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Independent families and resolvability

S. García-Ferreira [a,1], M. Hrušák [a,2], Á. Tamariz-Mascarúa [b,*,3]

[a] Instituto de Matemáticas (Campus Morelia), Universidad Nacional Autónoma de México, Apartado Postal 61-3, Xangari, 58089 Morelia, Michoacán, Mexico
[b] Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, Mexico D.F., 04510 Mexico

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Let τ and γ be infinite cardinal numbers with $\tau \leqslant \gamma$. A subset Y of a space X is called C_τ-compact if $f[Y]$ is compact for every continuous function $f : X \to \mathbb{R}^\tau$. We prove that every C_τ-compact dense subspace of a product of γ non-trivial compact spaces each of them of weight $\leqslant \tau$ is 2^τ-resolvable. In particular, every pseudocompact dense subspace of a product of non-trivial metrizable compact spaces is $\mathfrak{c}$-resolvable. As a consequence of this fact we obtain that there is no σ-independent maximal independent family. Also, we present a consistent example, relative to the existence of a measurable cardinal, of a dense pseudocompact subspace of $\{0,1\}^{2^\lambda}$, with $\lambda = 2^{\omega_1}$, which is not maximally resolvable. Moreover, we improve a result by W. Hu (2006) [17] by showing that if maximal θ-independent families do not exist, then every dense subset of $\square_\theta\{0,1\}^\gamma$ is ω-resolvable for a regular cardinal number θ with $\omega_1 \leqslant \theta \leqslant \gamma$. Finally, if there are no maximal independent families on κ of cardinality γ, then every Baire dense subset of $\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$ and every Baire dense subset of $[0,1]^\gamma$ of cardinality $\leqslant \kappa$ are ω-resolvable.

1. Introduction

In this paper, we use the Greek letters α, β, η, ξ and ζ to denote ordinals, while γ, λ, κ, τ and θ will denoted cardinals.

Every space in this article is Tychonoff and crowded (that is, without isolated points). E. Hewitt [16] called a space X *resolvable* if it contains two dense disjoint subsets and a space which is not resolvable is called *irresolvable*. A space that has θ-many pairwise disjoint dense subsets, for a cardinal number $\theta \geqslant 2$, is called θ-*resolvable*. The *dispersion character* $\Delta(X)$ of a space X is the minimum of the cardinalities of nonempty open subsets of X. If X is $\Delta(X)$-resolvable, then we say that X is *maximally resolvable*. It is shown in [16] (for a proof see [5]) that metric spaces and compact spaces are maximally resolvable.

Later, El'kin and Malyhin published a number of papers on this subject and their connections with various topological problems. One of the problems considered by Malyhin in [24] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. Afterwards, Kunen, Symański and Tall in [21] proved (see [22] as well):

(1) If we assume $V = L$, there is no Baire irresolvable space.
(2) The statements "there is a measurable cardinal" and "there is a Baire irresolvable space" are equiconsistent.

* Corresponding author.
E-mail addresses: sgarcia@matmor.unam.mx (S. García-Ferreira), michael@matmor.unam.mx (M. Hrušák), atamariz@servidor.unam.mx (Á. Tamariz-Mascarúa).

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R. Bolstein introduced in [3] the *almost resolvable* spaces as those spaces which are the union of a countable collection of subsets with void interior. He proved in [3] that every resolvable space is almost resolvable and also showed that a space is almost resolvable iff it is possible to define a real-valued everywhere discontinuous function with countable range. A space that is not almost resolvable is called *almost irresolvable*. V.I. Malykhin [23] (see also [24]) established the existence of a model of ZFC in which every topological space is almost resolvable.

Almost-ω-resolvable spaces were introduced in [27]; these are spaces X which can be covered by a countable collection $\{X_n\colon n<\omega\}$ of subsets in such a way that for each $m<\omega$, $\mathrm{int}(\bigcup_{i\leqslant m} X_i)=\emptyset$. So every almost-$\omega$-resolvable space is almost resolvable and every ω-resolvable space is almost-ω-resolvable. Moreover, every almost resolvable space is infinite, and every T_1 separable space is almost-ω-resolvable. It was also proved in [27] that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space which is not almost-ω-resolvable, and that, on the contrary, if $V=L$ then every crowded space is almost-ω-resolvable. Later, the following result was pointed out in [2]:

Theorem 1.1. *Resolvability and almost resolvability are equivalent in the class of Baire spaces. So, every Baire almost-ω-resolvable space is resolvable.*

It is unknown if every Baire almost-ω-resolvable space is 3-resolvable. With respect to this problem we have the following theorems.

Theorem 1.2. *([25]) Gödel's axiom of constructibility, $V=L$, implies that every Baire space is ω-resolvable.*

Following a similar proof to that given for Theorem 5.9 in [2], we obtain:

Theorem 1.3. *Every T_1 Baire space such that each of its dense subsets is almost-resolvable is ω-resolvable.*

These last two results raise the problem of finding subclasses of the class of Baire spaces such that each dense subset of each of their elements is almost-resolvable, assuming axioms consistent with ZFC which contrast with $V=L$. Of course, a classic subclass of Baire spaces is that of pseudocompact Tychonoff spaces. Related with this problem, W.W. Comfort and S. García-Ferreira [5] proved that countably compact spaces are ω-resolvable. Thus, the main general problem, still open, that will be discussed in this article is:

Question 1.4. Is every Tychonoff pseudocompact space resolvable in ZFC?

W.W. Comfort and S. García-Ferreira posed in [5] this question which appears as Question 9 in O. Pavlov's article in Open Problems in Topology II [26]. A natural related problem was posed in [2]:

Question 1.5. Is every Tychonoff pseudocompact space almost-ω-resolvable in ZFC?

Independent families were first considered by G. Fichtenholz and L. Kantorovich in [13], and they were initially used in relation to irresolvable spaces in [21] and [9]. Afterwards, several authors as F.G. Eckertson, W.W. Comfort and W. Hu, I. Juhasz, L. Soukup and S. Szetmiklossy have also studied the relations between independent families and resolvability (see [10,6,7,14,17,19]).

In this article, we also use independent families in order to obtain partial answers to Questions 1.4 and 1.5. In the second section, we list some known and some new results on the relationship between independent families $\mathcal{A}=\{A_\xi\colon \xi<\gamma\}$ and dense subsets of the Cantor cube $\{0,1\}^\gamma$, we prove that if there is no maximal independent family of cardinality γ on κ, every dense Baire subset of $\{0,1\}^\gamma$ of cardinality $\leqslant\kappa$ is ω-resolvable. In Section 3, we improve on a result by W. Hu in [17] by proving that every dense subset of the box product $\square_\theta\{0,1\}^\gamma$ is ω-resolvable, assuming that maximal θ-independent families do not exist, where θ is a regular cardinal number with $\omega_1\leqslant\theta\leqslant\gamma$. The fourth section is devoted to prove that for infinite cardinal numbers τ and γ with $\tau\leqslant\gamma$, every C_τ-compact dense subspace of a product of γ non-trivial compact spaces of weight $\leqslant\tau$ is 2^τ-resolvable. In particular, we obtain that every pseudocompact dense subspace of a product of γ non-trivial metrizable compact spaces is $\mathfrak{c}$-resolvable; as a consequence of this fact we obtain that there are no σ-independent maximal independent families. Furthermore, we show a σ-independent family $\mathcal{L}$ for which the dense subspace $D(\mathcal{L})$ of a Cantor cube related with it is pseudocompact and it is not maximally resolvable. Finally, in Section 5, we prove that if there are no maximal independent families on κ of cardinality γ, then every Baire dense subset of $[0,1]^\gamma$ of cardinality $\leqslant\kappa$ is ω-resolvable.

We would like to thank M. Tkachenko for pointing out to us the proof of Theorem 4.3 which simplifies the one we gave in a previous version of this article.

2. Independent families and irresolvable spaces

In this section, we are going to prove that every Baire dense subspace of a Cantor cube $\{0,1\}^\gamma$ of cardinality κ is ω-resolvable if there is no maximal independent family of cardinality γ on κ. We begin by giving the definition of an independent family.

Let θ be an infinite regular cardinal. Let τ be a cardinal number different from 0. Given an infinite cardinal number κ, a family $\mathcal{A}$ of subsets of κ is called (θ,τ)-*independent* on κ if for each pair of disjoint subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ such that $|\mathcal{A}_0 \cup \mathcal{A}_1| < \theta$, we have that

$$\left|\left(\bigcap_{A\in\mathcal{A}_0} A\right) \cap \left(\bigcap_{A\in\mathcal{A}_1} (\kappa\setminus A)\right)\right| \geqslant \tau.$$

A (θ,κ)-independent family on κ is also called *uniform θ-independent*. A $(\theta,1)$-independent family is called, simply, *θ-independent*. An ω-independent family is called *independent*, and an ω_1-independent family is called *σ-independent*. A θ-independent family $\mathcal{A}$ on κ is *maximal* if each family of subsets of κ which contains $\mathcal{A}$ properly is not θ-independent. It is not difficult to construct a θ-independent family $\mathcal{A}$ on κ such that $|\mathcal{A}| < \theta$ and $|\bigcap\mathcal{A}| = 1$. This θ-independent family is maximal. Hence, to avoid trivial cases we shall assume that $|\mathcal{A}| \geqslant \theta$ for each θ-independent family $\mathcal{A}$. It is known that Zorn's Lemma implies the existence of maximal independent families on ω (for a proof see [18]). Moreover, K. Kunen [20] proved that the existence of a maximal σ-independent family implies CH and there is a weakly inaccessible cardinal between ω_1 and 2^{ω_1}. The existence of such a family is equiconsistent with the existence of a measurable cardinal.

To each independent family $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ we are going to associate a dense subset $D(\mathcal{A}) = \{r_\zeta\colon \zeta < \kappa\}$ of $\{0,1\}^\gamma$ where, for each $\zeta < \kappa$, r_ζ is defined by

$$r_\zeta(\xi) = \begin{cases} 0 & \text{if } \zeta \notin A_\xi, \\ 1 & \text{if } \zeta \in A_\xi, \end{cases}$$

for every $\xi \in \gamma$.

The following notation is useful to analyze the relations between $\mathcal{A}$ and $D(\mathcal{A})$:

For $A \subseteq \kappa$, we set $A^0 = \kappa \setminus A$ and $A^1 = A$. Moreover, for $k < \omega$, $\xi_1,\dots,\xi_k < \gamma$ and $\epsilon_1,\dots,\epsilon_k \in \{0,1\}$, the symbol $[\xi_1,\xi_2,\dots,\xi_k;\epsilon_1,\epsilon_2,\dots,\epsilon_k]$ will denote the basic open subset $\{f \in \{0,1\}^\gamma\colon \forall i \in \{1,\dots,k\}\ (f(\xi_i) = \epsilon_i)\}$ of $\{0,1\}^\gamma$. In particular, if $\epsilon_1 = \dots = \epsilon_k = 0$ (resp., $\epsilon_1 = \dots = \epsilon_k = 1$), then we will write $[\xi_1,\dots,\xi_k;0]$ (resp. $[\xi_1,\dots,\xi_k;1]$) instead of $[\xi_1,\xi_2,\dots,\xi_k;\epsilon_1,\epsilon_2,\dots,\epsilon_k]$. In a more general form, if $(\xi_s)_{s\in S}$ is an S-sequence (a function with domain S) in γ where S is a non-empty set, and $(\epsilon_s)_{s\in S}$ is an S-sequence in $\{0,1\}$, then the symbol $[(\xi_s)_{s\in S};(\epsilon_s)_{s\in S}]$ will stand for the set $\{f \in \{0,1\}^\gamma\colon \forall s \in S,\ f(\xi_s) \in \epsilon_s\}$.

The basic relations between $\mathcal{A}$ and $D(\mathcal{A})$ are listed in the next lemma.

Lemma 2.1. *Let κ and γ be two infinite cardinals. Let $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$, $(\xi_s)_{s\in S}$ be an S-sequence in γ, $(\epsilon_s)_{s\in S}$ be an S-sequence in $\{0,1\}$, $M \subseteq \kappa$ and $U \subseteq \{0,1\}^\gamma$. Then*:

(1) *$\zeta \in \bigcap_{s\in S} A_{\xi_s}^{\epsilon_s}$ if and only if $r_\zeta \in [(\xi_s)_{s\in S};(\epsilon_s)_{s\in S}]$; that is, $\bigcap_{s\in S} A_{\xi_s}^{\epsilon_1} = \{\zeta < \kappa\colon r_\zeta \in [(\xi_s)_{s\in S};(\epsilon_s)_{s\in S}]\}$;*
(2) *$[(\xi_s)_{s\in S};(\epsilon_s)_{s\in S}] \cap D(\mathcal{A}) = \{r_\zeta\colon \zeta \in \bigcap_{s\in S} A_{\xi_s}^{\epsilon_s}\}$;*
(3) *$\bigcap_{s\in S} A_{\xi_s}^{\epsilon_s} \subseteq M$ if and only if $[(\xi_s)_{s\in S};(\epsilon_s)_{s\in S}] \cap D(\mathcal{A}) \subseteq \{r_\zeta\colon \zeta \in M\}$; and*
(4) *$[(\xi_s)_{s\in S};(\epsilon_s)_{s\in S}] \cap D(\mathcal{A}) \subseteq U$ if and only if $\bigcap_{s\in S} A_{\xi_s}^{\epsilon_s} \subseteq \{\zeta < \kappa\colon r_\zeta \in U\}$.*

Proof. The proof of each statement follows from the following observation: $\zeta \in A_{\xi_i}^{\epsilon_i}$ and $\epsilon_i = 0$ if and only if $\zeta \in \kappa \setminus A_{\xi_i}$, if and only if $r_\zeta(\xi_i) = 0 = \epsilon_i$; and $\zeta \in A_{\xi_i}^{\epsilon_i}$ and $\epsilon_i = 1$ if and only if $\zeta \in A_{\xi_i}$, if and only if $r_\zeta(\xi_i) = 1 = \epsilon_i$.

Now, we can think of the inverse process: Let γ be an infinite cardinal. Now suppose that $D = \{r_\zeta\colon \zeta < \kappa\}$ is a dense subset of $\{0,1\}^\gamma$. For each $\xi < \gamma$, we define $A_\xi = \{\zeta < \kappa\colon r_\zeta(\xi) = 1\} \in \mathcal{P}(\kappa)$. The family $\mathcal{A}(D) = \{A_\xi\colon \xi < \gamma\}$ is independent on κ.

It is not difficult to prove the following lemma (see Lemma 2.2 in [14]).

Lemma 2.2. *Let $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ be an independent family. Then,*

(1) *$\mathcal{A}(D(\mathcal{A})) = \mathcal{A}$;*
(2) *if D is a dense subset of $\{0,1\}^\gamma$, then $D(\mathcal{A}(D)) = D$.*

Recall that, for an infinite regular cardinal θ, a set F of a space X is a *G_θ-set* if there is a collection $\mathcal{U}$ of open subsets of X such that $|\mathcal{U}| < \theta$ and $F = \bigcap\mathcal{U}$. A subspace Y of a space X is said to be *G_θ-dense* if every nonempty G_θ-set intersects Y. As usual, we say *G_δ-set* and *G_δ-dense* instead of G_{ω_1}-set and G_{ω_1}-dense, respectively. A set Y of X is dense if it is G_θ-dense when $\theta = \omega$. The following result can be proved without difficulty.

Proposition 2.3. *Let θ be an infinite regular cardinal and κ an infinite cardinal. A family $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ is θ-independent on κ if and only if $D(\mathcal{A})$ is a G_θ-dense subset in $\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$.*

As a particular case of the previous result we have that a family $\mathcal{A} = \{A_\xi\colon \xi < \gamma\}$ of subsets of κ is independent iff $D(\mathcal{A})$ is dense in $\{0,1\}^\gamma$.

Now, we will present a well-known result (a proof is available in [6]).

Theorem 2.4. *A collection $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ is maximal independent if and only if $D(\mathcal{A})$ is a dense irresolvable subset of $\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$.*

The authors in [1] constructed by transfinite recursion, a countable dense irresolvable subspace of $\{0,1\}^{\mathfrak{c}}$. Theorem 2.4 says that there are dense irresolvable subspaces in $\{0,1\}^{2^\kappa}$ of cardinality κ for every $\kappa \geqslant \omega$ because for every infinite cardinal number κ there are maximal independent families of cardinality 2^κ on κ (see for example [8, Theorem 3.16]).

In order to get Theorem 2.8 below, which partially generalizes Theorem 2.4, we introduce some definitions.

Definition 2.5. Let κ be an infinite cardinal number and let $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ be an independent family on κ.

(1) We say that $\mathcal{A}$ is *ai-maximal independent* if for every partition $\{B_n\colon n < \omega\}$ of κ, there are two disjoint finite subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ and $m < \omega$ such that

$$\bigcap_{A \in \mathcal{A}_0} A \cap \bigcap_{A \in \mathcal{A}_1} (\kappa \setminus A) \subseteq B_m.$$

(2) We say that $\mathcal{A}$ is *aωi-maximal independent* if for every partition $\{B_n\colon n < \omega\}$ of κ, there are two disjoint finite subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ and $m < \omega$ such that

$$\bigcap_{A \in \mathcal{A}_0} A \cap \bigcap_{A \in \mathcal{A}_1} (\kappa \setminus A) \subseteq \bigcup_{i \leqslant m} B_i.$$

Every *ai*-maximal independent family is *aωi*-maximal independent and maximal independent.

Proposition 2.6. *Let κ be an infinite cardinal number and let $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$. Then, $\mathcal{A}$ is ai-maximal independent iff $D(\mathcal{A})$ is a dense almost-irresolvable subspace of $\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$.*

Proof. Necessity. Assume that $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ is *ai*-maximal independent. By Theorem 2.3, $D(\mathcal{A})$ is dense in $\{0,1\}^\gamma$. Now, we take a partition $\{U_n\colon n < \omega\}$ of $D(\mathcal{A})$. For each $n < \omega$, we define $B_n = \{\zeta < \kappa\colon r_\zeta \in U_n\}$. Hence, $\{B_n\colon n < \omega\}$ is a partition of κ. By hypothesis, there are two disjoint finite subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ and $m < \omega$ such that

$$\bigcap_{A \in \mathcal{A}_0} A \cap \bigcap_{A \in \mathcal{A}_1} (\kappa \setminus A) \subseteq B_m.$$

This implies that if $\mathcal{A}_0 = \{A_{\xi_1}, \ldots, A_{\xi_s}\}$ and $\mathcal{A}_1 = \{A_{\eta_1}, \ldots, A_{\eta_t}\}$, then $\emptyset \neq D(\mathcal{A}) \cap [\xi_1, \ldots, \xi_s; 1] \cap [\eta_1, \ldots, \eta_t; 0] \subseteq U_k$ (see Lemma 2.1). Since the partition $\{U_n\colon n < \omega\}$ of $D(\mathcal{A})$ was chosen arbitrarily, $D(\mathcal{A})$ cannot be almost resolvable.

Sufficiency. Now, assume that $D(\mathcal{A})$ is a dense almost-irresolvable subspace of $\{0,1\}^\gamma$. In particular, $D(\mathcal{A})$ is a dense irresolvable subspace of $\{0,1\}^\gamma$. In order to prove that $\mathcal{A}$ is *ai*-maximal independent, we take a partition $\{M_n\colon n < \omega\}$ of κ. For each $n < \omega$, we define U_n to be the subset $\{r_\zeta\colon \zeta \in M_n\}$ of $D(\mathcal{A})$. The collection $\{U_n\colon n < \omega\}$ is a partition of $D(\mathcal{A})$. Since $D(\mathcal{A})$ is almost-irresolvable, there is $k < \omega$ such that $int_{D(\mathcal{A})} U_k \neq \emptyset$. That is, there are $\xi_1, \ldots, \xi_s < \gamma$ and $\epsilon_1, \ldots, \epsilon_s \in \{0,1\}$ such that $\emptyset \neq [\xi_1, \ldots, \xi_s; \epsilon_1, \ldots, \epsilon_s] \cap D(\mathcal{A}) \subseteq U_k$. From this and Lemma 2.1, we can infer that $A_{\xi_1}^{\epsilon_1} \cap \cdots \cap A_{\xi_s}^{\epsilon_s} \subseteq M_k$. This finishes the proof. $\square$

By using a proof analogous to the one of Proposition 2.6, we obtain:

Proposition 2.7. *A collection $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ is aωi-maximal independent if and only if $D(\mathcal{A})$ is a dense almost-ω-irresolvable subspace of $\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$.*

Theorem 2.8. *Every Baire dense subset D of $\{0,1\}^\gamma$ of cardinality less or equal to κ is ω-resolvable if there is not an ai-maximal independent family of cardinality γ in κ.*

Proof. Assume that $D \subseteq \{0,1\}^\gamma$ is a Baire dense subset of cardinality less or equal to κ. Let D' be a dense subset of D. So, D' is dense in $\{0,1\}^\gamma$ and $|D'| \leqslant \kappa$. Because of our hypothesis, D' is almost resolvable (Propositions 2.2 and 2.6). By Theorem 1.3, D is ω-resolvable. $\square$

Corollary 2.9. *Let κ be an infinite cardinal and assume $V = L$. Then, there is no $a\omega i$-maximal family on κ. In particular, in this case, there is no ai-maximal family on κ.*

Proof. Since we are assuming $V = L$, every crowded space is almost-ω-resolvable (see [2, Theorem 5.11]). The conclusion follows by applying Propositions 2.6 and 2.7 □

Questions 2.10.

(1) Is there an *ai*-maximal independent family?
(2) Is every $a\omega i$-maximal independent family either *ai*-maximal independent or maximal independent?

Because of Lemma 2.2 we obtain:

Proposition 2.11. *A subset D of $\{0,1\}^\gamma$ is G_θ-dense iff $\mathcal{A}(D)$ is θ-independent. In particular, $D \subseteq \{0,1\}^\gamma$ is dense and pseudocompact iff $\mathcal{A}(D)$ is σ-independent.*

Proposition 2.12. *A dense subset D of $\{0,1\}^\gamma$ is irresolvable (resp., almost irresolvable, almost-ω-irresolvable) iff $\mathcal{A}(D)$ is maximal independent (resp., ai-maximal independent, $a\omega i$-maximal independent).*

3. Dense subspaces of $\square_\theta\{0,1\}^\gamma$

Recall that for a collection $\{X_s\colon s \in S\}$ of spaces and an infinite cardinal number θ with $\theta \leqslant |S|^+$, a set of the form $\prod_{s\in S} G_s$ is called an *open θ-box* of $X = \prod_{s\in S} X_s$ if G_s is an open set in X_s and $|\{s \in S\colon G_s \neq X_s\}| < \theta$. The θ-box topology on $\prod_{s\in S} X_s$ is the topology generated by the base of all the open θ-boxes. $\square_\theta X$ designates $\prod_{s\in S} X_s$ with the θ-box topology. Of course, $\square_\omega X$ is the Cartesian product $\prod_{s\in S} X_s$ with the Tychonoff topology. For uncountable cardinal numbers γ and θ with $\theta \leqslant \gamma^+$, we have

$$w\big(\square_\theta\{0,1\}^\gamma\big) \leqslant \gamma^{<\theta} \leqslant 2^\gamma = \Delta\big(\square_\theta\{0,1\}^\gamma\big).$$

So, by a well-known result of A.G. El'kin (see [11]), $\square_\theta\{0,1\}^\gamma$ is maximally resolvable.

It is well known that every dense subset of $\square_\theta\{0,1\}^\gamma$ is a Baire space when $cf(\theta) > \omega$. The next result is a corollary of Theorem 2.4 in [17].

Theorem 3.1. *Let θ be a regular cardinal with $\omega < \theta \leqslant \gamma$. A collection $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ is (maximal) θ-independent if and only if $D(\mathcal{A})$ is a dense Baire (irresolvable) subspace of the θ-box product $\square_\theta\{0,1\}^\gamma$.*

By Lemma 2.2, we obtain:

Theorem 3.2. *Let θ be a regular cardinal with $\omega < \theta \leqslant \gamma$. A dense subset D of $\square_\theta\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$ is irresolvable if and only if $\mathcal{A}(D)$ is a maximal θ-independent family on κ of cardinality γ.*

Remark 3.3. Let θ be an infinite cardinal. Let τ and κ be two infinite cardinal numbers such that $\tau \leqslant \kappa$. Let $\mathcal{A} = \{A_\xi\colon \xi < \gamma\}$ be a maximal θ-independent family of cardinality γ in τ. Considering every A_ξ as a subset of κ, the collection $\mathcal{A}' = \{A_\xi\colon \xi < \gamma\}$ is a maximal θ-independent family on κ of cardinality γ.

The following improves Theorem 2.4 from [17].

Theorem 3.4. *Let θ be a regular cardinal with $\omega < \theta \leqslant \gamma$. The following are equivalent:*

(1) *there are no maximal θ-independent families of cardinality γ on κ;*
(2) *every dense subspace of $\square_\theta\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$ is resolvable;*
(3) *every dense subspace of $\square_\theta\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$ is ω-resolvable.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is Theorem 2.4 from [17]. We have only to prove (1) $\Rightarrow$ (3). Let D' be a dense subset of $\square_\theta\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$. Let D be a dense subset of D' of cardinality τ. Then $\mathcal{A}(D)$ is a θ-independent family of cardinality γ on τ. By Remark 3.3 and our hypothesis, $\mathcal{A}(D)$ is not a maximal θ-independent family on τ. So, $D(\mathcal{A}(D)) = D$ is resolvable. Then, D' is Baire and every dense subset of D' is resolvable. Thus, by Theorem 1.3, D' is ω-resolvable. □

Here, we remark that if $\mathcal{A}$ is maximal θ-independent, then $D(\mathcal{A})$ is Baire and irresolvable. This offers an alternative example to that offered in [21].

The following result is a consequence of Theorem 1 of [20] and Theorem 3.4.

Corollary 3.5. *Let γ be greater or equal to ω_1. If we assume either $\neg CH$ or "there is not a weakly inaccessible cardinal number between ω_1 and 2^{ω_1}", then every dense subspace of $\square_{\omega_1}\{0,1\}^\gamma$ is ω-resolvable.*

Proof. Let D be a dense subset of $\square_{\omega_1}\{0,1\}^\gamma$. Our hypothesis implies that there are no maximal σ-independent families of cardinality γ on $|D|$. So, By Theorem 3.4, D is ω-resolvable. $\square$

Now, we introduce two new definitions (compare them with those given in Definition 2.5).

Definition 3.6. Let κ be an infinite cardinal number and let $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ be a σ-independent family on κ.

(1) We say that $\mathcal{A}$ is *ai-maximal σ-independent* if for every partition $\{B_n\colon n<\omega\}$ of κ, there are two disjoint countable subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ and $m<\omega$ such that

$$\bigcap_{A\in\mathcal{A}_0} A \cap \bigcap_{A\in\mathcal{A}_1} (\kappa\setminus A) \subseteq B_m.$$

(2) We say that $\mathcal{A}$ is *aωi-maximal σ-independent* if for every partition $\{B_n\colon n<\omega\}$ of κ, there are two disjoint countable subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ and $m<\omega$ such that

$$\bigcap_{A\in\mathcal{A}_0} A \cap \bigcap_{A\in\mathcal{A}_1} (\kappa\setminus A) \subseteq \bigcup_{i\leqslant m} B_i.$$

It is clear that if $\mathcal{A}$ is *ai*-maximal σ-independent, then it is maximal σ-independent and *aωi*-maximal σ-independent.

Theorem 3.7. *Let $\mathcal{A}=\{A_\xi\colon \xi<\gamma\}$ be a σ-independent family on a cardinal number κ. Then, the following assertions are equivalent:*

(1) *$\mathcal{A}$ is maximal σ-independent,*
(2) *$\mathcal{A}$ is ai-maximal σ-independent.*

Proof. Assume that $\mathcal{A}$ is maximal σ-independent. By Theorem 3.1, $D(\mathcal{A})$ is a Baire irresolvable subspace of $\square_{\omega_1}\{0,1\}^\gamma$. It follows, by Corollary 1 in [12], that $D(\mathcal{A})$ is almost irresolvable being a Baire space. Let $\{M_n\colon n<\omega\}$ be a partition of subsets of κ. For each $n<\omega$, let $T_n=\{r_\zeta\colon \zeta\in M_n\}$ (see the paragraph before Lemma 2.1). Then, $\{T_n\colon n<\omega\}$ is a partition of $D(\mathcal{A})$. Since $D(\mathcal{A})$ is an almost-irresolvable subspace of $\square_{\omega_1}\{0,1\}^\gamma$, there is a non-empty canonical open set $W=[\xi_0,\dots,\xi_n,\dots;\epsilon_0,\dots,\epsilon_n,\dots]$ in $\square_{\omega_1}\{0,1\}^\gamma$ and there is $k<\omega$ such that $W\cap D(\mathcal{A})\subseteq T_k$. But this means that

$$\emptyset \neq \bigcap_{i<\omega} A_{\xi_i}^{\epsilon_i} \subseteq M_k.$$

So, $\mathcal{A}$ is *ai*-maximal σ-independent. $\square$

Question 3.8. Is every *aωi*-maximal σ-independent family maximal σ-independent?

It is not difficult to prove the following proposition.

Proposition 3.9. *A collection $\mathcal{A}=\{A_\xi\colon \xi<\gamma\}\subseteq\mathcal{P}(\kappa)$ is aωi-maximal σ-independent (resp., ai-maximal σ-independent) if and only if $D(\mathcal{A})$ is a dense (and Baire) almost-ω-irresolvable (resp., almost-irresolvable) subspace of $\square_{\omega_1}\{0,1\}^\gamma$.*

A consequence of Lemma 2.2 and Proposition 3.9 is:

Proposition 3.10. *A subset D of $\square_{\omega_1}\{0,1\}^\gamma$ is dense and almost-ω-irresolvable (resp., almost-irresolvable) of cardinality $\leqslant\kappa$ if and only if $\mathcal{A}(D)$ is aωi-maximal (resp., ai-maximal) σ-independent on κ of cardinality γ.*

4. C_τ-compact dense subspaces of a product of compact spaces

Corollary 4.2 in [4] says that, under Souslin Hypothesis (SH), every Baire space with countable cellularity is ω-resolvable. (Recently, I. Juhasz communicated to us that he and Z. Szentmiklossy proved that if the continuum is less than the first weakly inaccessible cardinal, then every space with countable celularity is ω-resolvable.) It is known that every dense subset of a cube $\{0,1\}^\gamma$ has countable cellularity. Then, assuming SH, every dense Baire subspace of $\{0,1\}^\gamma$ is ω-resolvable. Recall that, for an infinite cardinal number τ, a subset Y of a space X is C_τ-compact if for every continuous function $f: X\to\mathbb{R}^\tau$, $f[Y]$ is compact. We will say that a subset Y of X is *C-compact* if it is a C_ω-compact subset of X. If X is C_τ-compact in itself, then we say that X is τ-pseudocompact. ω-pseudocompactness coincide with pseudocompactness and every C_λ-compact subset is C_τ-compact when $\tau \leqslant \lambda$. In this section, we are going to prove, in ZFC, that every dense C_τ-compact subspace of a product of γ non-trivial compact spaces of weight less or equal to τ with $\tau \leqslant \gamma$ is 2^τ-resolvable. A subspace D of a product $\prod_{s\in S} X_s$ is called *τ-dense* if for every $J \subset S$ with $|J| \leqslant \tau$, the J-projection $\pi_J : \prod_{s\in S} X_s \to \prod_{s\in J} X_s$ is surjective. A function $h : X \to Y$ is called *semi-open* if for each open set U of X, $\mathrm{int}_Y(h[U]) \neq \emptyset$.

First we recall two well-known facts (see, for instance, [15] and [2], respectively):

Lemma 4.1. *Let $\{X_s\colon s \in S\}$ be a family of non-trivial compact spaces each of them with weight less or equal to τ and $\omega \leqslant \tau \leqslant |S|$. A dense subset D of the topological product $X = \prod_{s\in S} X_s$ is C_τ-compact if and only if D is G_τ-dense in X, if and only if D is τ-dense in X.*

Lemma 4.2. *Let $\theta \geqslant 2$ and let Y be a θ-resolvable (resp., almost-resolvable, almost-ω-resolvable) space. If $f : X \to Y$ is a semi-open and onto function, then X is θ-resolvable (resp., almost-resolvable, almost-ω-resolvable).*

In the proof of the next result, we follow a suggestion made by M. Tkachenko which simplifies a previous version of this article.

Theorem 4.3. *Let $\{X_s\colon s \in S\}$ be a family of non-trivial compact spaces with weight less or equal to τ and $\omega \leqslant \tau \leqslant |S|$. Then, every dense C_τ-compact subspace D of $X = \prod_{s\in S} X_s$ is 2^τ-resolvable.*

Proof. Since D is a dense C_τ-compact subspace of X, if $J \subseteq S$ with $|J| = \tau$, $\pi_J[D] = \prod_{s\in J} X_s = X_J$. The space X_J is maximally resolvable (that is, X_j is $\Delta(X_J)$-resolvable). Each open subset of this last space contains a copy of the Cantor cube 2^τ. So $\Delta(X_J) \geqslant 2^\tau$. Because of Lemma 4.2, D is 2^τ-resolvable. $\square$

Corollary 4.4. *Let $\{X_s\colon s \in S\}$ be a family of non-trivial compact metrizable spaces where S is infinite. Then, every dense C-compact subspace D of $X = \prod_{s\in S} X_s$ is $\mathfrak{c}$-resolvable. In particular, in this case, every dense pseudocompact subspace D of $X = \prod_{s\in S} X_s$ is $\mathfrak{c}$-resolvable.*

Because of Lemma 4.1 and Proposition 2.11, we have:

Corollary 4.5. *For infinite cardinal numbers τ and γ with $\tau \leqslant \gamma$, a dense subset $D \subseteq \{0,1\}^\gamma$ is a C_τ-compact subset of $\{0,1\}^\gamma$ if and only if $\mathcal{A}(D)$ is τ-independent.*

We have the following consequence of Lemma 4.1, Theorem 4.4 and Propositions 2.11 and 2.12.

Theorem 4.6. *Let $\mathcal{A}$ be a σ-independent family on κ. Then $\mathcal{A}$ is not maximal independent.*

A combinatorial proof of this theorem is worth mentioning:

Proof. Let $\mathcal{A}' = \{A_n\colon n < \omega\}$ be a subfamily of pairwise different elements of $\mathcal{A}$. For each $f \in 2^\omega$, we take $J_f = \bigcap_{n<\omega} A_n^{f(n)}$. We can take a dense subset G of the Cantor cube 2^ω such that $2^\omega \setminus G$ is dense in 2^ω too. Consider the set $Z = \bigcup_{f\in G} J_f$. Observe that $\kappa = \bigcup_{f\in 2^\omega} J_f$ and $\kappa \setminus Z = \bigcup_{f\in 2^\omega\setminus G} J_f$. First, we have that Z cannot be equal to A_n or to $\kappa \setminus A_n$ for any $n < \omega$, because there is an $f \in G \cap [n;0]$ and there is $h \in G \cap [n;1]$; so, if $\zeta \in J_f$, then $\zeta \in Z \cap (\kappa \setminus A_n)$, and if $\zeta \in J_h$, ζ is an element of $Z \cap A_n$. Moreover, the set Z does not belong to $\mathcal{A} \setminus \mathcal{A}'$ because $\mathcal{A}$ is σ-independent and if $g \in (2^\omega \setminus G)$, $J_g \cap Z = \emptyset$. Nevertheless, $\mathcal{A} \cup \{Z\}$ is an independent family on κ. Indeed, take a finite subcollection $\mathcal{B} = \{B_0,\dots,B_n\}$ of $\mathcal{A} \setminus \mathcal{A}'$ and let $\epsilon_0,\dots,\epsilon_n \in \{0,1\}$. Let $\mathcal{A}_0 = \{A_{n_1},\dots,A_{n_k}\}$ and $\mathcal{A}_1 = \{A_{s_1},\dots,A_{s_m}\}$ be two disjoint finite subsets of $\mathcal{A}'$. Since G is dense, there is $f \in G$ which belongs to $[s_1,\dots,s_m;1] \cap [n_1,\dots,n_k;0]$. Since $\mathcal{A}$ is σ-independent, $J_f \cap \bigcap_{A\in\mathcal{A}_0}(\kappa \setminus A) \cap \bigcap_{A\in\mathcal{A}_1} A \cap \bigcap_{i\leqslant n} B_i^{\epsilon_i} \neq \emptyset$. Moreover,

$$J_f \cap \bigcap_{A\in\mathcal{A}_0}(\kappa \setminus A) \cap \bigcap_{A\in\mathcal{A}_1} A \cap \bigcap_{i\leqslant n} B_i^{\epsilon_i} \subseteq Z \cap \bigcap_{A\in\mathcal{A}_0}(\kappa \setminus A) \cap \bigcap_{A\in\mathcal{A}_1} A \cap \bigcap_{i\leqslant n} B_i^{\epsilon_i}. \qquad \square$$

From Corollary 4.4 the following question arises: Is every dense pseudocompact subspace of $\{0,1\}^\gamma$ maximally resolvable? Next, with respect to this question, we are going to show a consistent example of a dense pseudocompact subspace of $\{0,1\}^{2^\lambda}$, with $\lambda = \omega_1$, which is not maximally resolvable (Example 4.8 below).

Let $\mathcal{A}$ be an independent family on an infinite cardinal κ, and let $\{A_\xi\colon \xi < \gamma\}$ be an enumeration of $\mathcal{A}$. We can consider the topology $\mathcal{T}(\mathcal{A})$ in κ defined by the collection

$$\mathcal{B}(\mathcal{A}) = \left\{ \bigcap_{A\in\mathcal{A}_0} A \cap \bigcap_{A\in\mathcal{A}_1} (\kappa \setminus A)\colon \mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{A}]^{<\omega},\ \text{and } \mathcal{A}_0 \cap \mathcal{A}_1 = \emptyset \right\}$$

as a base. This topology will be Hausdorff when the independent family $\mathcal{A}$ on κ is *separated*; that is, if for every $\{\zeta, \xi\} \in [\kappa]^2$, there exists $A \in \mathcal{A}$ such that $|A \cap \{\zeta, \xi\}| = 1$.

Proposition 4.7. *Let $\mathcal{A} = \{A_\xi\colon \xi < \gamma\} \subseteq \mathcal{P}(\kappa)$ be a separated independent family. Then, the space $(\kappa, \mathcal{T}(\mathcal{A}))$ is homeomorphic to the subspace $D(\mathcal{A})$ of $\{0,1\}^\gamma$.*

Proof. Indeed, let $\phi(\zeta) = r_\zeta$ as was defined before Lemma 2.1. Since $\mathcal{A}$ is separated, ϕ is injective, and ϕ is continuous and open because

$$\phi\big[A^1_{\xi_1} \cap \cdots \cap A^1_{\xi_k} \cap A^0_{\eta_1} \cap \cdots \cap A^0_{\eta_m}\big] = [\xi_1, \ldots, \xi_k; 1] \cap [\eta_1, \ldots, \eta_k; 0] \cap D(\mathcal{A}). \qquad \square$$

Kunen constructed in [20] a maximal σ-independent family $\mathcal{L} \subseteq \mathcal{P}(2^{\omega_1})$ from a model in which the Continuum Hypothesis holds, and, if $\lambda = 2^{\omega_1}$, there is an ω_2-saturated λ-complete ideal $\mathcal{F}$ over λ (which is equiconsistent with the existence of a measurable cardinal) with $\mathcal{P}(\lambda)/\mathcal{F}$ isomorphic to the complete Boolean algebra $\mathcal{B}(Fn(2^\lambda, 2, \omega_1))$ into which $Fn(2^\lambda, 2, \omega_1)$ is densely embedded, where $Fn(2^\lambda, 2, \omega_1)$ is the set

$$\big\{p\colon\ p \text{ is a function},\ \operatorname{dom}(p) \subseteq 2^\lambda,\ \operatorname{ran}(p) \subseteq \{0,1\} \text{ and } |p| < \omega_1\big\}.$$

Recall that an ideal $\mathcal{I}$ of a $\mathcal{P}(\kappa)$ is θ-saturated if every collection $\mathcal{M}$ of elements in $\mathcal{P}(\kappa) \setminus \mathcal{I}$ such that $A \cap B \in \mathcal{I}$ for every two different elements A and B in $\mathcal{M}$, has cardinality strictly less than θ.

Example 4.8. Let $\lambda = 2^{\omega_1}$. Let $\mathcal{L} = \{L_\xi\colon \xi < 2^\lambda\} \subseteq \mathcal{P}(\lambda)$ be the maximal σ-independent family constructed by Kunen in [20]. Then, $D(\mathcal{L})$ is a dense pseudocompact subset of $\{0,1\}^{2^\lambda}$ which is ω_1-resolvable and it is not maximally resolvable.

Proof. K. Kunen constructed the family as follows: In the ground model M, assume that λ is measurable and CH holds. Let $P = Fn(\lambda, 2, \omega_1)$, and let $\mathcal{U}$ be a normal ultrafilter over λ. Let G be P-generic over M. Then, in $M[G]$, CH holds, $2^{\omega_1} = \lambda$, and if

$$\mathcal{F} = \big\{X \subseteq \lambda\colon \exists Y \in \mathcal{U}\ (X \cap Y = \emptyset)\big\},$$

then $\mathcal{F}$ is λ-complete and ω_2-saturated. Moreover, there is an isomorphism ψ from $\mathcal{B}(Fn(2^\lambda, 2, \omega_1))$ into $\mathcal{P}(\lambda)/\mathcal{F}$. For $\delta < 2^\lambda$, let $[A_\delta] = \psi(\{(\delta, 1)\})$; here $A_\delta \subseteq \lambda$ and $[A_\delta] \in \mathcal{P}(\lambda)/\mathcal{F}$ is its equivalence class. Let $\mathcal{F} = \{C_\delta\colon \delta < 2^\lambda\}$ where each $C \in \mathcal{F}$ is listed at least ω_1 times. Let $A'_\delta = A_\delta \setminus C_\delta$, and $\mathcal{L} = \{A'_\delta\colon \delta < 2^\lambda\}$. Then, as was proved by Kunen, $\mathcal{L}$ is a maximal ω_1-independent family on λ. Without loss of generality, we can assume that $\mathcal{L}$ is separated and uniform. Let $\mathcal{T}(\mathcal{L})$ be the topology in λ generated by $\mathcal{L}$ as was defined at the beginning of this section. Because of Proposition 4.7, Theorem 2.3 and Corollary 4.4, $X = (\lambda, \mathcal{T}(\mathcal{L}))$ is a pseudocompact ω_1-resolvable space. Observe that any dense subset of X belongs to $\mathcal{F}$; hence, since $\mathcal{F}$ is ω_2-saturated, X cannot be partitioned in ω_2 pairwise dense subsets. But this means, since $\mathcal{L}$ is uniform, that X is not maximally resolvable. Therefore, $D(\mathcal{L}) \cong (\lambda, \mathcal{T}(\mathcal{L}))$ is a dense pseudocompact subset of $\{0,1\}^{2^\lambda}$ which is ω_1-resolvable and is not maximally resolvable. $\square$

The results shown above lead us to ask the following:

Questions 4.9.

(1) Is there in ZFC a dense pseudocompact subspace of $\{0,1\}^\gamma$ which is not maximally resolvable?
(2) Is every dense Baire subspace of $\{0,1\}^\gamma$ resolvable?

5. Baire dense subspaces of $[0,1]^\gamma$

In this section, we shall study the Baire dense subspaces of $[0,1]^\gamma$. To make this possible we shall transfer information from the Cantor cube $\{0,1\}^\gamma$ to $[0,1]^\gamma$, via a semi-open function. The main result will follow from a sequences of claims. The first claim is easy to prove and well-known:

Claim 1. *For $\theta \geqslant 2$, if X is θ-resolvable (resp., almost-resolvable, almost-ω-resolvable) and $f : X \to Y$ is a bijective continuous function, then Y is θ-resolvable (resp., almost-resolvable, almost-ω-resolvable).*

Claim 2. *Let $f : X \to Y$ be a semi-open function, and let D be a dense subset of Y. Then $f^{-1}[D]$ is dense in X.*

Proof. Let U be a non-empty open subset of X. We have that $\operatorname{int}(f[U]) \neq \emptyset$. So there is $y \in \operatorname{int}(f[U]) \cap D$. Thus, there is $x \in U$ such that $f(x) = y \in \operatorname{int}(f[U]) \cap D \subseteq f[U] \cap D$. So, $x \in f^{-1}[D] \cap U$. $\square$

Claim 3. *If $f : X \to Y$ is a continuous semi-open and injective function, and D is a dense subset of X, then $f \restriction D : D \to f[D]$ is continuous semi-open and bijective.*

Proof. We are only going to verify that $f \restriction D : D \to f[D]$ is semi-open. Let U be an open subset of D. There is an open set U' in X such that $U' \cap D = U$. Since f is semi open, $W = int_Y f[U']$ is not empty. Since f is continuous, $f[D]$ is dense in Y. Then, $W \cap f[D]$ is a non-empty open subset of $f[D]$. Now, by using the fact that f is an injective function, we obtain $f[U] = f[U' \cap D] = f[U'] \cap f[D]$ and this set contains $W \cap f[D]$; that is, $int_{f[D]} f[U] \neq \emptyset$. □

Notation. Let τ be a cardinal number different to 0. Let $\{X_\xi\colon \xi < \tau\}$ be a family of topological spaces. Let $X = \prod_{\xi<\tau} X_\xi$ be the Tychonoff product of the family $\{X_\xi\colon \xi < \tau\}$. Let $(\xi_s)_{s\in S}$ be an injective S-sequence in τ where S is a non-empty set (in particular, $\xi_s \neq \xi_t$ if $s \neq t$). For each $s \in S$, let U_s be a subset of X_{ξ_s}. The symbol $[(\xi_s)_{s\in S}; (U_s)_{s\in S}]$ will represent the set $\{f \in X\colon \forall s \in S,\ f(\xi_s) \in U_s\}$. Observe that $[(\xi_s)_{s\in S}; (U_s)_{s\in S}]$ is not empty if and only if each U_s is not empty. For every family of functions $\{\phi_\xi : X_\xi \to Y_\xi \mid \xi < \tau\}$, $\Phi : \prod_{\xi<\tau} X_\xi \to \prod_{\xi<\tau} Y_\xi$ will be the product function of the family $\{\phi_\xi : X_\xi \to Y_\xi \mid \xi < \tau\}$; that is,

$$\Phi\big((x_\xi)_{\xi<\tau}\big) = \big(\phi_\xi(x_\xi)\big)_{\xi<\tau}.$$

Claim 4. *Let $\tau > 0$. Let $\{\phi_\xi : X_\xi \to Y_\xi \mid \xi < \tau\}$ be a family of functions. Let $(\xi_s)_{s\in S}$ be an injective S-sequence of τ, and for each $s \in S$ let U_s be a subset of X_{ξ_s}. Then:*

(i) $\Phi[[(\xi_s)_{s\in S}; (U_s)_{s\in S}]] \subseteq [(\xi_s)_{s\in S}; (\phi_{\xi_s}[U_s])_{s\in S}]$,
(ii) *if each ϕ_ξ is surjective (resp., injective, bijective), then Φ is surjective (resp., injective, bijective),*
(iii) *if each ϕ_ξ is surjective, then*

$$\Phi\big[\big[(\xi_s)_{s\in S}; (U_s)_{s\in S}\big]\big] = \big[(\xi_s)_{s\in S}; \big(\phi_{\xi_s}[U_s]\big)_{s\in S}\big], \quad \textit{and}$$

(iv) *if each ϕ_ξ is surjective and semi-open, then Φ is semi-open.*

Proof. (i) Let $x \in [(\xi_s)_{s\in S}, (U_s)_{s\in S}]$. $\Phi(x) = ((\phi_\xi(x_\xi))_{\xi<\tau}$. If $s \in S$, $x_{\xi_s} \in U_s$, hence $\phi_{\xi_s}(x_{\xi_s}) \in \phi_{\xi_s}[U_s]$. Then, $\Phi(x) \in [(\xi_s)_{s\in S}, (\phi_{\xi_s}[U_s])_{s\in S}]$.

(ii) This is obvious.

(iii) Let $y \in [(\xi_s)_{s\in S}, (\phi_{\xi_s}[U_s])_{s\in S}]$. Since each ϕ_ξ is surjective, for each $\xi < \tau$, there is $x_\xi \in X_\xi$ such that $\phi_\xi(x_\xi) = y_\xi$; and for each $s \in S$ we can choose x_{ξ_s} in U_s. Therefore, $(x_\xi)_{\xi<\tau} \in [(\xi_s)_{s\in S}; (U_s)_{s\in S}]$.

(iv) Let U be a non-empty open subset of $X = \prod_{\xi<\tau} X_\xi$. There are $n < \omega$, $\xi_1, \dots, \xi_n < \tau$ and non-empty open sets $U_1, \dots, U_n$ of $X_{\xi_1}, \dots, X_{\xi_n}$ respectively, such that $\emptyset \neq [\xi_1, \dots, \xi_n; U_1, \dots, U_n] \subseteq U$. Because of (iii), $\Phi[[\xi_1, \dots, \xi_n; U_1, \dots, U_n]] = [\xi_1, \dots, \xi_n; \phi_{\xi_1}[U_1], \dots, \phi_{\xi_n}[U_n]]$. Since each ϕ_ξ is semi-open, $\mathrm{int}(\phi_{\xi_i}[U_i]) \neq \emptyset$ for all $i \in \{1, \dots, n\}$. So,

$$\big[\xi_1, \dots, \xi_n; \mathrm{int}\big(\phi_{\xi_1}[U_1]\big), \dots, \mathrm{int}\big(\phi_{\xi_n}[U_n]\big)\big]$$

is not void. Therefore,

$$\big[\xi_1, \dots, \xi_n; \mathrm{int}\big(\phi_{\xi_1}[U_1]\big), \dots, \mathrm{int}\big(\phi_{\xi_n}[U_n]\big)\big]$$

is a non-empty open set contained in $\Phi[[\xi_1, \dots, \xi_n; U_1, \dots, U_n]] \subseteq \Phi[U]$. □

Let ϕ be the function from $\{0,1\}^\omega$ to $[0,1]$ defined as

$$\phi(x) = \sum_{i<\omega} \frac{x(i)}{2^{i+1}}.$$

Claim 5. *ϕ is an onto continuous, semi-open function and each fiber of ϕ has cardinality $\leqslant 2$.*

Proof. It is a well-known fact that ϕ is an onto continuous function each of its fibers having cardinality $\leqslant 2$.

Furthermore, ϕ is semi open because for $m < \omega$ and $\epsilon_0, \dots, \epsilon_m \in \{0,1\}$,

$$\phi\big[[0, \dots, m; \epsilon_0, \dots, \epsilon_m]\big] = \left[\sum_{i=0}^{i=m} \frac{\epsilon_i}{2^{i+1}}, \left(\sum_{i=0}^{i=m} \frac{\epsilon_i}{2^{i+1}}\right) + \frac{1}{2^{m+1}}\right]. \qquad \square$$

We denote by F the set $\{x \in \{0,1\}^\omega\colon$ for every $n < \omega$ there is $s > n$ such that $x(s) = 0\} \cup \{\tilde{1}\}$ where $\tilde{1}(n) = 1$ for all $n < \omega$.

Claim 6. *$H = \phi \restriction F : F \to [0,1]$ is a bijective continuous and semi-open function.*

Proof. Since $\phi : \{0,1\}^\omega \to [0,1]$ is continuous, so is $\phi \restriction F : F \to [0,1]$. The bijectivity of H follows from the following remarks: if $x, y \in \{0,1\}^\gamma$ with $x \neq y$, then $\phi(x) = \phi(y)$ if and only if when k is the first natural number n such that $x(n) \neq y(n)$, we have either $x(k) = 0$ and $x(i) = 1$ for all $i > k$ and $y(i) = 0$ for all $i > k$, or $x(k) = 1$ and $x(i) = 0$ for all $i > k$ and $y(i) = 1$ for all $i > k$. This fact follows from the equality $\sum_{i=k+1}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^{k+1}}$.

Finally, the function H is semi open because for $m < \omega$ and $\epsilon_0, \ldots, \epsilon_m \in \{0, 1\}$,

$$\left(\sum_{i=0}^{i=m} \frac{\epsilon_i}{2^{i+1}}, \left(\sum_{i=0}^{i=m} \frac{\epsilon_i}{2^{i+1}} \right) + \frac{1}{2^{m+1}} \right) \subseteq H\big[[0, \ldots, m; \epsilon_0, \ldots, \epsilon_m] \cap F\big]. \qquad \square$$

Claim 7. *For a cardinal number $\gamma > 0$, F^γ is a dense subset of $(\{0,1\}^\omega)^\gamma$.*

Next, we are going to obtain some results about resolvability for dense subspaces of $[0,1]^\gamma$ when γ is uncountable.

Proposition 5.1. *Let γ and κ be uncountable cardinal numbers. If there are no ai-maximal independent families of cardinality γ on κ, then every dense subset of $[0,1]^\gamma$ of cardinality $\leqslant \kappa$ is almost-ω-resolvable.*

Proof. Let $D \subseteq [0,1]^\gamma$ be a dense subset of cardinality $\leqslant \kappa$. Since $H : F \to [0,1]$ is a bijective continuous and semi-open function (Claim 6), $H^\gamma : F^\gamma \to [0,1]^\gamma$ is bijective continuous and semi-open (Claim 4). Thus, $(H^\gamma)^{-1}[D]$ is a dense subset of F^γ of cardinality $\leqslant \kappa$ (Claim 2). By Claim 7, $(H^\gamma)^{-1}[D]$ is dense in $(\{0,1\}^\omega)^\gamma \cong \{0,1\}^\gamma$. By Remark 3.3 and Proposition 2.12, $(H^\gamma)^{-1}[D]$ is almost-ω-resolvable. Moreover, $H^\gamma \restriction (H^\gamma)^{-1}[D] : (H^\gamma)^{-1}[D] \to D$ is a continuous and bijective function; thus, by Claim 1, D is almost-ω-resolvable. $\square$

Theorem 5.2. *Let γ and κ be uncountable cardinal numbers. If there are no ai-maximal independent families of cardinality γ on κ, every dense Baire subset of $[0,1]^\gamma$ of cardinality $\leqslant \kappa$ is ω-resolvable.*

Proof. Let D be a dense Baire subspace of $[0,1]^\gamma$ of cardinality $\leqslant \kappa$. Let $E \subseteq D$ be a dense subset of D. By Proposition 5.1, E is almost-ω-resolvable. By Theorem 1.3, D is ω-resolvable. $\square$

Remarks 5.3.

(1) Let $f : X \to Y$ be a continuous semi-open and onto function. If X is a Baire space, then Y is a Baire space.

Proof. Let $\{U_n: n < \omega\}$ be a countable family of open and dense subsets of Y. For each $n < \omega$, $f^{-1}[U_n]$ is open and dense because f is continuous and semi-open (Claim 2). Since X is a Baire space, then $D = \bigcap_{n<\omega} f^{-1}[U_n]$ is a dense subset of X. Then, $f[D]$ is dense in Y and is contained in $\bigcap_{n<\omega} U_n$. $\square$

(2) Let γ be an uncountable cardinal number. If every dense (and Baire) subset of $[0,1]^\gamma$ is θ-resolvable (resp., almost resolvable, almost-ω-resolvable), then every dense (and Baire) subset of F^γ is θ-resolvable (resp., almost resolvable, almost-ω-resolvable).

Proof. Let D be a dense (and Baire) subspace of F^γ. Then $(H^\gamma)[D]$ is dense (and Baire by (1)) in $[0,1]^\gamma$. By hypothesis, $(H^\gamma)[D]$ is κ-resolvable (resp., almost resolvable, almost-ω-resolvable). Moreover, $H^\gamma \restriction D : D \to (H^\gamma)[D]$ is a continuous semi-open and bijective function (Claim 3); thus, by Lemma 4.2, D is κ-resolvable (resp., almost resolvable, almost-ω-resolvable). $\square$

(3) The non-existence of *ai*-maximal independent families of cardinality γ on κ, implies that every Baire subset of $\{0,1\}^\gamma$ of cardinality $\leqslant \kappa$ contained densely in some subspace of $\{0,1\}^\gamma$ homeomorphic to F^γ is ω-resolvable.

We call an onto function $f : X \to Y$ *semicontinuous* if $\mathrm{int}(f^{-1}[U])$ is not empty for every open set U of Y. A space X is *semicompact* if every semicontinuous function $f : X \to \mathbb{R}$ is bounded. So, we have:

(4) Let γ be an uncountable cardinal number. Then, every dense semicompact subspace D of $[0,1]^\gamma$ is $\mathfrak{c}$-resolvable.

Proof. Let D be a dense semicompact subspace of $[0,1]^\gamma$. Then $(H^\gamma)^{-1}[D]$ is dense and pseudocompact in F^γ because for every continuous function $f : (H^\gamma)^{-1}[D] \to \mathbb{R}$, the composition $(H^\gamma)^{-1} \circ f : D \to \mathbb{R}$ is semicontinuous. By Corollary 4.4, $(H^\gamma)^{-1}[D]$ is $\mathfrak{c}$-resolvable. Therefore, D is $\mathfrak{c}$-resolvable (Lemma 4.2). $\square$

Questions 5.4.

(1) Is the statement "*every dense subset of $[0,1]^\gamma$ is almost-ω-resolvable*" equivalent to "*every dense subset of $\{0,1\}^\gamma$ is almost-ω-resolvable*"?
(2) Is the statement "*every Baire dense subset of $[0,1]^\gamma$ is ω-resolvable*" equivalent to "*every Baire dense subset of $\{0,1\}^\gamma$ is ω-resolvable*"?

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