})$ instead of $\sigma_n^1(\mathcal{A})$. Thus, by Proposition 3.4:

**Lemma 8.1.** *If $\mathcal{A}$ is a Mrówka mad family then $C_p(\Psi(\mathcal{A}), 2)$ is Menger if and only if $\sigma_n(\mathcal{A})$ is Menger for each $n \in \omega$.*

To characterize when $\sigma_n(\mathcal{A})$ is Menger, we need certain terminology and notation. For $a, b \in \mathcal{P}(\omega)$, $a \triangle b$ will denote their symmetric difference; that is $a \triangle b = (a \cup b) \setminus (a \cap b)$. Given a mad family $\mathcal{A}$ and $\mathcal{Y} \subset \mathcal{P}(\omega)$, we will say that $\mathcal{A}^n$ is *concentrated* on $\mathcal{Y}$ [10], if for each open $U$ of the Cantor set $2^\omega$ containing $\chi_\mathcal{Y} = \{\chi_y : y \in \mathcal{Y}\}$, there is a countable $\mathcal{B} \subset \mathcal{A}$ such that $\chi_{\cup x} \in U$ for all $x \in [\mathcal{A} \setminus \mathcal{B}]^n$. And we will say that $\mathcal{A}^n + [\omega]^{<\omega}$ is *concentrated* on $\mathcal{Y}$ if for each open subset $U$ of $2^\omega$ containing $\chi_\mathcal{Y}$, there is a countable subset $\mathcal{B} \subset \mathcal{A}$ such that $\chi_{(\cup x)\triangle b} \in U$ for all $x \in [\mathcal{A} \setminus \mathcal{B}]^n$ and for all $b \in [\omega]^{<\omega}$.

**Lemma 8.2.** *Let $\mathcal{A}$ be a mad family. If $\mathcal{A}^{n+1} + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ and $\sigma_n(\mathcal{A})$ is Menger, then $\sigma_{n+1}(\mathcal{A})$ is Menger.*

**Proof.** The proof depends on two claims.

**Claim 1.** *If $V$ is an open subset of $\sigma_{n+1}(\mathcal{A})$ containing $\sigma_n(\mathcal{A})$, then there is a countable subset $\mathcal{B} \subset \mathcal{A}$ such that $f^{-1}(1) \cap \mathcal{B} \neq \emptyset$ for any $f \in \sigma_{n+1}(\mathcal{A}) \setminus V$.*

Indeed, since $\sigma_0(\mathcal{A})$ is a countable subset of $\sigma_n(\mathcal{A})$, we can choose a sequence of finite functions $s_k \subset \Psi(\mathcal{A}) \times 2$ such that $\sigma_0(\mathcal{A}) \cap [s_k] \neq \emptyset$ and $\sigma_0(\mathcal{A}) \subset \bigcup_{k\in\omega}[s_k] \subset V$, where $[s_k] = \{f \in \sigma_{n+1}(\mathcal{A}) : s_k \subset f\}$ for each $k \in \omega$. Note that $s_k^{-1}(1) \subset \omega$ and $s_k \restriction \mathcal{A}$ is the constant zero for each $k \in \omega$. We define the open subset $U$ of $2^\omega$ to be $\bigcup_{k\in\omega}\{f \in 2^\omega : s_k \restriction \omega \subset f\}$ and note that $\chi_{[\omega]^{<\omega}} \subset U$. Then, by hypothesis, there is a countable subset $\mathcal{B}' \subset \mathcal{A}$ such that $\chi_{\bigcup x \triangle b} \in U$ for all $x \in [\mathcal{A} \setminus \mathcal{B}']^{n+1}$ and for all $b \in [\omega]^{<\omega}$. Let $\mathcal{B} = \mathcal{B}' \cup \bigcup_{k\in\omega}(s_k^{-1}(0) \cap \mathcal{A})$ and show that $\mathcal{B}$ is the required set by Claim 1. Let $f \in \sigma_{n+1}(\mathcal{A}) \setminus V$ and $x = f^{-1}(1) \cap \mathcal{A}$. Since $V$ contains $\sigma_n(\mathcal{A})$, $|x| = n + 1$. Proceed by contradiction, suppose that $x \cap \mathcal{B} = \emptyset$. We choose $b \in [\omega]^{<\omega}$ such that $f^{-1}(1) \cap \omega = \bigcup x \triangle b$. By the choice of $\mathcal{B}$, $\chi_{\bigcup x \triangle b} \in U$ and consequently, there is $k \in \omega$ such that $s_k^{-1}(1) \subset \bigcup x \triangle b = \omega \cap f^{-1}(1)$ and $s_k^{-1}(0) \cap \omega \subset \omega \setminus (\omega \cap f^{-1}(1)) = f^{-1}(0) \cap \omega$. Given that $x \cap s_k^{-1}(0) = \emptyset$, $s_k^{-1}(0) \subset f^{-1}(0)$. Then $f \in [s_k]$ which is a contradiction, and Claim 1 is proved.

**Claim 2.** *If $V$ is an open subset of $\sigma_{n+1}(\mathcal{A})$ containing $\sigma_n(\mathcal{A})$, then there is a countable subset $Y$ of $\sigma_1(\mathcal{A})$ such that $\sigma_{n+1}(\mathcal{A}) \setminus V \subset \bigcup_{h\in Y}(h + \sigma_n(\mathcal{A}))$, where $h + \sigma_n(\mathcal{A}) = \{h + g : g \in \sigma_n(\mathcal{A})\}$.*

Let $\mathcal{B}$ be the countable subset of $\mathcal{A}$ given by Claim 1, and define $Y = \{f \in \sigma_1(\mathcal{A}) : f^{-1}(1) \cap \mathcal{A} \subset \mathcal{B}\}$. Then $Y$ is countable. Let $f \in \sigma_{n+1}(\mathcal{A}) \setminus V$. Again, by the choice of $\mathcal{B}$, there is an element $a \in f^{-1}(1) \cap \mathcal{B}$. We define a continuous function $g : \Psi(\mathcal{A}) \to 2$ as follows

$$g(x) = \begin{cases} 1, & \text{if } x \in a \cup \{a\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $g \in Y$ and $f + g \in \sigma_n(\mathcal{A})$ and consequently $f = g + (f + g) \in \bigcup_{h \in Y}(h + \sigma_n(\mathcal{A}))$. This concludes the proof of Claim 2.

Now, we are going to finish the proof of our lemma. Let $\langle \mathcal{U}_k : k \in \omega \rangle$ be a sequence of covers of $\sigma_{n+1}(\mathcal{A})$. Since $\sigma_n(\mathcal{A})$ is Menger, there is a finite subset $\mathcal{F}'_k \subset \mathcal{U}_k$ for each $k \in \omega$ such that $\sigma_k(\mathcal{A}) \subset \bigcup \bigcup_{k \in \omega} \mathcal{F}'_k$. Then, by Claim 2, there is a countable subset $Y \subset \sigma_1(\mathcal{A})$ such that $\sigma_{k+1}(\mathcal{A}) \setminus \bigcup \bigcup_{k \in \omega} \mathcal{F}'_k \subset \bigcup_{h \in Y}(h + \sigma_n(\mathcal{A}))$. Since $\sigma_n(\mathcal{A})$ is homeomorphic to $h + \sigma_n(\mathcal{A})$ for each $h \in Y$ and $Y$ is countable, $\bigcup_{h \in Y}(h + \sigma_n(\mathcal{A}))$ is Menger. Then, there is a finite subset $\mathcal{F}''_k \subset \mathcal{U}_k$ for each $k \in \omega$ such that $\bigcup_{k \in \omega} \mathcal{F}''_k$ is a cover of $\sigma_{n+1}(\mathcal{A}) \setminus \bigcup \bigcup_{k \in \omega} \mathcal{F}'_k$. Therefore, the sequence $\langle \mathcal{F}'_k \cup \mathcal{F}''_k : k \in \omega \rangle$ is the required choice. $\square$

As we have already mentioned in the previous paragraphs, $\sigma_0(\mathcal{A})$ is countable. Then, by Lemma 8.2, if $\mathcal{A}^k + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ for each $k \leq n$, then $\sigma_n(\mathcal{A})$ is Menger. However, in [10, Corollary 4.3] M. Hrušák, P.J. Szeptycki and Á. Tamariz-Mascarúa prove the following two results.

**Proposition 8.3 ([10]).** *Let $\mathcal{A}$ be a mad family and $n \in \omega$. Then, $\mathcal{A}^n + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ if and only if $\mathcal{A}^n$ is concentrated on $[\omega]^{<\omega}$.*

**Corollary 8.4 ([10]).** *Suppose that $\mathcal{A}$ is a mad family and $n \in \omega$. Then, $\sigma_n(\mathcal{A})$ is Lindelöf if and only if $\mathcal{A}^k$ is concentrated on $[\omega]^{<\omega}$ for all $k \leq n$.*

These last two results with Lemma 8.2 imply the following result.

**Proposition 8.5.** *Let $\mathcal{A}$ be a mad family and $n \in \omega$. Then the following statements are equivalent.*

(a) *$\sigma_n(\mathcal{A})$ is Lindelöf;*
(b) *$\sigma_n(\mathcal{A})$ is Menger;*
(c) *$\mathcal{A}^k$ is concentrated on $[\omega]^{<\omega}$ for every $k \leq n$.*

**Proof.** It is clear that (b) implies (a) and by Corollary 8.4, (a) implies (c). Finally, Lemma 8.2 and Proposition 8.3 prove that (c) implies (b). $\square$

A corollary of the previous result is:

**Theorem 8.6.** *Let $\mathcal{A}$ be a Mrówka mad family. Then the following are equivalent.*

(a) *$C_p(\Psi(\mathcal{A}), 2)$ is Lindelöf;*
(b) *$C_p(\Psi(\mathcal{A}), 2)$ is Menger;*
(c) *$\mathcal{A}^n$ is concentrated on $[\omega]^{<\omega}$ for every $n \in \omega$.*

Theorem 4.5 in [10] shows, assuming CH, the existence of a Mrówka mad family $\mathcal{A}$ for which $\mathcal{A}^n$ is concentrated on $[\omega]^{<\omega}$ for all $n \in \omega$. Then we have the following:

**Theorem 8.7 (CH).** *There is a mad family $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}), 2)$ is Menger.*

**Problem 8.8.** Let $\mathcal{A}$ be the Mrówka mad family whose existence is guaranteed by Theorem 4.5 in [10]. Is $C_p(\Psi(\mathcal{A}), 2)^n$ Menger for every $n \geq 2$?

**Problem 8.9.** Are there a topological space $X$ and a natural number $n > 2$ such that $C_p(X, 2)$ is Menger and $C_p(X, 2)^n$ is not Menger?

## References


[1] A.V. Arhangel'skiĭ, Topological Function Spaces, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers, 1992.
[2] D. Bernal-Santos, The Rothberger property on $C_p(X, 2)$, manuscript.
[3] A. Blass, M. Hrušák, J. Verner, On strong $P$-points, Proc. Am. Math. Soc. 141 (8) (2013) 2875–2883.
[4] R.Z. Buzyakova, In search for Lindelöf $C_p$'s, Comment. Math. Univ. Carol. 45 (1) (2004) 145–151.
[5] R.Z. Buzyakova, Spaces of continuous characteristics functions, Comment. Math. Univ. Carol. 47 (4) (2006) 599–608.
[6] R.Z. Buzyakova, Function spaces over $GO$-spaces: Part I, Topol. Appl. 154 (2007) 917–924.
[7] A. Contreras-Carreto, Á. Tamariz-Macarúa, On some Generalizations of compactness in spaces $C_p(X, 2)$ y $C_p(X, \mathbb{Z})$, Bol. Soc. Mat. Mexicana (3) 9 (2003).
[8] A. Dow, P. Simon, Spaces of continuous functions over $\Psi$-space, Topol. Appl. 153 (13) (2006) 2260–2271.
[9] J. Gerlits, Zs. Nagy, Some properties of $C(X)$, I, Topol. Appl. 14 (1982) 151–161.
[10] M. Hrušák, P.J. Szeptycki, Á. Tamariz-Mascarúa, Spaces of functions defined on Mrówka spaces, Topol. Appl. 148 (2005) 239–252.
[11] W. Hurewicz, Über eine Verallgemeinerung des Borelschen Theorems, Math. Z. 24 (1926) 401–421.
[12] Lj.D.R. Kocinac, Selected results on selection principles, in: Sh. Rezapour (Ed.), Procceedings of the 3rd Seminar on Geometry and Topology, July 15–17, Tabriz, Iran, 2004, pp. 71–104.
[13] C. Laflamme, Forcing with filters and complete combinatorics, Ann. Pure Appl. Log. 42 (2) (1989) 125–163.
[14] A. Lelek, Some covering properties of spaces, Fundam. Math. 64 (2) (1969) 209–218.
[15] M.K. Menger, Einige Überdeckungssätze der Punktmengenlehre, Sitzungsber. Wien. Akad. 133 (1924) 421–444.
[16] N. Noble, Products with closed projections, Trans. Am. Math. Soc. 140 (1969) 381–391.
[17] F. Rothberger, Sur des families indenombrables de suites de nombres naturels, et le problémes concernant la propieté $C$, Math. Proc. Camb. Philos. Soc. 37 (1941) 109–126.
[18] M. Sakai, Property $C''$ and function spaces, Proc. Am. Math. Soc. 104 (1988) 917–919.
[19] M. Sakai, Spetial subsets of reals characterizing local properties of function spaces, in: L. Kočinac (Ed.), Selection Principles and Covering Properties in Topology, in: Quaderni di Matematica, vol. 18, Seconda Universita de Napoli, Caserta, 2006, pp. 195–225.
[20] M. Sakai, Selective separability of Pixley Roy hiperespaces, Topol. Appl. 159 (2012) 1591–1598.
[21] M. Scheepers, Selection principles and covering properties in topology, Note Mat. 22 (2003) 3–41.
[22] R. Telgaŕsky, Concerning product of paracompact spaces, Fundam. Math. 74 (1972) 153–159.
[23] B. Tsaban, Some new directions in infinite-combinatorial topology, in: J. Bagaria, S. Todorÿcevic (Eds.), Set Theory, in: Trends in Mathematics, Birkhäuser, 2006, pp. 225–255.
[24] B. Tsaban, Selections principles and special sets of reals, in: E. Pearl (Ed.), Open Problems in Topology II, Elsevier B.V., 2007, pp. 91–108.
[25] B. Tsaban, L. Zdomskyy, Hereditarily Hurewicz spaces and Arhangel'skiĭ sheaf amalga mations, arXiv:1004.0211v1 [math.GN], 1 Apr 2010.
[26] L. Wingers, Box products and Hurewicz spaces, Topol. Appl. 64 (1995) 9–21.