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EXAMPLE OF A T_1 TOPOLOGICAL SPACE WITHOUT A NOETHERIAN BASE

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ABSTRACT. A Noetherian base $\mathscr{B}$ of a topological space X is a base for the topology of X which has the following property: If $B_1 \subset B_2 \subset \cdots$ is a nondecreasing sequence of elements of $\mathscr{B}$, then $\{B_n\}_{n\in\mathbf{N}}$ is finite. In this article we give an example of a T_1 topological space without a Noetherian base.

I. Introduction.

DEFINITION 1.1. A collection $\mathscr{C}$ of subsets of a set X is *Noetherian* if $\mathscr{C}$ does not contain a strictly increasing infinite chain.

There are large classes of topological spaces which have a Noetherian base (see [**3**]), for example if X is a normed linear space, the collection of open balls of radius $1/n$ ($n \in \mathbf{N}$) constitutes a Noetherian base of X. On the other hand, $\mathbf{R}$ with the topology $\tau = \{\varnothing, \mathbf{R}\} \cup \{(a,\infty): a \in \mathbf{R}\}$ is a non T_1-space with no Noetherian base.

An important unsolved problem is the following:

Does Con(ZFC) imply that Con (ZFC+ there exists a T_2-space without a Noetherian base)?

However, the following is known:

THEOREM 1.2[**1** and **4**]. *Let α be an ordinal. The space α has a Noetherian base if and only if $\alpha + 1$ does not contain a strongly inaccessible cardinal.*

In the section that follows we give an example, in ZFC, of a T_1-space that has no Noetherian base.

II. A T_1 topological space with no Noetherian base.

DEFINITION 2.1. A topological space X is *Noetherianly refinable* or in abbreviated notation, *N-refinable*, if each open covering has a Noetherian open refinement.

It is easy to see that if X has a Noetherian base then it is N-refinable and that X is N-refinable if and only if each open cover has a refinement which is an antichain of open sets.

LEMMA 2.2 [**2**]. *Let α be an uncountable regular cardinal. Let $E \subset \alpha$ be a stationary subset of α and let $\phi: E \to \alpha$ be a regressive function. Then, there is $\xi < \alpha$ such that $|\phi^{-1}(\xi)| = \alpha$.*

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For each $\lambda \leq \omega_1$, let $\mathscr{B}_\lambda = \{A \subset \lambda \colon |\lambda - A| < \aleph_0\}$. If $B_1, B_2 \in \mathscr{B} = \bigcup_{\lambda \leq \omega_1} \mathscr{B}_\lambda$, then $B_1 \cap B_2 \in \mathscr{B}$. Therefore $\mathscr{B}$ is a base for a topology τ in ω_1.

REMARK 2.3. $A \in \tau - \mathscr{B}$ if and only if $A = \lambda - C$, where $\lambda < \omega_1$ and C is a cofinal subset in λ of order type ω (o.t. $C = \omega$).

THEOREM 2.4. *(ω_1, τ) is a T_1-space which is not N-refinable (and therefore, (ω_1, τ) does not have a Noetherian base).*

PROOF. Let us suppose that $\mathscr{A} \subset \tau$ is a refinement of $\mathscr{C} = \{\lambda + 1 \colon \lambda \in \omega_1\}$. Let $\lambda_0 = 0$ and let $A_0 \in \mathscr{A}$ be such that $\lambda_0 \in A_0$. Then, there is $\lambda_1 \in \omega_1$ such that $A_0 = \lambda_1 - C_1$, where C_1 is either finite or is an infinite cofinal subset of λ_1 of order type ω (see 2.3). Let $A'_0 = A_0 \cup \{\eta \in C_1 \colon \eta > \lambda_0\}$. Let $A_1 \in \mathscr{A}$ be such that $\lambda_1 \in A_1$. There is $\lambda_2 \in \omega_1$ such that $A_1 = \lambda_2 - C_2$, where C_2 is finite or is an infinite cofinal subset of λ_2 of order type ω. Let $A'_1 = A_1 \cup \{\eta \in C_2 \colon \eta > \lambda_1\}$.

Let us suppose that for some $\gamma < \omega_1$, we have chosen the collections: $\{\lambda_\beta\}_{\beta<\gamma} \subset \omega_1, \{A_\beta\}_{\beta<\gamma} \subset \mathscr{A}$ and $\{A'_\beta\}_{\beta<\gamma}$, such that $\lambda_\beta \in A_\beta = \lambda_{\beta+1} - C_{\beta+1}$, where $C_{\beta+1}$ is either finite or is an infinite cofinal subset of $\lambda_{\beta+1}$ of order type ω. Moreover, for each $\beta < \gamma$, $A'_\beta = A_\beta \cup \{\eta \in C_{\beta+1} \colon \eta > \lambda_\beta\}$.

We construct, inductively, $\lambda_\gamma \in \omega_1$, $A_\gamma \in \mathscr{A}$ and A'_γ:

If γ is a nonlimit ordinal and $\gamma - 1$ is the immediate predecessor of γ, then there exist $\lambda_\gamma < \omega_1$ such that $A_{\gamma-1} = \lambda_\gamma - C_\gamma$. If γ is a limit ordinal, let $\lambda_\gamma = \sup\{\lambda_\beta \colon \beta < \gamma\}$. In both cases, let $A_\gamma \in \mathscr{A}$ such that $\lambda_\gamma \in A_\gamma$. There is $\lambda_{\gamma+1} \in \omega_1$ such that $A_\gamma = \lambda_{\gamma+1} - C_{\gamma+1}$ where $C_{\gamma+1}$ is either finite or is an infinite cofinal subset of $\lambda_{\gamma+1}$ of order type ω (see 2.3). Let $A'_\gamma = A_\gamma \cup \{\eta \in C_{\gamma+1} \colon \eta > \lambda_\gamma\}$.

By the inductive construction, $\{\lambda_\beta\}_{\beta<\omega_1}$ is cofinal in ω_1.

Let $\mathscr{A}' = \{A'_\beta \colon \beta < \omega_1\}$. It is easy to see that each A'_β is an open set. In fact, $A'_\beta \in \mathscr{B}$ for each $\beta < \omega_1$.

We claim that:

(1) If $\mathscr{A}$ is an antichain, then $\mathscr{A}'$ is also an antichain.

In fact, let $A'_\gamma, A'_\beta \in \mathscr{A}'$ where $\gamma < \beta$. $A'_\gamma = A_\gamma \cup \{\eta \in C_{\gamma+1} \colon \eta > \lambda_\gamma\}$ and $A'_\beta = A_\beta \cup \{\eta \in C_{\beta+1} \colon \eta > \lambda_\beta\}$. A'_γ does not contain A'_β since $\lambda_\beta \in A'_\beta - A'_\gamma$. On the other hand, if $\eta_0 \in A_\gamma - A_\beta$, then $\eta_0 \in A'_\gamma - A'_\beta$. Therefore $\mathscr{A}'$ is an antichain.

(2) Let $E' = \bigcup_{\gamma<\omega_1} A'_\gamma$ and let $G = \omega_1 - E'$. Then, the set G is empty or has order type $\leq \omega$. Furthermore $E = \{\alpha \in E' \colon \alpha \text{ is a limit ordinal}\}$ is a stationary subset of ω_1.

In fact, let us suppose that G is a subset of ω_1 such that o.t. $G > \omega$. Let $\eta_0 \in G$ be such that o.t. $\{\eta \in G \colon \eta < \eta_0\} > \omega$. Since $\{\lambda_\gamma\}_{\gamma<\omega_1}$ is a cofinal subset in ω_1, then, there is λ_ξ such that $\eta_0 < \lambda_\xi$. But $\lambda_\xi \in A'_\xi = \lambda_{\xi+1} - C'_{\xi+1}$, where $C'_{\xi+1} = \{\eta \in C_{\xi+1} \colon \eta < \lambda_\xi\}$ and o.t. $C'_{\xi+1} \leq \omega$. Therefore $A'_\xi \cap G \neq \varnothing$. This contradiction proves that o.t. $G \leq \omega$. As an immediate consequence the set $E = \{\alpha \in E' \colon \alpha \text{ is a limit ordinal}\}$ is a stationary subset of ω_1.

For each $\eta \in E$, let $g(\eta)$ be the smallest γ such that $\gamma \in A'_\gamma = \lambda_{\gamma+1} - C'_{\gamma+1}$. If $T_\eta = \{\xi < \omega_1 \colon \lambda_\xi \leq \eta\}$, then $g(\eta) = \sup T_\eta$ and therefore $\lambda_{g(\eta)} \leq \eta$. Since η is a limit ordinal, $\lambda_{g(\eta)} \leq \eta$ and $\eta \in A'_{g(\eta)} = \lambda_{g(\eta)+1} - C'_{g(\eta)+1}$ (where $C'_{g(\eta)+1} \subset \lambda_{g(\eta)}$ is a finite set) there is $a_\eta < \eta$ such that $C'_{g(\eta)+1} \subset a_\eta$. The function $\phi(\eta) = a_\eta$ is a regressive function. Since E is a stationary subset in ω_1, there is $\xi < \omega_1$ such that $|\phi^{-1}(\eta)| = \omega_1$ (Lemma 2.2). Let $M = \phi^{-1}(\xi)$. Since $|M| = \omega_1$ and

$|\xi| = \omega$, there exist an infinite subset K of M and a finite subset $C \subset \xi$, such that $A'_{g(k)} = \lambda_{g(k)+1} - C$ for each $k \in K$. Therefore $\{A'_{g(k)} : k \in K\}$ is an infinite strictly increasing chain of elements of $\mathscr{A}'$. It follows from (1) that $\mathscr{A}$ is not an antichain, that is, (ω_1, τ) is not N-refinable.

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