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Extensions of functions in Mrówka–Isbell spaces

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Abstract

For an almost disjoint family (a.d.f.) Σ of subsets of ω, let $\Psi(\Sigma)$ be the Mrówka–Isbell space on Σ. In this article we will analyze the following problem: given an a.d.f. Σ and a function $\phi:\Sigma\to\{0,1\}$ (respectively $\phi:\Sigma\to\mathbb{R}$) is it possible to extend ϕ continuously to a big enough subspace $\Sigma\cup N$ of $\Psi(\Sigma)$ for which $\mathrm{cl}_{\Psi(\Sigma)}N\supset\Sigma$? Such an extension is called essential. We will prove that: (i) for every a.d.f. Σ of cardinality $2^{\aleph_0}$ we can find a function $\phi:\Sigma\to\{0,1\}$ without essential extensions; (ii) for every m.a.d. family Σ there exists a function $\phi:\Sigma\to\mathbb{R}$ that has no essential extension; and (iii) there exists a Mrówka–Isbell space $\Psi(\Sigma)$ of cardinality $\aleph_1$ such that every function $\phi:\Sigma\to\mathbb{R}$ with at least two different uncountable fibers, has no full extension. On the other hand, under Martin's Axiom every function $\phi:\Sigma\to\{0,1\}$ (respectively $\phi:\Sigma\to\mathbb{R}$) has an essential extension if $|\Sigma|<2^{\aleph_0}$. Finally, we analyze these questions under **CH** and by adding new Cohen reals to a ground model $\mathfrak{M}$ showing that the existence of an uncountable a.d.f. Σ for which every onto function $\phi:\Sigma\to\{0,1\}$ with infinite fibers has no essential extensions is consistent with **ZFC**. © 1997 Elsevier Science B.V.

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1. Introduction

All spaces below will be Tychonoff. If X is a space and $A\subset X$, then $\mathrm{cl}_X A$ is the closure of A in X, and $\mathcal{P}(X)$ is the collection of all subsets of X. For a function ϕ, we denote its domain as $\mathrm{dom}(\phi)$ and its range as $\mathrm{rng}(\phi)$; furthermore, if $Y\subset\mathrm{dom}(\phi)$, $\phi|_Y$

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denotes the restriction of ϕ to Y. As usual, ω is the set of natural numbers, ω^* is the set of all free ultrafilters on ω, and if $A \subset \omega$ and $\Sigma \subset \mathcal{P}(\omega)$, then

$$A^* = \{\mathcal{F} \in \omega^*\colon\ A \in \mathcal{F}\} \quad \text{and} \quad \Sigma^* = \{S^*\colon\ S \in \Sigma\}.$$

ω_1 is the set of all countable ordinals, $\mathfrak{c}$ is the cardinality of the continuum, and $[\omega]^\omega$ will denote the collection of all infinite subsets of ω. Finally, for a space X, $C_p(X)$ is the space of all continuous real-valued functions defined on X considered with the pointwise convergence topology. Let X be a space, Y a subspace of X, and let $\phi\colon Y \to W$ be a continuous function. We say that a continuous function $\widehat{\phi}\colon Z \to W$ is an *essential* (this term was suggested by M.G. Tkachenko) *extension* of ϕ if $Y \subset Z \subset X$, $\widehat{\phi}|_Y = \phi$ and each $y \in Y$ is a limit point of Z. If there exists a continuous function $\widehat{\phi}\colon X \to W$ with $\widehat{\phi}|_Y = \phi$ we will say that $\widehat{\phi}$ is a *full extension* of ϕ. If W is a subspace of some larger space T, we can use the term "essential extension of $\phi\colon Y \to W$ into T" considering $\widehat{\phi}$ as a function from Z to T.

As usual, we will call a collection Σ of infinite subsets of ω an *almost disjoint family* (a.d.f.) if for every two different elements A, B of Σ we have $|A \cap B| < \aleph_0$. A *maximal almost disjoint family* (m.a.d.f.) is a collection which is maximal with respect to the almost disjoint property. The symbols $A \subset^* B$ and $A =^* B$ mean that $|A \setminus B| < \aleph_0$ and $|(A \setminus B) \cup (B \setminus A)| < \aleph_0$, respectively.

Let Σ be an almost disjoint family of subsets of ω, and let us consider the following topology on $\Psi(\Sigma) = \omega \cup \Sigma$: each $n \in \omega$ is an isolated point, and a neighborhood of a point $A \in \Sigma$ is any set containing A and all of the points of A but a finite number. Such a space is called a *Mrówka–Isbell space* (also known as a *Ψ-space* [3]. These spaces were first considered by Mrówka in [12] and by Isbell).

A Mrówka–Isbell space $\Psi(\Sigma)$ is a first countable, locally compact and, if Σ is infinite, noncountably compact space; ω is dense in $\Psi(\Sigma)$, and Σ is closed and discrete. Moreover, $\Psi(\Sigma)$ is pseudocompact iff Σ is a m.a.d. family; so, in this case $\Psi(\Sigma)$ is not normal (for a more detailed analysis on basic properties of Mrówka–Isbell spaces, see [4,3]).

In this paper we will study essential and full extensions of real-valued functions defined on the subset of nonisolated points of Mrówka–Isbell spaces. In Section 2 we will show some negative results obtained in **ZFC**; in Section 3 we prove some consistency results by adding a collection of Cohen reals to a ground model $\mathfrak{M}$; Sections 4 and 5 are devoted to an analysis of essential extensions under **CH** and Martin's Axiom.

If Σ is an a.d.f. on ω of cardinality $\leqslant \aleph_0$, then it is not difficult to prove that every function $\phi\colon \Sigma \to \{0,1\}$ (respectively $\phi\colon \Sigma \to \mathbb{R}$) has a full extension. So, from now on, if nothing is said to the contrary, a.d.f. will mean uncountable a.d.f.

2. Some negative results

Lemma 2.1. *Let W be a space, Σ an a.d.f., and let $f, \phi\colon \Sigma \to W$ be two different functions. If $N \subset \omega$ and $\hat{f}, \widehat{\phi}\colon \Sigma \cup N \to W$ are essential extensions of f and ϕ, respectively, then $\hat{f}|_N \neq \widehat{\phi}|_N$.*

Proof. There is $A \in \Sigma$ such that $w_1 = f(A) \neq \phi(A) = w_2$. Since W is Hausdorff (every space is assumed to be even Tychonoff), there are disjoint open neighborhoods W_1, W_2 of w_1 and w_2. Since $\hat{f}$ and $\widehat{\phi}$ are continuous, there are open neighborhoods V_1, V_2 of the point A such that $\hat{f}(V_1) \subset W_1$, $\widehat{\phi}(V_2) \subset W_2$. But $N \cap V_1 \cap V_2 \neq \emptyset$. For every point x from this set we have $\hat{f}(x) \neq \widehat{\phi}(x)$. □

Observe that if W is a space with cardinality $\leqslant 2^{\aleph_0}$, then $|\{f : N \to W\colon N \subset \omega\}| = 2^{\aleph_0}$. So, by Lemma 2.1 we have:

Proposition 2.2. *Let W be a space and let Σ be an a.d.f. such that $|W| \leqslant 2^{\aleph_0} < 2^{|\Sigma|}$. Then there exists a function $\phi : \Sigma \to W$ without essential extensions.*

The following proposition is a corollary of Proposition 2.2; we include an alternative constructive proof.

Proposition 2.3. *If Σ is an almost disjoint family of cardinality $2^{\aleph_0}$, then there exists a function $\phi : \Sigma \to \{0,1\}$ (respectively $\phi : \Sigma \to \mathbb{R}$) that has no essential extension.*

Proof. Let $\Sigma = \{A_\alpha\colon \alpha < \mathfrak{c}\}$. Let $\{f_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all $\{0,1\}$-valued (respectively real-valued) functions f with the domain contained in ω and such that $\Sigma \subset \mathrm{cl}_{\Psi(\Sigma)}(\mathrm{dom}(f))$. Let $\alpha < \mathfrak{c}$; if $\lim_{n\to\infty} f_\alpha(a_n)$ does not exist for a sequence $(a_n)_{n<\omega}$ in $A_\alpha \cap \mathrm{dom}(f_\alpha)$, then we define $\phi(A_\alpha) = 0$. Otherwise, we define $\phi(A_\alpha)$ to be an $r \in \{0,1\}$ (respectively $r \in \mathbb{R}$) which is not equal to any of these limits. Such an r exists, because the set

$$L = \Big\{ \lim_{n\to\infty} f_\alpha(a_n)\colon\ (a_n)_{n<\omega} \text{ is a sequence in } A_\alpha \cap \mathrm{dom}(f_\alpha) \Big\}$$

has cardinality equal to 1 (respectively L is bounded). ϕ is the required function. □

The following result relates essential extensions of real-valued functions to those of $\{0,1\}$-valued functions.

Lemma 2.4. *Let Σ be an a.d.f. Then*

(1) *If $\phi : \Sigma \to \{0,1\}$ has an essential extension $\widehat{\psi} : \Sigma \cup N \to \mathbb{R}$, then ϕ has an essential extension $\widehat{\phi} : \Sigma \cup N \to \{0,1\}$.*

(2) *If every function $\psi : \Sigma \to \mathbb{R}$ has an essential extension, then every function $\phi : \Sigma \to \{0,1\}$ has an essential extension into $\{0,1\}$.*

(3) *If $|\Sigma| < 2^{\aleph_0}$ and each onto function $\phi : \Sigma \to \{0,1\}$ with infinite fibers has no essential extensions into $\{0,1\}$ then each $\psi : \Sigma \to \mathbb{R}$ with at least two infinite fibers has no essential extension.*

Proof. (2) is a consequence of (1), and the proof of this is similar to the one we are going to give for (3). Let $\psi : \Sigma \to \mathbb{R}$ be a function with at least two infinite fibers, and suppose that $\widehat{\psi} : \Sigma \cup N \to \mathbb{R}$ is an essential extension of ψ. Fix $a < b$ such that $\psi^{-1}(a)$ and $\psi^{-1}(b)$ are infinite. Since $|\Sigma| < 2^{\aleph_0}$, there is $c \in (a,b) \setminus \widehat{\psi}(\Sigma \cup N)$. Then, the function $\widehat{\phi}: \Sigma \cup N \to \{0,1\}$ defined by $\widehat{\phi}(x) = 0$ if $\widehat{\psi}(x) < c$, and $\widehat{\phi}(x) = 1$ if $\widehat{\psi}(x) > c$ is an essential extension of $\widehat{\phi}|_\Sigma : \Sigma \to \{0,1\}$; a contradiction. □

When Σ is an m.a.d. family we also obtain some negative results:

Proposition 2.5. *For every infinite m.a.d. family Σ on ω there exists a function $\phi: \Sigma \to \{0,1\}^\omega$ (= The Cantor set) that has no essential extension.*

Proof. There exists $\mathcal{F} \subset \{0,1\}^\omega$ such that

(1) $|\mathcal{F}| = |\Sigma|$ and

(2) for every $n < \omega$, $|\{f \in \mathcal{F}\colon f(n) = i\}| = |\Sigma|$ for $i \in \{0,1\}$.

Thus, we can index Σ in a one-to-one and onto fashion with $\mathcal{F}$:

$$\Sigma = \{A_f\colon f \in \mathcal{F}\}.$$

Consider the function $\phi: \Sigma \to \{0,1\}^\omega$ defined by $\phi(A_f) = f$. Let N be a subset of ω such that $\mathrm{cl}_{\Psi(\Sigma)} N \supset \Sigma$. For each $n < \omega$ and $i \in \{0,1\}$, put $\Sigma_{n,i} = \{A_f\colon f(n) = i\}$.

It is not difficult to prove the following assertion and we do not include its proof (see the proof of the Theorem in [15]).

Claim. *There exist $n_0 < \omega$ and $X \subset N$ such that*

$$\Sigma(X) = \{X \cap A\colon A \in \Sigma \text{ and } |X \cap A| = \aleph_0\}$$

is an infinite m.a.d. family, and for every $Y \subset X$, if

$$\Sigma_{n_0,i}(Y) = \{Y \cap A\colon A \in \Sigma_{n_0,i} \text{ and } |Y \cap A| = \aleph_0\}$$

is a m.a.d. family for some $i \in \{0,1\}$, then $\Sigma_{n_0,i}(Y)$ is finite.

Now suppose that $\widehat{\phi}: \Sigma \cup N \to \{0,1\}^\omega$ is an essential extension of ϕ. Define $\phi_n : \Sigma \to \{0,1\}$ as $\phi_n = \pi_n \circ \phi$ where π_n is the projection to the nth factor, and $\widehat{\phi}_n : \Sigma \cup N \to \{0,1\}$ by $\widehat{\phi}_n(x) = \widehat{\phi}(x)(n)$. If $A \in \Sigma_{n_0,i}$ is such that $|A \cap X| = \aleph_0$, then, because of the continuity of $\widehat{\phi}_{n_0}$, we have $|(\widehat{\phi}_{n_0}|_X)^{-1}(i) \cap A| = \aleph_0$. Hence, since $\Sigma = \Sigma_{n_0,0} \cup \Sigma_{n_0,1}$, either

$$S_0 = \{(\widehat{\phi}_{n_0}|_X)^{-1}(0) \cap A\colon A \in \Sigma_{n_0,0}\} \quad \text{or}$$

$$S_1 = \{(\widehat{\phi}_{n_0}|_X)^{-1}(1) \cap A\colon A \in \Sigma_{n_0,1}\}$$

is an infinite m.a.d. family; but this contradicts the claim. □

Corollary 2.6. *For every infinite m.a.d. family Σ there exists a function $\phi: \Sigma \to \mathbb{R}$ that has no essential extension.*

Proof. Let $\psi: \Sigma \to \{0,1\}^\omega$ be a function without an essential extension (Proposition 2.5). Suppose that there exists a continuous function $\widehat{\psi}: \Sigma \cup N \to \mathbb{R}$ that extends ψ. Since $|N| < 2^{\aleph_0}$, there exists a continuous function $h: \widehat{\psi}(\Sigma \cup N) \to \{0,1\}^\omega$ such that for every $A \in \Sigma$, $h(\widehat{\psi}(A)) = \widehat{\psi}(A) = \psi(A)$; hence $h \circ \widehat{\psi}: \Sigma \cup N \to \{0,1\}^\omega$ is an essential extension of ψ, which is impossible. □

Corollary 2.7. *For every infinite m.a.d. family Σ there exists a function $\phi : \Sigma \to \{0,1\}$ without full extension.*

Proof. Assume that for every $\phi : \Sigma \to \{0,1\}$ there exists a continuous extension $\widehat{\phi} : \Sigma \cup \omega \to \{0,1\}$ of ϕ. Let $\psi : \Sigma \to \{0,1\}^\omega$ be a function such that ψ has no essential extensions. By our assumption, for every $n < \omega$, $\psi_n = \pi_n \circ \psi : \Sigma \to \{0,1\}$ has a full extension $\widehat{\psi}_n : \Sigma \cup \omega \to \{0,1\}$. So $\widehat{\psi} : \Sigma \cup \omega \to \{0,1\}^\omega$ defined by $\widehat{\psi}(x)(n) = \widehat{\psi}_n(x)$ is a continuous extension of ψ. This is a contradiction. □

Corollary 2.8. *Let $X \subset \omega$ and let Σ be an a.d.f. If*

$$\Sigma(X) = \{A \cap X \colon\ A \in \Sigma \quad \text{and} \quad |A \cap X| = \aleph_0\}$$

is an infinite m.a.d. family, then there exists a function $\phi : \Sigma \to \mathbb{R}$ (respectively $\psi : \Sigma \to \{0,1\}$) without any essential (respectively full) extension.

Problem 2.9. Is there, for each m.a.d. family Σ, a function $\phi : \Sigma \to \{0,1\}$ without essential extension?

Let Σ_0 and Σ_1 be two families of infinite subsets of ω. We say that Σ_0 and Σ_1 are *separated* if there exists a set $S \subset \omega$ such that $A \subset^* S$ for each $A \in \Sigma_0$ and $B \cap S =^* \emptyset$ for each $B \in \Sigma_1$; in this case, we say that S *separates* Σ_0 *from* Σ_1. A pair (Σ_0, Σ_1) forms a *Luzin gap* if for every uncountable subsets Σ_0' and Σ_1' of Σ_0 and Σ_1, Σ_0' and Σ_1' are not separated.

Remark 2.10. N.N. Luzin constructed an almost disjoint family Σ of cardinality $\aleph_1$ such that every two disjoint uncountable subfamilies are not separated (see, e.g., [9] or [3, p. 124]).

The concept of essential extension of real-valued functions defined on the subset of all nonisolated points of Mrówka–Isbell spaces is related with that of Luzin gaps as follows:

Lemma 2.11. *Let Σ be an almost disjoint family and $N \subset \omega$ such that $\mathrm{cl}_{\Psi(\Sigma)} N \supset \Sigma$. Let $\phi : \Sigma \to \{0,1\}$ be a function, $\Sigma_0 = \phi^{-1}(0)$ and $\Sigma_1 = \phi^{-1}(1)$. Then the following statements are equivalent*

(a) *ϕ has an essential extension on $\Sigma \cup N$.*
(b) *$\Sigma_0' = \{A \cap N \colon\ A \in \Sigma_0\}$ and $\Sigma_1' = \{A \cap N \colon\ A \in \Sigma_1\}$ are separated.*
(c) *There exist two disjoint sets $P, Q \subset N$ such that $\mathrm{cl}_{\Psi(\Sigma)} P \supset \Sigma_0$ and $\mathrm{cl}_{\Psi(\Sigma)} Q \supset \Sigma_1$, but $\mathrm{cl}_{\Psi(\Sigma)} P \cap \Sigma_1 = \emptyset$ and $\mathrm{cl}_{\Psi(\Sigma)} Q \cap \Sigma_0 = \emptyset$.*
(d) *There exist two disjoint sets $P, Q \subset N$ such that, for every $A \in \Sigma_0$ and $B \in \Sigma_1$, $A \cap P$ and $B \cap Q$ are infinite, but $A \cap Q$ and $B \cap P$ are finite.*

Proof. (a) $\Rightarrow$ (b) Let $\widehat{\phi} : \Sigma \cup N \to \{0,1\}$ be an essential extension of ϕ, and let $S = \widehat{\phi}^{-1}(0) \cap \omega$. Then, S separates Σ_0' from Σ_1'.

(b) ⇒ (c) If S separates Σ_0' from Σ_1', then the sets $P = S \cap N$ and $Q = N \setminus S$ satisfy the requirement.

(c) ⇒ (a) The function $\widehat{\phi}: \Sigma \cup N \to \{0,1\}$ defined by $\widehat{\phi}(A) = i$ if $A \in \Sigma_i$, $\widehat{\phi}(n) = 0$ if $n \in P$ and $\widehat{\phi}(n) = 1$ if $n \in N \setminus P$ is an essential extension of ϕ.

(c) ⇔ (d) This is trivial. □

Observe that for an a.d.f. Σ, a function $\phi: \Sigma \to \mathbb{R}$ trivially has a full extension over $\Psi(\Sigma)$ if there exists an $r \in \mathbb{R}$ such that $|\{A \in \Sigma: \phi(A) \neq r\}| < \aleph_0$.

Corollary 2.12. *Let Σ be an almost disjoint family, and let (Σ_0, Σ_1) be a partition of Σ. The pair (Σ_0, Σ_1) is a Luzin gap iff every function ϕ from $\Sigma' \subset \Sigma \to \mathbb{R}$ with at least two uncountable fibers, one contained in Σ_0 and the other in Σ_1, has no full extension.*

Proof. (⇒) Let $\phi: \Sigma' \to \mathbb{R}$ be a function such that $\Gamma_0 = \phi^{-1}(r_0) \subset \Sigma_0$ and $\Gamma_1 = \phi^{-1}(r_1) \subset \Sigma_1$ are uncountable where $r_0 \neq r_1$. Let $\Gamma = \Gamma_0 \cup \Gamma_1$, and let $\psi: \Gamma \to \{r_0, r_1\}$ be a function defined as $\psi(A) = \phi(A)$ for $A \in \Gamma$. If ϕ has a full extension $\widehat{\phi}: \Sigma' \cup \omega \to \mathbb{R}$, then $\widehat{\phi}|_{\Gamma \cup \omega}: \Gamma \cup \omega \to \mathbb{R}$ is a full extension of ψ. By Lemma 2.4, there exists a full extension $\widehat{\psi}: \Gamma \cup \omega \to \{r_0, r_1\}$ of ψ. By Lemma 2.11, Γ_0 and Γ_1 are separated. Therefore (Σ_0, Σ_1) is not a Luzin gap.

(⇐) If (Σ_0, Σ_1) is not a Luzin gap, then there exists two uncountable sets $\Sigma_0' \subset \Sigma_0$ and $\Sigma_1' \subset \Sigma_1$ which are separated. Let $\Sigma' = \Sigma_0' \cup \Sigma_1'$ and let $\phi: \Sigma' \to \{0,1\}$ defined by $\phi(A) = 0$ if $A \in \Sigma_0'$ and $\phi(A) = 1$ if $A \in \Sigma_1'$. Because of Lemma 2.11, ϕ has a full extension. □

The following proposition is a consequence of Remark 2.10 and Corollary 2.12.

Proposition 2.13. *There exists a Mrówka–Isbell space $\Psi(\Sigma)$ of cardinality $\aleph_1$ such that every function $\phi: \Sigma \to \mathbb{R}$ with at least two different uncountable fibers has no full extension.*

Definition 2.14 [6]. Let $\mathcal{F}$ be an ultrafilter on ω. We say that $\mathcal{F}$ is an ***ω_1-p-ultrafilter*** if there exists a sequence $(p_\xi)_{\xi<\omega_1}$ of infinite subsets of ω such that for all $\xi < \eta < \omega_1$,

(1) $p_\eta \subset^* p_\xi$,

(2) $|p_\xi \setminus p_\eta| = \aleph_0$ and

(3) $\forall B \in \mathcal{F}\ \exists\, \xi < \omega_1\ (p_\xi \subset^* B)$.

The existence of an ω_1-p-ultrafilter is a consequence of **CH**, and is also consistent with the negation of **CH**, but it is not a theorem of **ZFC**; in fact:

Proposition 2.15. *Let $\mathcal{F} \in \omega^*$. $\mathcal{F}$ is an ω_1-p-ultrafilter if and only if $\mathcal{F}$ is a P-point with character equal to $\aleph_1$.*

Proof. (⇒) Let $(p_\xi: \xi < \omega_1)$ be a sequence of infinite subsets of ω witnessing that $\mathcal{F}$ is an ω_1-p-ultrafilter. Hence, $\mathcal{V} = \{p_\xi^*: \xi < \omega_1\}$ is a π-local base for $\mathcal{F}$; so $\mathcal{V}$ is a local base for $\mathcal{F}$ because each p_ξ^* is clopen and $\mathcal{V}$ is a decreasing chain. These facts also imply that $\mathcal{F}$ is a P-point.

($\Leftarrow$) Let $\{W_\xi\colon \xi < \omega_1\}$ be a local base of $\mathcal{F}$ in ω^*. Since $\mathcal{F}$ is a P-point and ω^* is zero-dimensional, we can assume that each W_ξ is clopen and for $\xi < \eta < \omega_1$, $W_\eta \subset W_\xi$ with $W_\eta \neq W_\xi$. Then, for each $\xi < \omega_1$ there is $p_\xi \subset \omega$ such that $p_\xi^* = W_\xi$. Thus, the sequence $(p_\xi\colon \xi < \omega_1)$ witnesses that $\mathcal{F}$ is an ω_1-p-ultrafilter. □

Recall that the existence of a free ultrafilter on ω which has the character equal to $\aleph_1$ and is a P-point, is consistent with any admissible cardinal arithmetics (see [7]).

Let $\mathcal{F}$ be an ω_1-p-ultrafilter and let $(p_\xi)_{\xi<\omega_1}$ be a sequence satisfying (1)–(3) in Definition 2.14. For each ξ, fix an infinite set $A_\xi \subset p_\xi \setminus p_{\xi+1}$. Note that the family $\Sigma(\mathcal{F}) = \{A_\xi\colon \xi < \omega_1\}$ is almost disjoint. An argument similar to the proof of claim 3.5 in [6] shows that if $N \subset \omega$ is such that $\Sigma(\mathcal{F}) \subset \mathrm{cl}_{\Psi(\Sigma(\mathcal{F}))} N$, then every continuous function $\phi\colon \Sigma(\mathcal{F}) \cup N \to \mathbb{R}$ is *eventually constant*, i.e., there is an $r \in \mathbb{R}$ such that $|\{A \in \Sigma(\mathcal{F})\colon \phi(A) \neq r\}| < |\Sigma(\mathcal{F})|$. So, we obtain

Proposition 2.16. *Let $\mathcal{F}$ be an ω_1-p-ultrafilter on ω. If $\phi\colon \Sigma(\mathcal{F}) \to \mathbb{R}$ is not eventually constant, then ϕ has no essential extension.*

From Proposition 4.2 in [6], Proposition 2.16 and Corollary 2.12 we conclude that every ω_1-p-ultrafilter provides an example of a Luzin gap (Σ_0, Σ_1) such that every function $\phi\colon \Sigma_0 \cup \Sigma_1 \to \mathbb{R}$ with two uncountable fibers has no essential extension.

Definition 2.17. Let $\mathcal{A}$ be a family of subsets of a space X. We say that $\mathcal{A}$ is an *arrow* in X if there exists an element $x \in X$ such that for each neighborhood V of x we have $|\{A \in \mathcal{A}\colon A \not\subset V\}| < |\mathcal{A}|$. If this is the case, we say that the arrow $\mathcal{A}$ *converges to* x, and we write $\mathcal{A} \to x$.

Observe that if $\mathcal{A} = \{A_n\colon n < \omega\}$ is an arrow of X, $x_n \in A_n$ for each $n < \omega$ and $\mathcal{A} \to x$, then the sequence $(x_n)_{n<\omega}$ converges to x. Since there is no convergent sequences in ω^*, every arrow in ω^* is uncountable.

Proposition 2.18. *Let Σ be an a.d.f. of regular cardinality α of infinite subsets of ω. If $\Sigma^* = \{A^*\colon A \in \Sigma\}$ is an arrow in ω^* and $\phi\colon \Sigma \to \mathbb{R}$ is not eventually constant, then ϕ has no essential extension.*

Proof. Let $\Sigma = \{A_\xi\colon \xi < \alpha\}$ be a faithful enumeration of Σ. Let N be a subset of ω such that $\mathrm{cl}_{\Psi(\Sigma)} N \supset \Sigma$, and suppose that $\widehat{\phi}\colon \Sigma \cup N \to \mathbb{R}$ is a continuous function. We are going to prove that $\widehat{\phi}$ is eventually constant. Since Σ^* is an arrow in ω^*, there exists $\mathcal{U} \in \omega^*$ such that $\Sigma^* \to \mathcal{U}$.

We now proceed as in [6, pp. 5, 6]: for every basic open interval $(a,b) \subset \mathbb{R}$, let $D(a,b) = \{n \in N\colon \widehat{\phi}(n) \in (a,b)\}$ and $E(a,b) = \{\xi \in \alpha\colon A_\xi \in \mathrm{cl}_{\Psi(\Sigma)} D(a,b)\}$. Because $\aleph_0 < |\Sigma|$, the following will complete the proof.

Claim. *If $a < b < c < d$ are real numbers, then at most one of the sets $E(a,b)$ and $E(c,d)$ has cardinality α.*

Proof. In fact, since $D(a,b) \cap D(c,d) = \emptyset$, at most one of them is an element of $\mathcal{U}$. Without loss of generality, suppose that $D(a,b) \notin \mathcal{U}$. Then there exists $U \in \mathcal{U}$ such that $D(a,b) \cap U = \emptyset$. Since Σ^* converges to $\mathcal{U}$ and $\mathrm{cof}(\alpha) = \alpha$, there exists $\xi_0 \in \alpha$ such that $A_\xi \subset^* U$ $\forall \xi \geqslant \xi_0$, hence $\xi \notin E(a,b)$ for $\xi \geqslant \xi_0$. That is, $|E(a,b)| < \alpha$. □

Proposition 2.19. *If $\mathcal{U} \in \omega^*$ has a base of cardinality $\aleph_1$, then there exists a disjoint clopen arrow $\mathcal{A}$ converging to $\mathcal{U}$.*

Proof. If $\mathcal{U}$ is not a P-point, then there exists a countable family of clopen subsets of ω^*, $\mathcal{T}$, such that $\mathcal{U} \notin T = \bigcup \mathcal{T}$, but $\mathcal{U} \in \mathrm{cl}_{\omega^*}(T)$. Let $\mathcal{B} = \{V_\lambda\colon \lambda < \omega_1\}$ be a local base of $\mathcal{U}$ in ω^*. We will construct $\mathcal{A}$ by transfinite induction. Let A_0 be a nonempty clopen subset of ω^* such that $A_0 \subset V_0$ and $\mathcal{U} \notin A_0$. Suppose that we have already defined two α-sequences $\{A_\lambda\colon \lambda < \alpha\}$ and $\{\eta_\lambda\colon \lambda < \alpha\}$ $(\alpha < \omega_1)$ such that

(a) A_λ is a nonempty clopen subset of ω^* with $A_\lambda \cap T = \emptyset$;

(b) $A_\lambda \cap A_\xi = \emptyset$ if $\lambda \neq \xi$;

(c) every η_λ is a countable ordinal, $0 = \eta_0$, and $\eta_\lambda < \eta_\gamma$ if $\lambda < \gamma < \alpha$;

(d) $A_\lambda \subset (\bigcap_{\xi \leqslant \eta_\lambda} V_\xi) \cap (\omega^* \setminus V_{\eta_\gamma})$ for every $\lambda < \gamma < \alpha$.

Let us make the following α-step. The sets $M_\alpha = \bigcup\{A_\lambda\colon \lambda < \alpha\}$ and T are two disjoint open F_σ subsets of ω^*. Since ω^* is a normal F-space, $\mathrm{cl}_{\omega^*} M_\alpha \cap \mathrm{cl}_{\omega^*} T = \emptyset$; but $\mathcal{U} \in \mathrm{cl}_{\omega^*} T$, so $\mathcal{U} \notin \mathrm{cl}_{\omega^*} M_\alpha$. Then there exists $\eta_\alpha < \omega_1$ such that $\eta_\alpha > \eta_\lambda$ for every $\lambda < \alpha$, and $V_{\eta_\alpha} \cap \mathrm{cl}_{\omega^*} M_\alpha = \emptyset$. Besides,

$$\mathcal{U} \in W_\alpha = \left(\bigcap_{\lambda \leqslant \eta_\alpha} V_\lambda \right) \cap (\omega^* \setminus \mathrm{cl}_{\omega^*} M_\alpha) \cap (\omega^* \setminus T),$$

and W_α is a nonempty G_δ subset of ω^*. Then W_α contains an infinite interior. Let $\mathcal{V} \neq \mathcal{U}$ and $\mathcal{V} \in \mathrm{int}\, W_\alpha$. Since ω^* is zero-dimensional, there exists a clopen set A_α such that $\mathcal{V} \in A_\alpha \subset \mathrm{int}\, W_\alpha$ and $\mathcal{U} \notin A_\alpha$.

In this manner we can obtain a family $\mathcal{A} = \{A_\lambda\colon \lambda < \omega_1\}$ that satisfies conditions (a)–(d). It can be proved that $\mathcal{A}$ is a disjoint clopen arrow converging to $\mathcal{U}$.

If $\mathcal{U}$ is a P-point in ω^*, we can construct $\mathcal{A}$ in a similar and easier way than before without the need of any auxiliary family $\mathcal{T}$. □

The following result is a consequence of the two previous propositions.

Corollary 2.20. [CH] *Let Σ be an almost disjoint family on ω, and let $\phi : \Sigma \to \mathbb{R}$ be a noneventually constant function. If Σ^* is an arrow, then ϕ has no essential extension.*

3. Adding new Cohen reals

In what follows, by a model we shall mean one that is standard and transitive for the axiom system **ZFC** of set theory (see [7]).

Lemma 3.1. *Let $\mathcal{A}$ be a countable a.d.f. on ω in a model $\mathfrak{M}$. If $\mathfrak{N}$ is a model obtained by adding a new Cohen real to $\mathfrak{M}$, then there exists $B \in [\omega]^\omega$ in $\mathfrak{N}$ such that*

(1) *$\mathcal{A} \cup \{B\}$ is an a.d.f., and*

(2) *$|B \cap (\bigcup \psi(\mathcal{A}_0))| = |B \cap (\bigcup \psi(\mathcal{A}_1))| = \aleph_0$ for every disjoint infinite partition $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ with $\mathcal{A}_0$, $\mathcal{A}_1 \in \mathfrak{M}$ and every function $\psi \in \mathfrak{M}$, $\psi : \mathcal{A} \to [\omega]^\omega$ with $\psi(A) \subset A$ $\forall A \in \mathcal{A}$.*

Proof. Let $\mathbb{P}$ be the set of all pairs of finite functions $\langle f, \mathcal{F}\rangle$ such that

(a) $\mathrm{dom}(f) \subset \omega$ and $\mathrm{rng}(f) \subset \{0,1\}$;

(b) $\mathrm{dom}(\mathcal{F}) \subset \mathcal{A}$ and $\mathrm{rng}(\mathcal{F}) \subset \omega$;

(c) $f(k) = 0$ if $k \in \bigcup_{A \in \mathrm{dom}(\mathcal{F})}[(A \cap \mathrm{dom}(f)) \setminus \mathcal{F}(A)]$.

We define in $\mathbb{P}$ the following relation: $\langle f, \mathcal{F}\rangle \leqslant \langle g, \mathcal{G}\rangle$ iff

(i) $f \supset g$, $\mathcal{F} \supset \mathcal{G}$, and

(ii) $f(k) = 0$ for every $k \in \bigcup\{[A \cap \mathrm{dom}(f)] \setminus \mathcal{G}(A)\colon\ A \in \mathrm{dom}(\mathcal{G})\}$.

Since $\mathbb{P}$ is countable, there exists an $\mathfrak{M}$-generic $\boldsymbol{G} \subset \mathbb{P}$. Let $\mathfrak{N} = \mathfrak{M}[\boldsymbol{G}]$,

$$F = \bigcup\{f\colon\ \exists \mathcal{F} \in \mathcal{A} \times \omega(\langle f, \mathcal{F}\rangle \in \boldsymbol{G})\}$$

and $B = F^{-1}(1)$. Now we will prove that B satisfies the requirements.

It is easy to prove that for each $a \in \omega$, the set $\mathcal{D}_a = \{\langle f, \mathcal{F}\rangle \in \mathbb{P}\colon\ a \in \mathrm{dom}(f)\}$ is dense in $\mathbb{P}$. Besides, for each $A \in \mathcal{A}$, the set $\mathcal{D}_A = \{\langle f, \mathcal{F}\rangle \in \mathbb{P}\colon\ A \in \mathrm{dom}(\mathcal{F})$ and $[\mathrm{dom}(f) \cap A] \setminus \mathcal{F}(A) \neq \emptyset\}$ is also dense. In fact, let $\langle g, \mathcal{G}\rangle \in \mathbb{P} \setminus \mathcal{D}_A$. We take $a_1, a_2 \in A$ such that $a_2 > a_1 > \max(\mathrm{dom}(g))$. If additionally $A \in \mathrm{dom}(\mathcal{G})$, we have to be careful to choose a_2 bigger than $\mathcal{G}(A)$. We define $\mathcal{F}$: $\mathrm{dom}(\mathcal{G}) \cup \{A\} \to \omega$ and $f : \mathrm{dom}(g) \cup \{a_2\} \to \{0,1\}$ as follows: $\mathcal{F}(G) = \mathcal{G}(G)$ if $G \in \mathrm{dom}(\mathcal{G})$ and $\mathcal{F}(A) = a_1$ if $A \notin \mathrm{dom}(\mathcal{G})$; and $f(n) = g(n)$ if $n \in \mathrm{dom}(g)$ and $f(a_2) = 0$. Then $\langle f, \mathcal{F}\rangle \in \mathcal{D}_A$ and $\langle f, \mathcal{F}\rangle \leqslant \langle g, \mathcal{G}\rangle$.

Let $A \in \mathcal{A}$ and take $\langle f, \mathcal{F}\rangle \in \mathcal{D}_A \cap \boldsymbol{G}$. Then there is $a_1 \in [\mathrm{dom}(f) \cap A] \setminus \mathcal{F}(A)$. For each $a > a_1$ we can find $\langle g, \mathcal{G}\rangle \in \mathcal{D}_a \cap \boldsymbol{G}$. If $\langle h, \mathcal{H}\rangle \in \boldsymbol{G}$ is a stronger condition than $\langle f, \mathcal{F}\rangle$ and $\langle g, \mathcal{G}\rangle$, then $h(a) = 0$. This means that $|A \cap B| < \aleph_0$. So, B satisfies condition (1).

On the other hand, if $\mathcal{C} \subset \mathcal{A}$ is infinite and $\psi : \mathcal{C} \to [\omega]^\omega$ is a function such that $\psi(C) \subset C$ for every $C \in \mathcal{C}$, then the set

$$\mathcal{D}(\mathcal{C}, \psi) = \left\{\langle f, \mathcal{F}\rangle \in \mathbb{P}\colon\ \exists m \in \bigcup_{C \in \mathcal{C}} \psi(C)\ \bigl(f(m) = 1\bigr)\right\}$$

is also dense in $\mathbb{P}$. Let us verify this: let $\langle g, \mathcal{G}\rangle \in \mathbb{P} \setminus \mathcal{D}(\mathcal{C}, \psi)$. Since $\mathcal{C}$ is infinite, there is $C_0 \in \mathcal{C} \setminus \mathrm{dom}(\mathcal{G})$. So, we can take $m \in \psi(C_0) \setminus [\mathrm{dom}(g) \cup \mathrm{rng}(\mathcal{G})]$. Let $f : \mathrm{dom}(g) \cup \{m\} \to \{0,1\}$ be the function that coincides with g in $\mathrm{dom}(g)$ and has the value 1 at m. Then $\langle f, \mathcal{G}\rangle \in \mathcal{D}(\mathcal{C}, \psi)$ and $\langle f, \mathcal{G}\rangle \leqslant \langle g, \mathcal{G}\rangle$.

Now, using the density of $\mathcal{D}(\mathcal{C},\psi)$, we can prove that B also satisfies condition (2). □

Proposition 3.2. *Let $\mathfrak{M}$ be a model. If one adds $\aleph_1$ new Cohen reals to $\mathfrak{M}$, then in a resulting model there is a Mrówka–Isbell space $\Psi(\Sigma)$ of cardinality $\aleph_1$ for which every onto function $\phi:\Sigma\to\{0,1\}$ with infinite fibers has no essential extension.*

Proof. Let $\mathbb{K}(\omega\times\alpha,2)$ be the partially ordered set of all finite functions from $\omega\times\alpha$ to $\{0,1\}$ with the relation $f\leqslant g$ iff $f\supset g$. Let $\mathfrak{M}_{\omega_1}$ be the generic extension of $\mathfrak{M}$ by means of an $\mathfrak{M}$-generic subset G of $\mathbb{K}(\omega\times\omega_1,2)$. In $\mathfrak{M}_{\omega_1}$ there is a transfinite sequence of models $\mathfrak{M}=\mathfrak{M}_0,\ldots,\mathfrak{M}_\alpha,\ldots,\mathfrak{M}_{\omega_1}$, where $\mathfrak{M}_\alpha=\mathfrak{M}[G\cap\mathbb{K}(\omega\times\alpha,2)]$. For every $\alpha\in\omega_1$ we can consider $\mathfrak{M}_{\alpha+1}$ as a model which is obtained from $\mathfrak{M}_\alpha$ by adding one new Cohen real; so, let $\Sigma=\{A_\alpha\colon \alpha<\omega_1\}$ be an a.d.f. in $\mathfrak{M}_{\omega_1}$ obtained by using Lemma 3.1 in each step of a transfinite induction.

Now, let $\phi:\Sigma\to\{0,1\}$ be any function satisfying $\min\{|\phi^{-1}(0)|,|\phi^{-1}(1)|\}\geqslant\aleph_0$. We will prove that ϕ has no essential extension. Indeed, let M_0 and M_1 be disjoint subsets of ω such that $\mathrm{cl}_{\Psi(\Sigma)}M_0\supset\phi^{-1}(0)$ and $\mathrm{cl}_{\Psi(\Sigma)}M_1\supset\phi^{-1}(1)$. There is an $\alpha<\omega_1$ such that $\phi\cap\mathfrak{M}_\alpha\in\mathfrak{M}_\alpha$, $M_0\cap\mathfrak{M}_\alpha\in\mathfrak{M}_\alpha$ and $M_1\cap\mathfrak{M}_\alpha\in\mathfrak{M}_\alpha$. Because of the way we defined $A_{\alpha+1}$ by using Lemma 3.1, we have that $|A_{\alpha+1}\cap M_0|=|A_{\alpha+1}\cap M_1|=\aleph_0$. But this means that we cannot extend ϕ essentially. □

Lemma 3.3. *Let $\mathfrak{M}$ be a countable model and $\mathfrak{N}$ be obtained from $\mathfrak{M}$ by adding one new Cohen real r. Let $\Sigma=\{A_n\colon n\in\omega\}$ be an a.d.f. in $\mathfrak{M}$. Let us define, in $\mathfrak{N}$, the function $\phi(A_n)=r(n)$ for every $n\in\omega$, and let $\widehat{\phi}$ be an essential extension of ϕ. Let $E_0=\widehat{\phi}^{-1}(0)\cap\omega$ and $E_1=\widehat{\phi}^{-1}(1)\cap\omega$. If E is an infinite subset of ω, $E\in\mathfrak{M}$ and $|(\mathrm{cl}_{\Psi(\Sigma)}E)\setminus E|=\aleph_0$ then $E\setminus E_i$ is infinite for $i=0,1$.*

Proof. Let us assume the contrary; say $E\setminus E_0$ is finite. By the assumption of the lemma, $Q=\{n\in\omega\colon |E\cap A_n|\geqslant\aleph_0\}$ is an infinite element of $\mathfrak{M}$. It is known that $|r\cap K|=\aleph_0=|K\setminus r|$ for each infinite set $K\subset\omega$, $K\in\mathfrak{M}$. So, there is an $n\in\omega$ such that $\phi(A_n)=1$ and $A_n\cap E_0$ is infinite; but this is in contradiction with the continuity of $\widehat{\phi}$. □

Proposition 3.4. *By adding $\aleph_1$ new Cohen reals to a model $\mathfrak{M}$, we obtain a model $\mathfrak{N}$ in which for each uncountable a.d.f. Σ there exists a function $\phi:\Sigma\to\{0,1\}$ without an essential extension in $\mathfrak{N}$.*

Proof. Let $\Sigma=\{A_\alpha\colon \alpha\in\omega_1\}$ be an uncountable a.d.f. in $\mathfrak{N}$. There exists a transfinite sequence of models $\mathfrak{M}=\mathfrak{M}_0\subset\cdots\subset\mathfrak{M}_\alpha\subset\cdots\subset\mathfrak{M}_{\omega_1}=\mathfrak{N}$, where $\mathfrak{M}_\alpha$ is the smallest model containing all preceding models if α is a limit ordinal, and $\mathfrak{M}_{\alpha+1}$ is obtained from $\mathfrak{M}_\alpha$ by adding one new Cohen real r_α.

Changing enumeration of models if necessary, we can assume that

$$\{A_\beta\colon \beta<\omega\cdot\alpha+\omega\}\in\mathfrak{M}_\alpha$$

for each $\alpha\in\omega_1$.

Now let us describe a desired function $\phi : \Sigma \to \{0, 1\}$. We put $\phi(A_{\omega\cdot\alpha+n}) = r_\alpha(n)$. This function has no essential extensions in $\mathfrak{N}$. Indeed, suppose $\widehat{\phi} \in \mathfrak{N}$ is an essential extension of ϕ. Then $E_0 = \widehat{\phi}^{-1}(0) \cap \omega$ and $E_1 = \widehat{\phi}^{-1}(1) \cap \omega$ belong to $\mathfrak{N}$. On the other hand, $\mathcal{P}_{\mathfrak{N}}(\omega) = \bigcup\{\mathcal{P}_\alpha(\omega): \alpha \in \omega_1\}$, where $\mathcal{P}_\alpha(\omega) = \mathcal{P}(\omega) \cap \mathcal{M}_\alpha$. So, there exists $\alpha < \omega_1$ such that $E_0, E_1 \in \mathfrak{M}_\alpha$. Let $\phi_\alpha = \phi|_{\{A_{\omega\cdot\alpha+n}:\ n\in\omega\}}$ and

$$\widehat{\phi}_\alpha : \{A_{\omega\cdot\alpha+n}:\ n \in \omega\} \cup \big(\mathrm{dom}(\widehat{\phi}) \cap \omega\big) \to \{0, 1\}$$

be defined by $\widehat{\phi}_\alpha(A_{\omega\cdot\alpha+n}) = \phi_\alpha(A_{\omega\cdot\alpha+n}) = r_\alpha(n)$ for every $n < \omega$, and $\widehat{\phi}_\alpha(n) = \widehat{\phi}(n)$ for every $n \in \mathrm{dom}(\widehat{\phi}) \cap \omega$. Then $\widehat{\phi}_\alpha$ is an essential extension of ϕ_α, and $E_0 = \widehat{\phi}_\alpha^{-1}(0) \cap \omega$ and $E_1 = \widehat{\phi}_\alpha^{-1}(1) \cap \omega$. But this is impossible because of Lemma 3.3. □

Corollary 3.5. *It is consistent with any admissible cardinal arithmetics that for every uncountable a.d.f. Σ there exists a function $\phi : \Sigma \to \{0, 1\}$ having no essential extension.*

Proposition 3.6. *If one adds $\aleph_2$ new Cohen reals to a model $\mathfrak{M}$ in which* **GCH** *is true, then in a resulting model $\mathfrak{N}$, for each uncountable a.d.f. Σ there exists a function $\phi : \Sigma \to \mathbb{R}$ having no essential extension.*

The following lemma plays a crucial role.

Lemma 3.7. *Let Σ be an a.d.f. of cardinality $\mathfrak{c}$ in a model $\mathfrak{M}$. If $\mathfrak{N}$ is a model obtained by adding one new Cohen real to $\mathfrak{M}$, then there exists a function $\phi : \Sigma \to \mathbb{R}$ such that ϕ has no essential extension in $\mathfrak{N}$.*

Proof. Let $\Sigma = \{A_\alpha:\ \alpha < \mathfrak{c}\}$, and let $\mathbb{K}$ be a partially ordered set for adding one Cohen real, $|\mathbb{K}| = \aleph_0$. We construct ϕ by transfinite induction. We may enumerate all the names for functions from ω to $\mathbb{R}$ by using ordinals less than $\mathfrak{c}$: $\{\dot{f}_\alpha:\ \alpha < \mathfrak{c}\}$. Suppose that our function ϕ is already defined on all A_β with $\beta < \alpha$. We are now going to define $\phi(A_\alpha)$.

Let $\mathcal{P} = \{p \in \mathbb{K}:\ p \Vdash \lim_{n\in A_\alpha, n\to\infty} \dot{f}_\alpha(n) = \dot{r}_p\}$. For each $p \in \mathcal{P}$ we can find some $q \leqslant p$ and some segment $[a_p, b_p]$ with rational ends whose length ε_p is as small as we wish so that $q \Vdash \dot{r}_p \in [a_p, b_p]$. As $|\mathcal{P}| = \aleph_0$, we can find $x_\alpha \in \mathbb{R} \setminus \bigcup\{[a_p, b_p]:\ p \in \mathcal{P}\}$. Put $\phi(A_\alpha) = x_\alpha$. □

It is not difficult to prove that

Lemma 3.8. *The function ϕ constructed in Lemma* 3.3 *has no essential extension after adding any Cohen reals.*

Proof of Proposition 3.6. Let $\mathfrak{M}$ be a ground model, and $\mathfrak{N}$ a resulting model obtained after adding $\aleph_2$ new Cohen reals. It is possible to prove that $\mathfrak{c} = \aleph_2$ in $\mathfrak{N}$ (see, for example, Lemma 5.14 in [7]). If $\Sigma \in \mathfrak{N}$ is an a.d.f. and $|\Sigma| = \aleph_2 = \mathfrak{c}$, then, according to Proposition 2.3, on this Σ there exists a function ϕ without essential extensions. Let $\Sigma \in \mathfrak{N}$ and $|\Sigma| = \aleph_1$. Then there exists an intermediate model $\mathfrak{C}$ with $\Sigma \in \mathfrak{C}$ and

$|\Sigma| = 2^{\aleph_0}$ in $\mathfrak{C}$. Now we may apply Lemmas 3.7 and 3.8 and define a function ϕ which has no essential extension in $\mathfrak{N}$. □

Corollary 3.9. *It is impossible to construct in* **ZFC** *or just in* **ZFC** + $\neg$**CH** *an uncountable a.d.f. Σ such that every function $\phi:\Sigma \to \mathbb{R}$ has an essential extension.*

In [6] it was noted that $C_p(\Psi(\Sigma))$ is not a normal space if Σ is an a.d.f. of cardinality $\aleph_1$ such that

(1) for each countable subset Σ' of Σ, every function $f:\Sigma' \to \mathbb{R}$ can be extended to a continuous function $g:\Psi(\Sigma) \to \mathbb{R}$ such that for an $r \in \mathbb{R}$, $|\{x \in \Psi(\Sigma)\colon\ g(x) \neq r\}| \leqslant \aleph_0$ (that is, g is eventually constant); and

(2) every $g \in C_p(\Psi(\Sigma))$ is eventually constant.

This is the case when $\Sigma = \Sigma(\mathcal{F})$ where $\mathcal{F}$ is an ω_1-p-ultrafilter, and also when Σ satisfies the conditions

($*$) every partition of Σ is a Luzin gap, and

($**$) every countable subset of Σ can be separated from its complement.

In [6] it was proved that there exists an a.d.f. Σ satisfying ($*$) and ($**$), and van Douwen [3] proved that it is consistent with **ZFC** that every a.d.f. with ($*$) also satisfies ($**$).

Remark 3.10. The Mrówka–Isbell space $\Psi(\Sigma)$ constructed in Proposition 3.2 has very interesting properties, namely:

(1) For each $f \in C_p(\Psi(\Sigma), \{0,1\})$, either $f^{-1}(0)$ or $f^{-1}(1)$ is finite; and only functions $\phi:\Sigma \to \{0,1\}$ with a finite fiber can be continuously extended to all $\Psi(\Sigma)$.

(2) If $\widehat{\phi}:\Psi(\Sigma) \to \mathbb{R}$ is continuous and $\phi = \widehat{\phi}|_{\Sigma}$, then there exists $r \in \mathbb{R}$ such that, either

 (a) $|\Sigma \setminus \phi^{-1}(r)| < \aleph_0$, or

 (b) $|\Sigma \setminus \phi^{-1}(r)| = \aleph_0$, $\operatorname{rng}(\phi) = \{r\} \cup \{r_n \in \mathbb{R}\colon\ n \in \mathbb{N}\}$, $(r_n)_{n\in\mathbb{N}}$ converges to r and $|\phi^{-1}(r_n)| < \aleph_0$ for each $n \in \mathbb{N}$. Besides,

(3) If $\phi:\Sigma \to \mathbb{R}$ is such that there is $r \in \mathbb{R}$ satisfying (a) or (b), then ϕ has a full extension $\widehat{\phi}:\Psi(\Sigma) \to \mathbb{R}$.

Problem 3.11. Let $\Psi(\Sigma)$ be the Mrówka–Isbell space constructed in Proposition 3.2. Is $C_p(\Psi(\Sigma))$ a normal space?

4. Essential extensions under CH

In this section we will prove that under **CH** we can also obtain an example of an a.d.f. Σ such that none of its onto $\{0,1\}$-valued functions with infinite fibers have an essential extension.

Proposition 4.1. [CH] *There exists an uncountable a.d.f. Σ on ω such that no function $\phi:\Sigma \to \{0,1\}$ for which $|\phi^{-1}(0)| \geqslant \aleph_0$ and $|\phi^{-1}(1)| \geqslant \aleph_0$, has an essential extension.*

Proof. We will construct $\Sigma = \{A_\beta\colon \beta < \omega_1\}$ by transfinite induction. Let $\{J_\alpha\colon \alpha < \omega_1\}$ be a collection of subsets of ω_1 such that $J_n = \emptyset$ for all $n < \omega$ and $\{J_\alpha\colon \omega \leqslant \alpha < \omega_1\}$ is a partition of ω_1 consisting of uncountable disjoint subsets. Let $\{A'_1, \dots, A'_n, \dots\}$ be a countable a.d.f. on ω. We are going to define the following sets for every $\beta < \omega_1$:

(1) $A_\beta \subset \omega$,

(2) a collection T_β, each of its elements being a two-element set consisting of disjoint subsets of ω,

(3) two infinite and disjoint subsets P_β and Q_β of ω,

(4) an ordinal $\omega \leqslant \alpha_\beta < \omega_1$, and

(5) two infinite and disjoint subsets a_0^β and a_1^β of β,

such that

(i) for every $n < \omega$, $A_n = A'_n$, $\alpha_n = 0$ and $T_n = P_n = Q_n = a_0^n = a_1^n = \emptyset$;

(ii) the collection $\{A_\beta\colon \beta < \omega_1\}$ is an a.d.f.;
and for every $\omega \leqslant \beta < \omega_1$:

(iii) $T_\beta = \{\{P, Q\}\colon P, Q \subset \omega, P \cap Q = \emptyset$ and

$$\min\Big\{\big|\{\lambda < \beta\colon |P \cap A_\lambda| = \aleph_0\}\big|, \big|\{\lambda < \beta\colon |Q \cap A_\lambda| = \aleph_0\}\big|\Big\} = \aleph_0;$$

(iv) $|\{\lambda < \beta\colon |P_\beta \cap A_\lambda| = \aleph_0\}| = |\{\lambda < \beta\colon |Q_\beta \cap A_\lambda| = \aleph_0\}| = \aleph_0$;

(v) either $\alpha_\beta \in [\omega, \omega_1) \setminus \{\alpha_\lambda\colon \lambda < \beta\}$ if $\beta \in \bigcup_{\lambda<\beta} J_{\alpha_\lambda}$ or α_β is such that $\beta \in J_{\alpha_\beta}$ if $\beta \notin \bigcup_{\lambda<\beta} J_{\alpha_\lambda}$;

(vi) $a_0^\beta = \{\lambda < \beta\colon |P_\beta \cap A_\lambda| = \aleph_0\}$, $a_1^\beta = \{\lambda < \beta\colon |Q_\beta \cap A_\lambda| = \aleph_0\}$; and

(vii) $|A_\beta \cap \bigcup_{\lambda \in a_0^\beta}(P_\beta \cap A_\lambda)| = \aleph_0$ and $|A_\beta \cap \bigcup_{\lambda \in a_1^\beta}(Q_\beta \cap A_\lambda)| = \aleph_0$.

Suppose we have already carried out everything we want to for every $\beta < \kappa < \omega_1$ with $\omega \leqslant \kappa$. Now, we are going to define all we need for the κ-step: Let α_κ be an ordinal $< \omega_1$ such that either $\kappa \in J_{\alpha_\kappa}$ if $\kappa \notin \bigcup_{\beta<\kappa} J_{\alpha_\beta}$, or α_κ is an ordinal different from any α_β ($\beta < \kappa$) if $\kappa \in \bigcup_{\beta<\kappa} J_{\alpha_\beta}$. Let

$$T_\kappa = \Big\{\{P, Q\}\colon\ P, Q \subset \omega,\ P \cap Q = \emptyset,\ \big|\{\lambda < \kappa\colon |P \cap A_\lambda| = \aleph_0\}\big| = \aleph_0 \text{ and } \big|\{\lambda < \kappa\colon |Q \cap A_\lambda| = \aleph_0\}\big| = \aleph_0\Big\}.$$

Observe that $|T_\kappa| = \mathfrak{c} = \aleph_1$. So, we can make an enumeration of the elements of T_κ by using a one-to-one and onto correspondence with the elements of J_{α_κ}: $T_\kappa = \{\{P_\lambda, Q_\lambda\}\colon \lambda \in J_{\alpha_\kappa}\}$; so, $\{P_\kappa, Q_\kappa\} \in \bigcup_{\lambda \leqslant \kappa} T_\lambda$. We set $a_0^\kappa = \{\lambda < \kappa\colon |P_\kappa \cap A_\lambda| = \aleph_0\}$ and $a_1^\kappa = \{\lambda < \kappa\colon |Q_\kappa \cap A_\lambda| = \aleph_0\}$; these are infinite subsets of κ, and we enumerate them as $a_0^\kappa = \{\lambda_n\colon n < \omega\}$, $a_1^\kappa = \{\xi_n\colon n < \omega\}$.

We are going to construct the set A_κ by induction. Take

$$x_0 \in P_\kappa \cap A_{\lambda_0}, \qquad y_0 \in (Q_\kappa \cap A_{\xi_0}) \setminus A_{\lambda_0},$$

and if we have already taken different elements $x_0, \dots, x_n$ and $y_0, \dots, y_n$, we can choose, since $\{A_\lambda\colon \lambda < \kappa\}$ is an a.d.f. and $P_\kappa \cap A_{\lambda_{n+1}}$, $Q_\kappa \cap A_{\xi_{n+1}}$ are infinite sets,

$$x_{n+1} \in (P_\kappa \cap A_{\lambda_{n+1}}) \setminus \Big(\bigcup_{i<n+1} A_{\lambda_i} \cup \bigcup_{i<n+1} A_{\xi_i}\Big) \quad \text{and}$$

$$y_{n+1} \in (Q_\kappa \cap A_{\xi_{n+1}}) \setminus \Big(\bigcup_{i \leqslant n+1} A_{\lambda_i} \cup \bigcup_{i<n+1} A_{\xi_i} \Big).$$

Let $A_\kappa = \{x_n\colon n < \omega\} \cup \{y_n\colon n < \omega\}$.

It happens that $\{A_\lambda\colon \lambda \leqslant \kappa\}$ is an a.d.f.,

$$\Big| A_\kappa \cap \bigcup_{\lambda \in a_0^\kappa} (P_\kappa \cap A_\lambda) \Big| = \aleph_0 \quad \text{and} \quad \Big| A_\kappa \cap \bigcup_{\lambda \in a_1^\kappa} (Q_\kappa \cap A_\lambda) \Big| = \aleph_0.$$

In this way, we can obtain all the sets listed in (1)–(5) with properties (i)–(vii).

Now we are going to prove that $\Sigma = \{A_\lambda\colon \lambda < \omega_1\}$ satisfies the requirement. Let $\phi\colon \Sigma \to \{0,1\}$ be a function where $\phi^{-1}(0)$ and $\phi^{-1}(1)$ are infinite subsets of Σ. Let $N \subset \omega$ be such that $\mathrm{cl}_{\Psi(\Sigma)} N \supset \Sigma$, and let $\widehat{\phi}\colon \Sigma \cup N \to \{0,1\}$ be a function that extends ϕ. We will prove that $\widehat{\phi}$ is not a continuous function. Take $M_0 = \widehat{\phi}^{-1}(0) \cap N$ and $M_1 = \widehat{\phi}^{-1}(1) \cap N$. If for an $i \in \{0,1\}$ there exists $A_\lambda \in \phi^{-1}(i)$ with $|M_i \cap A_\lambda| < \aleph_0$, then $\widehat{\phi}$ is not continuous at the point A_λ. Now assume the contrary: $|M_i \cap A_\lambda| = \aleph_0$ for all $A_\lambda \in \phi^{-1}(i)$ and $i \in \{0,1\}$. Let $Z_i = \{\lambda < \omega_1\colon A_\lambda \in \phi^{-1}(i)\}$ and let Z_i' be a countable and infinite subset of Z_i for $i \in \{0,1\}$. If $\kappa_0 = \sup(Z_0' \cup Z_1')$, then $\{M_0, M_1\}$ belongs to T_{κ_0}; so there exists $\gamma_0 \in J_{\alpha_{\kappa_0}}$ such that $\{M_0, M_1\} = \{P_{\gamma_0}, Q_{\gamma_0}\}$. Thus $|A_{\gamma_0} \cap M_0| = \aleph_0 = |A_{\gamma_0} \cap M_1|$ because of property (vii). But this means that $\widehat{\phi}$ is not continuous at the point A_{γ_0}. □

5. Essential extensions and Martin's Axiom

In contrast with the previous sections we will now see that the scenery changes when Martin's Axiom is assumed.

The following assertion, which is called Booth's Lemma (BL), is a consequence of Martin's Axiom (**MA**) and was first proved by D. Booth in [2]:

BL. *If Σ is a family of subsets of ω with $|\Sigma| < 2^{\aleph_0}$ and $|\bigcap \Sigma'| = \aleph_0$ for every finite subfamily Σ' of Σ, then there exists an infinite subset B of ω such that $|B \setminus A| < \aleph_0$ for every $A \in \Sigma$.*

This combinatorial principle (also denoted by $P(\mathfrak{c})$ or $\mathfrak{p} = \mathfrak{c}$) is equivalent to a strict weakening of **MA**, the so-called Martin's Axiom for σ-centered partially ordered sets (see [1,8]).

BL has significant and interesting consequences (see for example [1,10]); among them is the following statement: *every m.a.d. family on ω has cardinality $\mathfrak{c}$.*

It is very possible that the following result belongs to the set-theoretical folklore (see, e.g., [11]).

Lemma 5.1. BL *is equivalent to the following statement:*

($*$) *If $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(\omega)$ are families of cardinality less than $2^{\aleph_0}$, and for all finite $\mathcal{C} \subset \mathcal{A}$ and $B \in \mathcal{B}$, $|B \setminus \bigcup \mathcal{C}| = \aleph_0$, then there is an $M \subset \omega$ such that $|B \setminus M| = \aleph_0$ for all $B \in \mathcal{B}$ and $|A \setminus M| < \aleph_0$ for all $A \in \mathcal{A}$.*

Proof. Theorem 7 of [13] guarantees that **MA** implies $(*)$. In order to obtain **BL** $\Rightarrow (*)$, we can use the same argument in proof of the cited theorem because the partial order $\mathbb{P}$ used there is σ-centered (see [1]).

The statement $(*) \Rightarrow$ **BL** is Corollary 8 in [13]. $\square$

Proposition 5.2. [BL] *Let Σ be an a.d.f. of cardinality $< \mathfrak{c}$. Then, every function $\phi : \Sigma \to \{0,1\}$ has an essential extension.*

Proof. Let $\phi : \Sigma \to \{0,1\}$ be a function and set $\Sigma_i = \phi^{-1}(i)$ $(i = 0,1)$. Because of Lemma 5.1, there are two sets M_0, $M_1 \subset \omega$ such that $|A \setminus M_0| = \aleph_0$ and $|A \setminus M_1| < \aleph_0$ for all $A \in \Sigma_0$ and $|B \setminus M_0| < \aleph_0$ and $|B \setminus M_1| = \aleph_0$ for all $B \in \Sigma_1$. Define $S' = \omega \setminus M_0$ and $T' = \omega \setminus M_1$. Then, for $A \in \Sigma_0$, $A \cap S' = A \setminus (\omega \setminus S') = A \setminus M_0$ is infinite. Similarly, for all $A \in \Sigma_0$, and for all $B \in \Sigma_1$, $T' \cap B$ is infinite and $S' \cap B$, $T' \cap A$ are finite. The sets $S = S' \setminus T'$ and $T = T'$ are disjoint and, for each $A \in \Sigma_0$ and each $B \in \Sigma_1$, $A \cap S$, $B \cap T$ are infinite and $A \cap T$, $B \cap S$ are finite. We define $\widehat{\phi}|_\Sigma = \phi$, $\widehat{\phi}(s) = 0$ for every $s \in S$ and $\widehat{\phi}(t) = 1$ for every $t \in T$. The function $\widehat{\phi}$ is an essential extension of ϕ. $\square$

The following definition and theorem appear in [14, p. 55]; we include their formulations for the sake of completeness.

Definition 5.3. An a.d.f. $(A_\lambda)_{\lambda<\alpha}$ is called *a tree* if for each $\beta < \alpha$ and every $\gamma, \xi < \alpha$, either $A_\beta \cap A_\gamma \subset A_\beta \cap A_\xi$ or $A_\beta \cap A_\xi \subset A_\beta \cap A_\gamma$.

Proposition 5.4. [MA] *If Σ is an a.d.f. of cardinality $< 2^{\aleph_0}$ which is a tree, then every function $\phi : \Sigma \to \{0,1\}$ has a full extension.*

For arbitrary a.d.f. of cardinality $< 2^{\aleph_0}$, **MA** also implies the real-valued essential version of Proposition 5.2:

Proposition 5.5. [MA] *Let Σ be an a.d.f. of cardinality $< 2^{\aleph_0}$. Then every function $\phi : \Sigma \to \mathbb{R}$ has an essential extension.*

Proof. Let $\mathbb{P}$ be the set of all pairs $p = \langle f_p, \mathcal{F}_p \rangle$, where f_p and $\mathcal{F}_p$ are functions satisfying the following conditions:

(1) $\mathrm{dom}(f_p) \subset \omega$ and $\mathrm{rng}(f_p) \subset \mathbb{Q}$;

(2) $|\mathrm{dom}(f_p) \cup \mathrm{dom}(\mathcal{F}_p)| < \aleph_0$;

(3) $\mathrm{dom}(\mathcal{F}_p) \subset \Sigma$, and $\mathcal{F}_p(A) = \langle u^p_A, r^p_A \rangle \in \omega \times \mathbb{Q}^+$ for each $A \in \mathrm{dom}(\mathcal{F}_p)$; and

(4) if $A \in \mathrm{dom}(\mathcal{F}_p)$ and $k \in (A \setminus u^p_A) \cap \mathrm{dom}(f_p)$, then

$$\left| f_p(k) - \phi(A) \right| < r^p_A.$$

($\mathbb{Q}$ and $\mathbb{Q}^+$ denote the sets of rational numbers and of positive rational numbers, respectively.)

We define in $\mathbb{P}$ the following partial order: $p \leqslant q$ iff the following conditions hold:

(i) $f_p \supset f_q$;

(ii) $\mathrm{dom}(\mathcal{F}_p) \supset \mathrm{dom}(\mathcal{F}_q)$;
and for each $A \in \mathrm{dom}(\mathcal{F}_q)$,
(iii) $u_A^p \geqslant u_A^q$, $r_A^p \leqslant r_A^q$;
(iv) $[\mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q)] \cap A \subset A \setminus u_A^q$;
(v) for each $k \in (A \setminus u_A^q) \cap \mathrm{dom}(f_p)$, $|f_p(k) - \phi(A)| < r_A^q$.

The pair $(\mathbb{P}, \leqslant)$ is a partially ordered set. Indeed, let us verify that the relation $\leqslant$ is transitive. Suppose that $p, q, s \in \mathbb{P}$ are such that $p \leqslant s$ and $s \leqslant q$. We are going to prove that $p \leqslant q$. It is easy to see that the conditions (i)–(iii) hold. We now prove the statement (iv). Let us suppose that $A \in \mathrm{dom}(\mathcal{F}_q)$ and $n_0 \in [\mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q)] \cap A$. We need to prove $n_0 \geqslant u_A^q$. But

$$\mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q) = [\mathrm{dom}(f_p) \setminus \mathrm{dom}(f_s)] \cup [\mathrm{dom}(f_s) \setminus \mathrm{dom}(f_q)].$$

If $n_0 \in \mathrm{dom}(f_p) \setminus \mathrm{dom}(f_s)$, then $n_0 \geqslant u_A^s \geqslant u_A^q$, and if $n_0 \in \mathrm{dom}(f_s) \setminus \mathrm{dom}(f_q)$, then $n_0 \geqslant u_A^q$ too.

It remains to prove that p and q satisfy (v). Let $A \in \mathrm{dom}(\mathcal{F}_q)$ and $k \in (A \setminus u_A^q) \cap \mathrm{dom}(f_p)$. Since $s \leqslant q$, if $k \in \mathrm{dom}(f_s)$ then

$$|f_s(k) - \phi(A)| = |f_p(k) - \phi(A)| < r_A^q.$$

If $k \notin \mathrm{dom}(f_s)$, then $k \in [\mathrm{dom}(f_p) \setminus \mathrm{dom}(f_s)] \cap A \subset A \setminus u_A^s$, so $|f_p(k) - \phi(A)| < r_A^s \leqslant r_A^q$. It is easy to verify that $\leqslant$ is reflexive and antisymmetric.

Now let us check that $c(\mathbb{P}) \leqslant \aleph_0$. Let $\mathcal{E}$ be a subset of $\mathbb{P}$ of cardinality $\aleph_1$. We can find an uncountable subset $\mathcal{E}_1 \subset \mathcal{E}$ such that $f_p = f_q$ for every $p, q \in \mathcal{E}_1$. Moreover, we can find a finite subset $\Delta \subset \Sigma$ and an uncountable $\mathcal{E}_2 \subset \mathcal{E}_1$ such that $\mathrm{dom}(\mathcal{F}_p) \cap \mathrm{dom}(\mathcal{F}_q) = \Delta$ for all different $p, q \in \mathcal{E}_2$ [5, p. 87]. As the range of all possible values of every $\mathcal{F}(A)$ is countable, there exists an uncountable $\mathcal{E}_3 \subset \mathcal{E}_2$ such that $\mathcal{F}_p(A) = \mathcal{F}_q(A)$ for each $A \in \Delta$ and all $p, q \in \mathcal{E}_3$. If $p, q \in \mathcal{E}_3$, then they are compatible: in fact, $\langle f_p \cup f_q, \mathcal{F}_p \cup \mathcal{F}_q \rangle$ extends p and q.

Now we will define a convenient system $\mathfrak{D}$ of dense subsets of $\mathbb{P}$.

Claim 1. *For each $(A, n, r) \in \Sigma \times \omega \times \mathbb{Q}^+$ the set*

$$\mathcal{D}(A, n, r) = \{p = \langle f, \mathcal{F} \rangle \in \mathbb{P} \colon\ A \in \mathrm{dom}(\mathcal{F}),\ \mathcal{F}(A) = \langle m, t \rangle \text{ and } m \geqslant n,\ t \leqslant r\}$$

is dense.

Indeed, let $\langle g, \mathcal{G} \rangle \in \mathbb{P} \setminus \mathcal{D}(A, n, r)$, and let $\mathcal{F} \colon \mathrm{dom}(\mathcal{G}) \cup \{A\} \to \omega \times \mathbb{Q}^+$ defined by the conditions:
- $\mathcal{F}(G) = \mathcal{G}(G)$ if $G \in \mathrm{dom}(\mathcal{G}) \setminus \{A\}$,
- $\mathcal{F}(A) = \langle u_0, r_0 \rangle$, where $u_0 = \max\{\mathrm{dom}(g) \cup \{m, n\}\} + 1$ and $r_0 = \min\{t, r\}$, if $A \in \mathrm{dom}(\mathcal{G})$ and $\mathcal{G}(A) = \langle m, t \rangle$, and
- $\mathcal{F}(A) = \langle \max\{\mathrm{dom}(g) \cup \{n\}\} + 1, r \rangle$ if $A \notin \mathrm{dom}(\mathcal{G})$.

It is easy to see that $\langle g, \mathcal{F}\rangle$ satisfies conditions (1)–(3) above; and since $\langle g, \mathcal{G}\rangle \in \mathbb{P}$ and $(A \setminus u_0) \cap \operatorname{dom}(g) = \emptyset$, $\langle g, \mathcal{F}\rangle$ satisfies (4) too; so $\langle g, \mathcal{F}\rangle \in \mathbb{P}$. Besides, it can be proved without difficulty that $\langle g, \mathcal{F}\rangle \in \mathcal{D}(A, n, r)$ and $\langle g, \mathcal{F}\rangle \leqslant \langle g, \mathcal{G}\rangle$.

Claim 2. *For each* $(A, m) \in \Sigma \times \omega$, *the set*

$$\mathcal{H}(A, m) = \{p = \langle f, \mathcal{F}\rangle\colon\ A \in \operatorname{dom}(\mathcal{F}) \text{ and } \operatorname{dom}(f) \cap (A \setminus m) \neq \emptyset\}$$

is dense.

Indeed, let $q = \langle g, \mathcal{G}\rangle \in \mathbb{P} \setminus \mathcal{H}(A, m)$. Since Σ is an a.d.f. of infinite subsets of ω, we can take $t \in A \setminus \bigcup\{G\colon\ G \in \operatorname{dom}(\mathcal{G}) \setminus \{A\}\}$ such that

$$t > \max\{m, \max\{u_G^q\colon\ G \in \operatorname{dom}(\mathcal{G})\}\}.$$

Let $f : \operatorname{dom}(g) \cup \{t\} \to \mathbb{Q}$ defined by $f(n) = g(n)$ if $n \in \operatorname{dom}(g)$ and $f(t) = \phi(A)$. On the other hand, if $A \in \operatorname{dom}(\mathcal{G})$, let $\mathcal{F} = \mathcal{G}$, and, if $A \notin \operatorname{dom}(\mathcal{G})$, let $\mathcal{F} : \operatorname{dom}(\mathcal{G}) \cup \{A\} \to \mathbb{Q}^+$ be defined by $\mathcal{F}(G) = \mathcal{G}(G)$ for all $G \in \operatorname{dom}(\mathcal{G})$ and $\mathcal{F}(A) = \langle u_A^p, r_A^p\rangle$ where $u_A^p > \max\{\operatorname{dom}(g)\}$ and $r_A^p \in \mathbb{Q}^+$. It can be proved that $\langle f, \mathcal{F}\rangle \in \mathcal{H}(A, m)$ and $\langle f, \mathcal{F}\rangle \leqslant \langle g, \mathcal{G}\rangle$.

So, the collection

$$\mathfrak{D} = \{\mathcal{D}(A, n, r)\colon\ A \in \Sigma,\ n \in \omega,\ r \in \mathbb{Q}^+\} \cup \{\mathcal{H}(A, m)\colon\ A \in \Sigma,\ m \in \omega\}$$

is a system of dense subsets of $\mathbb{P}$. Since $|\mathcal{D}| < 2^{\aleph_0}$, and using **MA**, there exists a $\mathfrak{D}$-generic $\boldsymbol{G} \subset \mathbb{P}$. Let

$$\psi = \bigcup\{f\colon\ \exists p \in \boldsymbol{G}(p = \langle f, \mathcal{F}\rangle)\} \quad \text{and}$$
$$M = \bigcup\{\operatorname{dom}(f)\colon\ \exists p \in \boldsymbol{G}(p = \langle f, \mathcal{F}\rangle)\}.$$

Since $\boldsymbol{G}$ is $\mathfrak{D}$-generic, we have

Claim 3. *For each* $(A, k, \varepsilon) \in \Sigma \times \omega \times \mathbb{Q}^+$ *there exist* $m(A, k, \varepsilon) \in (M \cap A) \setminus k$ *and* $\langle f, \mathcal{F}\rangle = p \in \boldsymbol{G}$ *such that* $m(A, k, \varepsilon) \in \operatorname{dom}(f)$ *and* $|f(m(A, k, \varepsilon)) - \phi(A)| < \varepsilon$.

Indeed, let $q = \langle f_q, \mathcal{F}_q\rangle \in \boldsymbol{G} \cap \mathcal{D}(A, k, \varepsilon)$. Hence, $A \in \operatorname{dom}(\mathcal{F}_q)$ and $\mathcal{F}_q(A) = \langle m_0, t_0\rangle$ with $m_0 \geqslant k$ and $t_0 \leqslant \varepsilon$. Let $s = \langle f_s, \mathcal{F}_s\rangle \in \boldsymbol{G} \cap \mathcal{H}(A, m_0)$ and let $p = \langle f, \mathcal{F}\rangle \in \boldsymbol{G}$ such that $p \leqslant q$ and $p \leqslant s$. By the assumption, $E = \operatorname{dom}(f_s) \cap (A \setminus m_0)$ is not empty. An element $m(A, k, \varepsilon)$ in E satisfies the above requirements.

For each $A \in \Sigma$ let $n(A, 0) = 0$ and $n(A, i+1) = m(A, n(A, i), 1/(i+1))$, and take $N = \{n(A, i)\colon\ A \in \Sigma, i \in \omega\}$. The function $\widehat{\phi} : \Sigma \cup N \to \mathbb{R}$ defined by $\widehat{\phi}|_\Sigma = \phi$ and $\widehat{\phi}(n) = \psi(n)$ for each $n \in N$ is an essential extension of ϕ. $\square$

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