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ON THE ČECH NUMBER OF $C_p(X)$, II

Ofelia T. Alas and Ángel Tamariz-Mascarúa

Abstract. The Čech number of a space Z, $\check{C}(Z)$, is the pseudocharacter of Z in βZ. In this article we obtain (in ZFC and assuming additional set theoretic consistent axioms) some upper and lower bounds of the Čech number of spaces of continuous functions defined on X and with values in $\mathbb{R}$ or in $[0,1]$ with the pointwise convergence topology – denoted by $C_p(X)$ and $C_p(X,I)$ respectively – when: (1) X has countable functional tightness, (2) X is an Eberlein-Grothendieck space, (3) X is a k-space and (4) X is a countable space. Also, we prove some results related to $kcov(C_p(X)) = min\{|\mathcal{K}| : \mathcal{K}$ is a compact cover of $C_p(X)\}$. And we answer several questions posed in [OT]; in particular, it is proved that $\check{C}(C_p(X)) = \mathfrak{c}$ for every metrizable space X with $|X| = \mathfrak{c}$, and a consistent example of a non-discrete space X with $\check{C}(C_p(X,I)) < \mathfrak{d}$ is provided.

1. Notations and Basic results

In this article, every space X is a Tychonoff space, and X' is the set of non isolated points in X. The symbols ω (or $\mathbb{N}$), $\mathbb{R}$, I, $\mathbb{Q}$ and $\mathbb{P}$ stand for the set of natural numbers, the real numbers, the segment $[0,1]$, the rational numbers and the irrational numbers, respectively. Given two spaces X and Y, we denote by $C(X,Y)$ the set of all continuous functions from X to Y, and $C_p(X,Y)$ stands for $C(X,Y)$ equipped with the topology of pointwise convergence, that is, the topology in $C(X,Y)$ of subspace of the Tychonoff product Y^X. We will denote by $[x_1,...,x_n;A_1,...,A_n]$ the canonical open subset $\{f \in C(X,Y) : f(x_i) \in A_i \ \forall i \in \{1,...,n\}\}$ of $C_p(X,Y)$ where $x_1,...,x_n \in X$, and $A_1,..,A_n$ are open subsets of Y; in particular, for $Y \subset \mathbb{R}$ and $\delta > 0$, we will denote the set of continuous functions f from X to Y such that $|f(x) - z| < \delta$ as $[x;(z-\delta, z+\delta)]$. The space $C_p(X,\mathbb{R})$ is denoted by $C_p(X)$. The restriction of a function f with domain X to $A \subset X$ is denoted by $f \restriction A$. For a space X, βX is its Stone-Čech compactification, and X^* is the subspace $\beta X \setminus X$ of βX.

Recall that for $X \subset Y$, the *pseudocharacter of* X *in* Y is defined as

$$\Psi(X,Y) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a family of open sets in } Y \text{ and } X = \bigcap \mathcal{U}\}.$$

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1.1. Definitions. *(1) The Čech number of a space Z is $\check{C}(Z) = \Psi(Z, \beta Z)$.*

(2) The k-covering number of a space Z is $kcov(Z) = min\{|\mathcal{K}| : \mathcal{K}$ is a compact cover of $Z\}$.

We have that (see Section 1 in [OT]): $\check{C}(Z) = 1$ if and only if Z is locally compact; $\check{C}(Z) \leq \omega$ if and only if Z is Čech-complete; $\check{C}(Z) = kcov(\beta Z \setminus Z)$; if Y is a closed subset of Z, then $kcov(Y) \leq kcov(Z)$ and $\check{C}(Y) \leq \check{C}(Z)$; if $f : Z \to Y$ is an onto continuous function, then $kcov(Y) \leq kcov(Z)$; if $f : Z \to Y$ is perfect and onto, then $kcov(Y) = kcov(Z)$ and $\check{C}(Y) = \check{C}(Z)$; if bZ is a compactification of Z, then $\check{C}(Z) = \Psi(Z, bZ)$.

We know that $\check{C}(C_p(X)) \leq \aleph_0$ if and only if X is countable and discrete ([LMc]), and $\check{C}(C_p(X, I)) \leq \aleph_0$ if and only if X is discrete ([T]).

For a space X, $ec(X)$ (*the essential cardinality of* X) is the smallest cardinality of a clopen subspace Y of X such that $X \setminus Y$ is discrete. Observe that, in this case, $\check{C}(C_p(X, I)) = \check{C}(C_p(Y, I))$. In [OT] it was pointed out that $ec(X) \leq \check{C}(C_p(X, I))$ and $\check{C}(C_p(X)) = |X| \cdot \check{C}(C_p(X, I))$ always hold. So, if X is discrete, $\check{C}(C_p(X)) = |X|$, and if $|X| = ec(X)$, $\check{C}(C_p(X)) = \check{C}(C_p(X, I))$.

Consider in the set of functions from ω to ω, ${}^{\omega}\omega$, the partial order $\leq^*$ defined by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A collection D of $({}^{\omega}\omega, \leq^*)$ is *dominating* if for every $h \in {}^{\omega}\omega$ there is $f \in D$ such that $h \leq^* f$. As usual, we denote by $\mathfrak{d}$ the cardinal number $\min\{|D| : D$ is a dominating subset of ${}^{\omega}\omega\}$. It is known that $\mathfrak{d} = kcov(\mathbb{P})$ (see [vD]); so $\mathfrak{d} = \check{C}(\mathbb{Q})$. Moreover, $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

We will denote a cardinal number τ with the discrete topology, simply, as τ; so, the space τ^{κ} is the Tychonoff product of κ copies of the discrete space τ. The cardinal number τ with the order topology will be symbolized as $[0, \tau)$.

In this article we will relate $\check{C}(C_p(X, I))$ with the functional tightness and the weak functional tightness of X (Section 3), and obtain some upper and lower bounds of $\check{C}(C_p(X, I))$ when X is one of the following: an Eberlein-Grothendieck space (Section 4), a k-space (Sections 5 and 6), a countable space (Section 7). We will relate $\check{C}(C_p(X, I))$ with the Novak numbers of $\mathbb{R}$ and I^{τ} (Section 8). Also, we will consider $\check{C}(C_p(X, I))$ when some consistent axioms are assumed such as MA and GCH (Section 9). Some results involving the compact covering of $C_p(X)$ are proved (Sections 2 and 4), and several questions posed in [OT] are answered; in particular, we show that the existence of a countable space X for which $\check{C}(C_p(X, I)) < \mathfrak{d}$ is consistent with ZFC .

For notions and concepts not defined here the reader can consult [Ar] and [E].

2. Conditions on X which are implied by $\check{C}(C_p(X, I))$

For an infinite cardinal number τ, let $\mathfrak{d}_\tau$ be the smallest cardinality of a dominating family of functions from ω to τ. We have that $\mathfrak{d}_\tau = cof(\tau)$ if $cof(\tau) > \omega$ and $\mathfrak{d}_\tau = \mathfrak{d}$ if $cof(\tau) = \omega$. $\mathfrak{d}_\tau$ concide with $kcov([0, \tau)^{\omega})$. Of course, $\mathfrak{d}_\omega = \mathfrak{d}$. A space X is a *P_τ-space* if the intersection of less than τ open sets in X is an open set yet. We say that X is a *P-space* if it is a P_{ω_1}-space. Observe that every topological space is a P_ω-space.

We will say that a space X is a *P_τ-space with respect to chains*, or briefly, *cP_τ-space*, if for every sequence $\{A_\lambda : \lambda < \tau\}$ of open subsets of X with $A_\lambda \subset A_\alpha$ if

$\alpha \leq \lambda$, we must have that $\bigcap_{\lambda<\tau} A_\lambda$ is open. So, we have: (1) X is a cP_τ-space iff it is $cP_{cof(\tau)}$-space, (2) every P_{τ^+}-space is a cP_τ-space, (3) a space X is a P-space iff it is a cP_ω-space, (4) if $|X| < cof(\tau)$, then X is a cP_τ-space. Furthermore, for a regular cardinal number $\tau > \omega$, $[0, \tau)$ is a cP_τ-space which is not a P_{τ^+}-space (and it is not cP_γ-space for all cardinal number $\gamma < \tau$), and the topological space $(\tau + 1, \mathcal{T})$ where $\mathcal{T} = \{A \subset \tau + 1 : \text{either } \tau \notin A \text{ or } \tau \in A \text{ and } |\tau + 1 \setminus A| < \tau\}$ is a P_τ-space (and, then, it is a cP_γ-space for every $\gamma < \tau$) which is not a cP_τ-space.

For a cardinal number κ, we say that a space X is *initially κ-compact* if every open cover of X of cardinality $\leq \kappa$ has a finite subcover. It is known that X is initially κ-compact if and only if every subset E of X of cardinality $\leq \kappa$ has a complete accumulation point x; that is, for every neigborhood V of x, $|V \cap E| = |E|$ (see [St], Theorem 2.2). Of course, (1) every closed subset of an initially κ-compact shares this property, (2) a space is countably compact iff it is initially ω-compact, and (3) every compact space is initially κ-compact for every κ.

2.1. Theorem. *Let X be a topological space and $\kappa < \mathfrak{d}_\tau$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) where:*

1. *$C_p(X)$ is the union of κ initially τ-compact subsets.*
2. *$C_p(X, I)$ is the union of κ initially τ-compact subsets.*
3. *X is a cP_τ-space.*

Proof. (1) $\Rightarrow$ (2): Of course, if $C_p(X)$ is the union of κ initially τ-compact subsets, then so is $C_p(X, I)$ because it is a closed subset of $C_p(X)$.

(2) $\Rightarrow$ (3): The proof follows the pattern of the proof of Corollary I.2.4 in [Ar]: Let $C_p(X, I) = \bigcup_{\alpha<\kappa} K_\alpha$, where each K_α is initially τ-compact. If X is not a cP_τ-space, there is an increasing sequence of closed sets $(F_\lambda)_{\lambda<\tau}$ and a point $y_* \in \text{cl}(\bigcup_{\lambda<\gamma} F_\lambda) \setminus \bigcup_{\lambda<\gamma} F_\lambda$. Denote by $K'_\alpha = \{f \in K_\alpha : f(y_*) = 0\}$ for each $\alpha < \kappa$. Since K'_α is a closed subset of K_α, K'_α is initially τ-compact.

Claim: For each $\alpha < \kappa$ and each $k < \omega$, there is $\lambda(\alpha, k) \in \tau$ such that for every $f \in K'_\alpha$, there exists $y_f \in F_{\lambda(\alpha,k)}$ for which $f(y_f) < 1/2^{k+1}$.

Indeed, assume the contrary of the conclution of this Claim. That is, there are $\alpha < \kappa$ and $k < \omega$ such that for each $\lambda < \tau$ there is $f_\lambda \in K'_\alpha$ with $f_\lambda(y) \geq 1/2^{k+1}$ for all $y \in F_\lambda$. Since K'_α is initially τ-compact, there is $f \in K'_\alpha$ which is a complete accumulation point of the set $\{f_\lambda : \lambda < \tau\}$ in $C_p(X, I)$. Take $y \in \bigcup_{\lambda<\tau} F_\lambda$; say $y \in F_{\lambda_0}$. If $f(y) < 1/2^{k+1}$, then $f \in [y; (f(y) - \delta, f(y) + \delta)] = \{g \in C_p(X, I) : |f(y) - g(y)| < \delta\}$ where $\delta = 1/2^{k+1} - f(y)$. Since f is a complete accumulation point of $\{f_\lambda : \lambda < \tau\}$, there is $\lambda \geq \lambda_0$ such that $f_\lambda \in [y; (f(y) - \delta, f(y) + \delta)]$. But, this implies that $f_\lambda(y) < 1/2^{k+1}$, which is not possible because $y \in F_{\lambda_0} \subset F_\lambda$ and then, by assumption, $f_\lambda(y) \geq 1/2^{k+1}$.

Thus, $f(y)$ must be $\geq 1/2^{k+1}$ for every $y \in \bigcup_{\lambda<\tau} F_\lambda$. Since f is continuous and $y_* \in \text{cl} \bigcup_{\lambda<\gamma} F_\lambda$, then $f(y_*) \geq 1/2^{k+1} > 0$. But, $f \in K'_\alpha$ which implies that $f(y_*) = 0$, a contradiction. This proves the Claim.

Hence, for each $\alpha < \kappa$, we have a function $h_\alpha : \omega \to \tau$ defined by $h_\alpha(k) = \lambda(\alpha, k)$. Since $|\{h_\alpha : \alpha < \kappa\}| < \mathfrak{d}_\tau$, $\{h_\alpha : \alpha < \kappa\}$ is not dominating in τ^ω, so there is $h : \omega \to \tau$ such that $h \not\leq^* h_\alpha$ for all $\alpha < \kappa$.

For each $k < \omega$, fix $g_k : X \to [0, 1/2^{k+1}]$, continuous, such that $g_k(y_*) = 0$ and $g_k(t) = 1/2^{k+1}$ when $t \in F_{h(k)}$. Let g be equal to $\Sigma_{k<\omega} g_k$. We have that $g \in C_p(X, I)$, so there is $\beta < \kappa$ for which $g \in K'_\beta$.

Since $h \not\leq^* h_\beta$, the set $\{n < \omega : h(n) > h_\beta(n)\}$ is infinite; fix m such that $h(m) > h_\beta(m)$. The relation $g \in K'_\beta$ implies that there is $y \in F_{h_\beta(m)}$ for which $g(y) < 1/2^{m+1}$. But $y \in F_{h_\beta(m)} \subset F_{h(m)}$ and $g(y) \geq 1/2^{m+1}$; a contradiction. □

2.2. Corollary. *Let X be a topological space, and let γ be the first cardinal number such that X is not a P_{γ^+}-space. Then, $C_p(X)$ cannot be equal to the union of κ initially γ-compact subsets for any $\kappa < \mathfrak{d}_\gamma$.*

2.3. Corollary. *Let X be a topological space and $\kappa < \mathfrak{d}$. Then (1) ⇒ (2) ⇒ (3), where:*

(1) *$C_p(X)$ is the union of κ countably compact subsets.*
(2) *$C_p(X, I)$ is the union of κ countably compact subsets.*
(3) *X is a P-space.*

For a cardinal number τ, we say that a space X is *weak-τ-pseudocompact* if every discrete collection of open sets in X has cardinality $< \tau$. Then, we have that a space X is pseudocompact if and only if it is weak-$\aleph_0$-pseudocompact.

2.4. Theorem. *Let X be a topological space and let $\kappa < kcov(\omega^\tau)$. Then (1) ⇒ (2) ⇒ (3), where:*

(1) *X is not discrete and $\mathbb{R}^X \setminus C_p(X)$ is contained in the union of κ countably compact subsets of $\mathbb{R}^X$.*
(2) *$C_p(X)$ is contained in the union of κ countably compact subsets of $\mathbb{R}^X$.*
(3) *X is weak-τ-pseudocompact.*

Proof. (1) ⇒ (2): Assume that $\mathbb{R}^X \setminus C_p(X)$ is contained in the union of κ countably compact subsets of $\mathbb{R}^X$. Since X is not discrete, there exists $f_0 \in \mathbb{R}^X \setminus C_p(X)$, and $C_p(X) + f_0 \subset \mathbb{R}^X \setminus C_p(X) \subset \bigcup_{\alpha<\kappa} K_\alpha$ where K_α is a countably compact subset of $\mathbb{R}^X$. Thus, $C_p(X) \subset \bigcup_{\alpha<\kappa}(K_\alpha - f_0)$. Moreover $K_\alpha - f_0$ is countably compact for every $\alpha < \kappa$.

(2) ⇒ (3): Now, assume that $C_p(X) \subset \bigcup_{\alpha<\kappa} K_\alpha \subset \mathbb{R}^X$, each K_α is countably compact, and assume that $(U_\lambda)_{\lambda<\tau}$ is a discrete collection of open nonempty subsets of X. For each $\lambda < \tau$ fix $b_\lambda \in U_\lambda$ and fix $h_\lambda : X \to [0, 1]$, continuous, such that $h_\lambda(b_\lambda) = 1$ and $h_\lambda(t) = 0$ if $t \in X \setminus U_\lambda$.

For each $\alpha < \kappa$ define $g_\alpha : \tau \to \omega$ as follows: For each $\lambda < \tau$ we define

$$g_\alpha(\lambda) = n^\alpha_\lambda \text{ where } \pi_{b_\lambda}[K_\alpha] \subset [-n^\alpha_\lambda, n^\alpha_\lambda].$$

For each $\alpha < \kappa$, we take $C_\alpha = \prod_{\lambda<\tau}[-n^\alpha_\lambda, n^\alpha_\lambda]$. Each C_α is a compact subset of ω^τ. Since $\kappa < kcov(\omega^\tau)$, there is $f : \tau \to \omega$ such that $f \notin \bigcup_{\alpha<\kappa} C_\alpha$.

Define $\phi : X \to \mathbb{R}$ as $\phi(t) = \Sigma_{\lambda<\tau} f(\lambda) \cdot h_\lambda(t)$. Since $(U_\lambda)_{\lambda<\tau}$ is discrete, ϕ is continuous. Hence, there is $\beta < \kappa$ such that $\phi \in K_\beta$. For each $\lambda < \tau$, $\phi(b_\lambda) = f(\lambda)$, and $f(\lambda) \in [-n^\beta_\lambda, n^\beta_\lambda]$, hence, $f \in C_\beta$, which is a contradiction. Then, every discrete collection of nonempty open subsets of X must be of cardinality $< \tau$. □

For a space X we denote by $\vartheta(X)$ the supremum of the set $\{|\mathcal{C}| : \mathcal{C}$ is a discrete collection of open sets of $X\}$. By Theorem 2.4, if X is not discrete, $\vartheta(X) \leq$

kcov($C_p(X)$). Observe that if X is discrete and infinite, then $\check{C}(C_p(X)) = \check{C}(\mathbb{R}^X) = |X|$ (Lemma 2.1 in [OT]) and $kcov(C_p(X)) = kcov(\mathbb{R}^X) = kcov(\omega^X)$. In fact, since $\mathbb{N}^X$ is a closed subset of $\mathbb{R}^X$, $kcov(\mathbb{R}^X) \geq kcov(\omega^X)$. On the other hand, $\mathbb{R}$ is a continuous image of ω^ω, so $\mathbb{R}^X$ is a continuous image of ω^X; then, $kcov(\mathbb{R}^X) \leq kcov(\omega^X)$.

Theorems 2.1 and 2.4 produce the following corollaries.

2.5. Corollary. *Let X be a topological space and let $\omega \leq \kappa < \mathfrak{d}$. Then $C_p(X)$ is contained in the union of κ countably compact subsets of $\mathbb{R}^X$ if and only if X is pseudocompact.*

Proof. If X is pseudocompact, then $C_p(X) \subset \bigcup_{n<\omega}[-n, n]^X \subset \mathbb{R}^X$.

The converse follows from Theorem 2.4. $\square$

D.B. Shakmatov and V.V. Tkachuk, and N.V. Velichko proved that $C_p(X)$ is σ-compact $\Leftrightarrow$ $C_p(X)$ is σ-countably compact $\Leftrightarrow$ X is finite (see [TS] and Corollary I.2.4 in [Ar]). The following result is a generalization of their result.

2.6. Corollary. *$kcov(C_p(X)) < \mathfrak{d}$ if and only if X is finite.*

Proof. If X is finite of cardinality n, then $kcov(C_p(X)) = kcov(\mathbb{R}^n) = \aleph_0 < \mathfrak{d}$.

If $kcov(C_p(X)) < \mathfrak{d}$ then X is a pseudocompact P-space, that is, X is finite. $\square$

2.7. Corollary. *If X is a non-discrete space, then kcov$(C_p(X)) \geq \mathfrak{d}$.*

We cannot have a similar result to the last corollary for $C_p(X, I)$. In fact, let $X = Y \cup \{p\}$ be the one point Lindelöfication of the discrete space $Y = \{y_\lambda : \lambda < \omega_1\}$, where $p \notin Y$ is the distinguished point. We have that $C_p(X, I)$ is equal to $\bigcup_{\alpha<\omega_1} F_\alpha$ where $F_\alpha = [0, 1]^{Y_\alpha} \times \Delta_\alpha$ where $Y_\alpha = \{y_\gamma : \gamma < \alpha\}$ and Δ_α is the diagonal in $[0, 1]^{X\setminus Y_\alpha}$. Each F_α is a closed subset of $[0, 1]^X$, so it is compact. Then $C_p(X, I)$ is the union of ω_1 compact subsets, and it is consistent with ZFC that $\omega_1 < \mathfrak{d}$.

Since $C_p(X, I)$ is a closed subset of $C_p(X)$ and $C_p(X) = \mathbb{R}^X \cap C_p(X, [-\infty, \infty])$, then $kcov(C_p(X, I)) \leq kcov(C_p(X)) \leq kcov(C_p(X, I)) \cdot kcov(\omega^{|X|})$ always happens. Moreover, if $X = \beta\omega$, then $kcov(C_p(X)) \leq 2^{d(X)} = \mathfrak{c} < 2^{\mathfrak{c}} = ec(X) < kcov(\omega^{|X|})$.

3. $\check{C}(C_p(X))$ for spaces X with countable functional tightness

Let τ be a cardinal number. A function $f : X \to Y$ is called *τ-continuous* if for every subspace A of cardinality $\leq \tau$, the restriction of f to A is continuous. The *functional tightness* $t_\theta(X)$ of a space X is the smallest infinite cardinal τ such that every realvalued τ-continuous function on X is continuous [A2]. A function $f : X \to Y$ is called *strictly-τ-continuous* if for every subspace A of cardinality $\leq \tau$, there is a continuous function $g : X \to Y$ for which $g \restriction A = f \restriction A$. The *weak functional tightness* $t_\mathbb{R}(X)$ of a space X is the smallest infinite cardinal τ such that every realvalued strictly-τ-continuous function on X is continuous [A1], [A2]. It is obvious that for every space X, $t_\mathbb{R}(X) \leq t_\theta(X)$, and $t_\mathbb{R}(X) = t_\theta(X)$ if X is normal (see [Ar], Proposition II.4.8). Also, note that if $t(X)$ denotes the tightness of X, then $t_\theta(X) \leq t(X)$. Besides, $t_\theta(X)$ is always $\leq$ to the density $d(X)$ of X (see Proposition II.4.2 in [Ar]). The topological cardinal function $t_\mathbb{R}$ measures some kind of degree of realcompactness of $C_p(X)$. In particular, $C_p(X)$ is realcompact if and only if $t_\mathbb{R}(X) = \aleph_0$, [Ar], Theorem II.4.16.

3.1. Proposition. *Let X be a space and κ be a cardinal number. Assume that $t_\theta(X) < cof(ec(X))$ and that for every subspace Z of X of cardinality $< ec(X)$, $\check{C}(C_p(Z,I)) \le \kappa$. Then $\check{C}(C_p(X,I)) \le cof(ec(X)) \cdot \kappa$.*

Proof. Let Y be a clopen subspace of X of cardinality $ec(X)$ such that $X \setminus Y$ is discrete. Then $t_\theta(Y) = t_\theta(X)$ and $\check{C}(C_p(X,I)) = \check{C}(C_p(Y,I))$. Let $\{y_\lambda : \lambda < ec(X)\}$ be a faithful enumeration of Y. For each ordinal number $\gamma < ec(X)$, let $Z_\gamma = \{y_\lambda : \lambda < \gamma\}$ and $C_\gamma = \{f \in I^Y : f \restriction Z_\gamma \in C_p(Z_\gamma, I)\}$. Let $\{\Omega_\alpha : \alpha < \kappa\}$ be a collection of open sets in I^{Z_γ} which satisfies $C_p(Z_\gamma, I) = \bigcap_{\alpha<\kappa} \Omega_\alpha$. Consider the set $W_\alpha = \Omega_\alpha \times I^{Y \setminus Z_\gamma}$ for each $\alpha < \kappa$. They are open in I^Y and $\bigcap_{\alpha<\kappa} W_\alpha = C_\gamma$. So $\check{C}(C_\gamma) \le \check{C}(C_p(Z_\gamma, I)) \le \kappa$. On the other hand, $C_\gamma^0 = \{f \in C_\gamma : f(y_\alpha) = 0$ for every $\alpha \ge \gamma\}$ is a closed subset of C_γ which is homeomorphic to $C_p(Z_\gamma, I)$. Thus, $\check{C}(C_\gamma) = \check{C}(C_p(Z_\gamma, I))$. Moreover, it is easy to prove that $C_p(Y,I) \subset \bigcap_{\gamma < ec(X)} C_\gamma$. Now, from the fact that $t_\theta(Y) = t_\theta(X) < cof(ec(X))$, we get $C_p(Y,I) = \bigcap_{\delta < cof(ec(X))} C_{\gamma(\delta)}$. Therefore, $\check{C}(C_p(X,I)) \le cof(ec(X)) \cdot \kappa$. $\square$

3.2. Theorem. *For every space X, $\check{C}(C_p(X,I)) \le ec(X)^{t_{\mathbb{R}}(X)}$.*

Proof. Let Y be a clopen subspace of X with $X \setminus Y$ discrete and $|Y| = ec(X)$. We are going to prove that $\check{C}(C_p(Y,I)) \le |Y|^{t_{\mathbb{R}}(Y)}$. Observe that $t_{\mathbb{R}}(Y) = t_{\mathbb{R}}(X)$. Let $\mathcal{A}$ denote the set of all infinite subsets of Y of cardinality $\le t_{\mathbb{R}}(Y)$. (So, $|\mathcal{A}| \le |Y|^{t_{\mathbb{R}}(Y)}$).

For each $A \in \mathcal{A}$ we consider the set $C_A = \{f \in C_p(A,I) : \exists\, g \in C_p(Y,I)$ with $g \restriction A = f \restriction A\}$. We have that $\check{C}(C_A) \le 2^{|A|} \le 2^{t_{\mathbb{R}}(Y)}$. Put $\lambda = 2^{t_{\mathbb{R}}(Y)}$ and fix a family $\{\Omega_\alpha^A : \alpha < \lambda\}$ of open subsets of I^A such that

$$\bigcap_{\alpha<\lambda} \Omega_\alpha^A = C_A. \tag{1}$$

Now define, for each $A \in \mathcal{A}$ and $\alpha < \lambda$, $W_\alpha^A = \Omega_\alpha^A \times I^{Y \setminus A}$ which is open in I^Y. We shall prove that

$$\bigcap_{A \in \mathcal{A}} \bigcap_{\alpha<\lambda} W_\alpha^A = C_p(Y,I). \tag{2}$$

Indeed, if $f \in C_p(Y,I)$, then $f \restriction A \in C_A$ for every $A \in \mathcal{A}$ and $f \restriction A \in \Omega_\alpha^A$, for every $\alpha < \lambda$, hence $f \in W_\alpha^A$ for every $A \in \mathcal{A}$ and every $\alpha < \lambda$. On the other hand, if $g : Y \to I$ is not continuous, there exists $A \subset Y$ of cardinality $\le t_{\mathbb{R}}(Y)$ such that $g \restriction A \notin C_A$; so, there is $\beta < \lambda$ for which $g \restriction A \notin \Omega_\beta^A$. Therefore, $g \notin W_\beta^A$. $\square$

3.3. Corollary. *For every space X with $ec(X) = ec(X)^{t_{\mathbb{R}}(X)}$, $\check{C}(C_p(X,I)) = ec(X)$.*

The following result, which follows from Corollary 3.3, answers Question 4.19 in [OT] in the affirmative.

3.4. Corollary. *Let X be a metrizable space with $ec(X) = \mathfrak{c}$. Then*

$$\check{C}(C_p(X,I)) = \mathfrak{c}$$

Theorem 3.2 produces:

3.5. Corollary. *For every space X,*

$$\check{C}(C_p(X,I)) \leq ec(X)^{t_\theta(X)} \leq min\{ec(X)^{t(X)}, ec(X)^{d(X)}\}.$$

We have that $ec(\beta\omega) = ec(\omega^*) = |\omega^*| = 2^{2^\omega}$ and $t(\beta\omega) = t(\omega^*) = 2^\omega$; thus $\check{C}(C_p(\beta\omega, I)) = \check{C}(C_p(\omega^*, I)) = 2^{2^\omega}$.

3.6. Corollary. *Let X be a topological space with $ec(X) = \mathfrak{c}$. If $C_p(X)$ is a realcompact space, then $\check{C}(C_p(X,I)) = \mathfrak{c}$.*

Proof. The hypothesis of this Corollary implies that $t_{\mathbb{R}}(X) = \aleph_0$ (see II.4.16 in [Ar]). Now, in order to finish the proof, we have only to apply Corollary 3.3. □

Even for collectionwise normal spaces X with essential cardinality $\mathfrak{c}$, $ec(X) = \mathfrak{c} = \check{C}(C_p(X,I))$ does not imply the realcompactness of $C_p(X)$ as was noted by O. Okunev. In fact, let $X = [0,\mathfrak{c}]$. We have that $\mathfrak{c} = ec([0,\mathfrak{c}]) \leq \check{C}(C_p([0,\mathfrak{c}], I) \leq kcov(\mathfrak{c}^\omega) = \mathfrak{c}$ (see [AT]). On the other hand, $t_{\mathbb{R}}([0,\mathfrak{c}]) = \mathfrak{c} > \aleph_0$, so $C_p([0,\mathfrak{c}])$ is not realcompact. Also observe that $\check{C}(C_p([0,\mathfrak{c}], I) = \mathfrak{c}$ is strictly less than $ec([0,\mathfrak{c}])^{t_{\mathbb{R}}([0,\mathfrak{c}])} = 2^{\mathfrak{c}}$.

Since, for every non-discrete space X, $t_{\mathbb{R}}(X) \leq ec(X)$, we have $ec(X)^{t_{\mathbb{R}}(X)} \leq 2^{ec(X)}$. Therefore, for every non-discrete space X, $ec(X) \leq \check{C}(C_p(X,I)) \leq 2^{ec(X)}$. In particular, GCH implies $ec(X) \leq \check{C}(C_p(X,I)) \leq ec(X)^+$ for every non-discrete space X.

4. $\check{C}(C_p(X))$ for Eberlein-Grothendieck spaces X

A space X is an *Eberlein-Grothendieck* space or an *EG*-space if it is a subspace of a space $C_p(Y)$ where Y is a compact space (or, equivalently, X is a subspace of $C_p(Y)$ for a σ-compact space Y, [Ar], Theorem. III.1.11).

The following lemma is proved in [OT, Lemma 4.3].

4.1. Lemma. *Let X be a subspace of $C_p(Y)$, K a compact set in X, and let C be the set of all functions in I^X that are continuous at every point of K. Then there is a family $\{ B_{mn} : m \in \mathbb{N}^+, n \in \mathbb{N}^+ \}$ of subsets of I^X such that*

(1) $C = \bigcap_{m\in\mathbb{N}^+} \bigcup_{n\in\mathbb{N}^+} B_{mn}$, *and*
(2) *for any $m, n \in \mathbb{N}^+$, B_{mn} is a continuous image of a closed subspace of $Y^n \times I^X$.*

We utilize Lemma 4.1 to prove the following result that generalizes Corollary 4.5 in [OT]. It is worth recalling here Definition 3.1 in [OT]: for two cardinal numbers $\tau \geq 1$ and λ, a space X is $K(\tau,\lambda)$-analytic if X is a continuous image of a closed subspace of a product of τ^λ and a compact space.

4.2. Theorem. *For every EG-space X,*

$$ec(X) \leq \check{C}(C_p(X,I)) \leq kcov(ec(X)^\omega).$$

Proof. Let Z be a compact space such that X is a subspace of $C_p(Z)$. Let Y be a subspace of X such that $X \setminus Y$ is clopen, discrete and $|Y| = ec(X)$. We have that $\check{C}(C_p(X,I)) = \check{C}(C_p(Y,I))$. So we are going to prove that $\check{C}(C_p(Y,I)) \leq kcov(|Y|^\omega)$. Let $\mathcal{C}$ be a compact cover of Y with minimal cardinality ($|\mathcal{C}| = kcov(Y)$). For each $K \in \mathcal{C}$ we take the set $C_K = \{f \in I^Y : f$ is continuous (as a function from Y to I) in each point of $K\}$. Lemma 4.1 guaranties that there exists a familly $\{B^K_{mn} : m, n < \omega\}$ of subsets of I^Y such that

(1) $C_K = \bigcap_{m<\omega} \bigcup_{n<\omega} B^K_{mn}$, and

(2) each B^K_{mn} is a continuous image of a closed subspace of $Z^n \times I^Y$.

Since Z is a compact space, each B^K_{mn} is compact, so $I^Y \setminus B^K_{mn}$ is an open subset of I^Y, but each open subset of I^Y is the union of $|Y|$ compact subsets because I^Y is a locally compact space with weight equal to $|Y|$. Say $I^Y \setminus B^K_{mn} = \bigcup_{\alpha<|Y|} C_\alpha$ where C_α is compact for every $\alpha < |Y|$. Therefore, $I^Y \setminus C_K = \bigcup_{m<\omega} \bigcap_{n<\omega} \bigcup_{\alpha<|Y|} C_\alpha$ is a $K(|Y|, \omega)$-analytic set. This means that $\check{C}(C_K) = kcov(I^Y \setminus C_K) \leq kcov(|Y|^\omega)$ ([OT], Proposition 3.2). For each $K \in \mathcal{C}$, let $\mathcal{A}_K$ be a familly of open subsets of I^Y with cardinality $\leq kcov(|Y|^\omega)$ satisfying $C_K = \bigcap \mathcal{A}_K$. We have that $C_p(Y,I) = \bigcap_{K\in\mathcal{C}} C_K$; so $C_p(Y,I) = \bigcap_{K\in\mathcal{C}} \bigcap \mathcal{A}_K$. Thus, $\check{C}(C_p(Y,I)) \leq kcov(Y) \cdot kcov(|Y|^\omega) = kcov(|Y|^\omega)$. $\square$

It was proved in [OT] that for every non-discrete space X the relation $ec(X) \leq \check{C}(C_p(X,I))$ holds; and Corollary 4.12, in the same article, states that if X contains a convergent sequence, then $\check{C}(C_p(X,I)) \geq \mathfrak{d}$. Thus, since every non-discrete metrizable space is an EG-space and contains a non trivial convergent sequence, we obtain for every non-discrete metrizable space X, $ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I)) \leq kcov(ec(X)^\omega)$. Moreover, if $\tau < \omega_\omega$, then $kcov(\tau^\omega) = \tau \cdot \mathfrak{d}$. So, if X is a metrizable non-discrete space and $ec(X) < \omega_\omega$, then $\check{C}(C_p(X,I)) = ec(X) \cdot \mathfrak{d}$.

For a cardinal number τ, a space X is a $K_{\sigma\tau}$-set if X is a subspace of a space Y and there are σ-compact subspaces Y_λ of Y ($\lambda < \tau$) such that $X = \cap_{\lambda<\tau} Y_\lambda$. If X is a $K_{\sigma\tau}$-set, then X is an $F_{\sigma\tau}$-set in every space where X is embedded.

4.3. Proposition. *Let X be a non-discrete EG-space and let $\tau = kcov(X)$. Then $C_p(X,I)$ is a $K_{\sigma\tau}$-set.*

Proof. Let $X = \bigcup_{\lambda<\tau} K_\lambda$ where K_λ is compact for all $\lambda < \tau$. By Lemma 4.1, for each $\lambda < \tau$, $C_\lambda = \{f : X \to I : f$ is continuous at each point in $K_\lambda\} = \bigcap_{m<\omega} \bigcup_{n<\omega} B^\lambda_{mn}$ where B^λ_{mn} is compact for each $m, n < \omega$ and each $\lambda < \tau$. Therefore, $C_p(X,I) = \bigcap_{\lambda<\tau} \bigcap_{m<\omega} \bigcup_{n<\omega} B^\lambda_{mn}$; that is $C_p(X,I)$ is a $K_{\sigma\tau}$-set. $\square$

4.4. Corollary. *If X is a non-discrete EG-space and $\tau = kcov(X)$, then*

$$kcov(C_p(X,I)) \leq kcov(\omega^\tau) \leq kcov(\omega^{|X|}).$$

Proof. By Proposition 4.5, $C_p(X,I)$ is an $F_{\sigma\tau}$-set in I^X. So, it is $K(\omega,\tau)$-analytic and $kcov(C_p(X,I)) \leq kcov(\omega^\tau)$ (see Corollary 3.4 in [OT]). The last inequality follows because $kcov(X) \leq |X|$ always holds. $\square$

5. $\check{C}(C_p(X))$ when X is a sequential space

5.1. Definitions. ([S]) (a) For a topological space X, a collection $\mathcal{P}$ of subsets of X is called *a sequential base* if for every point $x \in X$ one can assign a collection $\mathcal{P}_x \subset \mathcal{P}^\omega$ such that

(1) if $(P_n)_{n<\omega} \in \mathcal{P}_x$, then $x \in \bigcap_{n<\omega} P_n$ and $P_{n+1} \subset P_n$ for each $n < \omega$,

(2) a set V is open in X iff for every point $x \in V$ and every $(P_n)_{n<\omega} \in \mathcal{P}_x$ there exists $m < \omega$ such that $P_m \subset V$.

(b) Let $\mathcal{P}$ be a sequential base of a space X. Then, the *fan number* of $\mathcal{P}$, denoted $fn(\mathcal{P})$, is the smallest cardinal τ such that for each $x \in X$, the collection $\mathcal{P}_x$ may be chosen with cardinality $\leq \tau$.

S.A. Svetlichny proved in [S] that a space X is sequential iff X has a sequential base.

In the sequel we will use the following

5.2. Notations. For each $n < \omega$, we will denote as $\mathcal{E}_n$ the collection of intervals

$$[0, 1/2^{n+1}), (1/2^{n+2}, 3/2^{n+2}), (1/2^{n+1}, 2/2^{n+1}), (3/2^{n+2}, 5/2^{n+2}), ...$$

$$..., ((2^{n+2}-2)/2^{n+2}, (2^{n+2}-1)/2^{n+2}), ((2^{n+1}-1)/2^{n+1}, 1].$$

Observe that $\mathcal{E}_n$ is an irreducible open cover of $[0,1]$ and each element in $\mathcal{E}_n$ has diameter $= 1/2^{n+1}$. For a set S and a point $y \in S$, we will use the symbol $[yS]^{<\omega}$ in order to denote the collection of finite subsets of S containing y.

5.3. Theorem. *For every non-discrete sequential space X with sequential base $\mathcal{P}$,*

$$ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I)) \leq fn(\mathcal{P}) \cdot kcov(ec(X)^\omega).$$

Proof. Let Y be a clopen subspace of X for which $X \setminus Y$ is discrete and $|Y| = ec(X)$. For each $y \in Y$, each $(T_n)_{n<\omega} \in \mathcal{P}_y$ and each $n < \omega$, we take $S_n = T_n \cap Y$. Let $\mathcal{P}'_y = \{(S_n)_{n<\omega} : (T_n)_{n<\omega} \in \mathcal{P}_y\}$. Observe that $\mathcal{P}' = \{S_m : m < \omega, (T_n)_{n<\omega} \in \mathcal{P}_y$ and $y \in Y\}$ is a sequential base for Y where for each $y \in Y$, $\mathcal{P}'_y = \{(S_n)_{n<\omega} : (T_n)_{n<\omega} \in \mathcal{P}_y\}$. Moreover $fn(\mathcal{P}') \leq fn(\mathcal{P})$.

For each $y \in Y'$, $s = (S_n)_{n<\omega} \in \mathcal{P}'_y$, $m, n < \omega$, $E \in \mathcal{E}_n$ and $F \in [yS_m]^{<\omega}$, we take the set

$$B(y, s, E, m, F) = \prod_{z \in Y} J_z$$

where $J_z = E$ if $z \in F$ and $J_z = I$ if $z \notin F$. Let

$$B(y, s, n, m, F) = \bigcup_{E \in \mathcal{E}_n} B(y, s, E, m, F).$$

We define

$$B(y, s, n, m) = \bigcap \{B(y, s, n, m, F) : F \in [yS_m]^{<\omega}\}.$$

Because $S_m \subset Y$, $B(y, s, n, m)$ is the intersection of $\leq ec(X)$ open subsets $B(y, s, n, m, F)$ of I^Y. Now we define $G(y, s, n) = \bigcup_{m<\omega} B(y, s, n, m)$, $G(y, s) = \bigcap_{n<\omega} G(y, s, n)$, $G(y) = \bigcap_{s \in \mathcal{P}'_y} G(y, s)$ and, finally, $G = \bigcap_{y \in Y'} G(y)$.

Claim: $G = C(Y, I)$.

Indeed, let $g \in C(Y, I)$, $y \in Y'$, $s = (S_n)_{n<\omega} \in \mathcal{P}'_y$ and $n < \omega$. We shall prove that $g \in G(y, s, n)$; that is, we are going to show that there is $m < \omega$ so that

$g \in B(y,s,n,m)$. There is $E \in \mathcal{E}_n$ such that $g(y) \in E$. Since g is a continuous function in y, there exists an open set V of Y containing y for which $g(V) \subset E$. By definition of sequential base, there exists $N < \omega$ such that $S_N \subset V$. For every $F \in [yS_N]^{<\omega}$, $g \in B(y,s,E,N,F)$. Then, $g \in B(y,s,n,N)$. Therefore, $C(Y,I) \subset G$.

Now, assume that $h \in I^Y \setminus C(Y,I)$. There are a point $y \in Y'$, $\epsilon > 0$ and a sequence $x_0, x_1, ..., x_n, ...$ in $Y \setminus \{y\}$ which converges to y and such that

$$(*) \qquad |h(x_n) - h(y)| \geq \epsilon$$

for all $n < \omega$. Let $k \geq 1$ such that $1/2^k < \epsilon$. The set $W' = Y \setminus (\{x_n : n < \omega\} \cup \{y\})$ is open, and $W = Y \setminus \{x_n : n < \omega\}$ is not open. This means that there exists $s = (S_n)_{n<\omega} \in \mathcal{P}'_y$ such that, for every $m < \omega$,

$$(**) \qquad S_m \cap \{x_n : n < \omega\} \neq \emptyset.$$

We want to show that $h \notin G(y,s,k)$. Assume that for an $m < \omega$, $h \in B(y,s,k,m)$. This means that $h \in \bigcap\{B(y,s,k,m,F) : F \in [yS_m]^{<\omega}\}$. Because of (**), there is $x_t \in S_m$. Take $F = \{x_t, y\}$. Our hypothesis implies that h must belong to $B(y,s,k,m,F)$. But this means that $|h(x_t) - h(y)| < 1/2^k$, which contradicts (*). So, we have to conclude that $G = C(X,I)$.

Thus, the complement of G in I^Y is equal to

$$\bigcup_{y \in Y'} \bigcup_{s \in \mathcal{P}'_y} \bigcup_{n<\omega} (I^Y \setminus G(y,s,n));$$

that is, $I^Y \setminus G$ is the union of $\leq |Y'| \cdot fn(\mathcal{P}') \cdot \omega$ sets M_α ($\alpha < |Y'| \cdot fn(\mathcal{P}') \cdot \omega$) each of them being the countable intersection of sets of the form $I^Y \setminus B(y,s,n,m)$.

We have that $I^Y \setminus B(y,s,n,m) = I^Y \setminus \bigcap\{B(y,s,n,m,F) : F \in [yS_m]^{<\omega}\} = \bigcup\{I^Y \setminus B(y,s,n,m,F) : F \in [yS_m]^{<\omega}\}$. That is, each M_α is a set of the form $\bigcap_{m<\omega} \bigcup_{F \in [yS_m]^{<\omega}} (I^Y \setminus B(y,s,n,m,F)$. But this means that M_α is an $F_{|Y|\delta}$-set in I^Y. Now, Corollary 3.4 in [OT] guaranties that $kcov(M_\alpha) \leq kcov(|Y|^\omega)$, so $I^Y \setminus G$ is the union of $\leq |Y'| \cdot fn(\mathcal{P}') \cdot kcov(|Y|^\omega)$ compact subsets of I^Y. That is,

$$\check{C}(C_p(X,I)) \leq |Y'| \cdot fn(\mathcal{P}') \cdot kcov(|Y|^\omega) \leq fn(\mathcal{P}) \cdot kcov(ec(X)^\omega). \quad \square$$

A space X is *weakly-quasi-first-countable* if it has a sequential base with fan number $\leq \aleph_0$. In particular, every first countable space is weakly-quasi-first-countable. In fact, for each $x \in X$, we take a countable local base $\mathcal{B}_x = \{B_n : n < \omega\}$ of x in X satisfying $B_{n+1} \subset B_n$ for every n. The collection $\mathcal{P} = \bigcup_{x \in X} \mathcal{B}_x$ plus the assignment $x \to \mathcal{B}_x$ constitute a sequential base for X, and $fn(\mathcal{P}) = 1$. Theorem 5.3 implies:

5.4. Corollary. *If X is a non-discrete weakly-quasi-first-countable space (in particular, if X is first countable), then*

$$ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I)) \leq kcov(ec(X)^\omega).$$

5.5. Corollary. *If X is a non-discrete weakly-quasi-first-countable space (in particular, if X is non-discrete and first countable) of cardinality $< \omega_\omega$, then*

$$\check{C}(C_p(X,I)) = ec(X) \cdot \mathfrak{d}.$$

Proof. This is a consequence of the previous result and Proposition 3.6, Corollaries 4.8 and 4.12 in [OT]. □

5.6. Problem. Does every Fréchet-Urysohn space X have a sequential base $\mathcal{P}$ with the property $fn(\mathcal{P}) \leq kcov(ec(X)^\omega)$?

The following result is Theorem 2.7 in [S].

5.7. Theorem. *The following are equivalent for a space X and any cardinal τ:*

(1) *X is a quotient of a metric space having cardinality τ.*
(2) *$|X| \leq \tau$ and X has a sequential base $\mathcal{P}$ such that $fn(\mathcal{P}) \leq \tau$.*

As a consequence of this last result and Theorem 5.3, we have:

5.8. Corollary. *If a non-discrete space X is the quotient of a metric space of cardinality τ, then*

$$ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I)) \leq \tau \cdot kcov(ec(X)^\omega).$$

Let X be a sequential space and $x \in X$. A *base of sequences centered in x* is a collection $\mathcal{S}$ of sequences in $X \setminus \{x\}$ converging to x and such that, for each sequence $(y_n)_{n<\omega}$ in $X \setminus \{x\}$ which converges to x, there is a sequence $(x_n)_{n<\omega} \in \mathcal{S}$ such that $|\{x_n : n < \omega\} \cap \{y_n : n < \omega\}| = \aleph_0$. Denote by $Seq(X,x)$ the minimum element in $\{|\mathcal{S}| : \mathcal{S}$ is a base of sequences centered in $x\}$. Let $Seq(X) = sup\{Seq(X,x) : x \in X\}$. It is possible to prove that every sequential space X has a sequential base with fan number $\leq Seq(X)$; so the following result follows from Theorem 5.3.

5.9. Corollary. *For every non-discrete sequential space X we have*

$$ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I) \leq Seq(X) \cdot kcov(ec(X)^\omega).$$

6. The Čech number of $C_p(X)$ when X is a k-space

6.1. Definition. A family $\mathcal{D}$ of closed subsets of a space X *determines the topology* of X if $F \subset X$ is closed in X iff $F \cap D$ is closed in D for every $D \in \mathcal{D}$.

Note that for a collection $\mathcal{D}$ of closed subsets which determines the topology of X, $X \setminus \bigcup \mathcal{D}$ is discrete and clopen, and every base for the closed subsets of X determines the topology of X. Furtheremore, $\mathcal{D} = \emptyset$ or $\mathcal{D} = \{\emptyset\}$ determines the topology of X iff X is discrete.

6.2. Lemma. *Let $\mathcal{D}$ be a collection of closed subsets of a topological space X which determines its topology. Let Y be a topological space. A function $f : X \to Y$ is continuous if and only if $f \restriction D$ is continuous for every $D \in \mathcal{D}$.*

Proof. Evidently, if $f : X \to Y$ is continuous, $f \restriction D$ is continuous. Now, assume that $g : X \to Y$ is a non-continuous function; so there exists a closed subset F of Y such that $g^{-1}[F]$ is not a closed subset of X. This means that there is $D \in \mathcal{D}$ for which $D \cap g^{-1}[F] = (g \restriction D)^{-1}[F]$ is not closed in D. But this implies that $g \restriction D$ is not a continuous function. □

6.3. Theorem. *Let $\mathcal{D}$ be a collection of closed subsets of a non-discrete space X which determines the topology of X. Then $\check{C}(C_p(X,I)) \leq |\mathcal{D}| \cdot sup_{D\in\mathcal{D}}\check{C}(C_p(D,I))$.*

Proof. For each $D \in \mathcal{D}$, denote by κ_D the cardinal number $\check{C}(C_p(D,I))$, and let $\{V^D_\alpha : \lambda < \kappa_D\}$ be a collection of open sets in I^D such that $C_p(D,I) = \bigcap_{\alpha<\kappa_D} V^D_\lambda$. For each $\alpha < \kappa_D$ we take the set $W^D_\alpha = I^{X\setminus D} \times V^D_\alpha$. Then, $C_p(X,I) = \bigcap_{D\in\mathcal{D}} \bigcap_{\alpha<\kappa_D} W^D_\alpha$. We obtain the conclusion of the Theorem because each W^D_α is an open subset of I^X. $\square$

For a k-space X we denote by $\mathcal{K}(X)$ a collection of compact subsets of X which determine the topology of X with the smallest possible cardinality, and we denote with $k(X)$ the cardinality of $\mathcal{K}(X)$.

6.4. Corollary. $\check{C}(C_p(X,I)) \leq k(X) \cdot sup_{K\in\mathcal{K}(X)}\check{C}(C_p(K,I))$ *for every non-discrete k-space X.*

If for each point x in a sequential space, $\mathcal{S}_x$ is a base of sequences centered in x, and $\mathcal{S} = \bigcup_{x\in X} \mathcal{S}_x$, then $\{\{x_n : n < \omega\} \cup \{x\} : (x_n)_{n<\omega} \in \mathcal{S}_x,\ x \in X\}$ determines the topology of X. Then:

6.5. Corollary. *For a non-discrete sequential space X,*

$$ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I)) \leq Seq(X) \cdot ec(X) \cdot \mathfrak{d}.$$

This last result improves Corollary 5.9. In particular, if Σ is an almost disjoint collection of subsets of ω, and $\Psi(\Sigma)$ is the Mrowka space determined by Σ, $\check{C}(C_p(\Psi(\Sigma),I)) = |\Sigma| \cdot \mathfrak{d}$. Also, we obtain $\check{C}(C_p(V(\aleph_0),I)) = \mathfrak{d}$ where $V(\aleph_0)$ is the Fréchet fan.

Every countable compact space is a separable metrizable space, so, by the remark made after Theorem 4.2, $\check{C}(C_p(K,I)) = \mathfrak{d}$ for any non-discrete countable compact space K. Also, by Theorem 4.2, every non-discrete countable and sequential EG-space X satisfies $\check{C}(C_p(X,I)) = \mathfrak{d}$. As was noted to the authors by O. Okunev, not every countable EG-space is sequential. In fact, in [Ar] it is proved that $C_p(I)$ is not sequential (Lemma II.3.1). In this proof, it was considered a countable discrete subspace $\{f_n : n < \omega\}$ of $C_p(I)$ such that $X = \{f_n : n < \omega\} \cup \{\bar{0}\}$ is not sequential, where $\bar{0}$ is the constant function equal to 0. So X is a countable EG-space which is not sequential. Even more: X does not contain a non-trivial convergent sequence. On the other hand, every countable k-space is sequential. Then, for every countable k-space we have:

6.6. Corollary. *For every non-discrete countable k-space X,*

$$\mathfrak{d} \leq \check{C}(C_p(X,I)) \leq k(X) \cdot \mathfrak{d} \leq Seq(X) \cdot \mathfrak{d}.$$

We define θ as the smallest infinite cardinal number such that if (X,τ) is a countable sequential non-discrete space, there is a sequential base $\mathcal{P}$ in X such that $fn(\mathcal{P}) \leq \theta$.

6.7. Proposition. *Let (X,τ) be a Fréchet-Urysohn space. There is a sequential base $\mathcal{P}$ of X such that $fn(\mathcal{P}) \leq \theta \cdot kcov(ec(X)^\omega)$.*

Proof. Let Y be a clopen subspace of X with $|Y| = ec(X)$ and such that $X \setminus Y$ is discrete. Considering Y with its discrete topology, Y^ω can be covered by $kcov(ec(X)^\omega) = \lambda$ compact subsets K_α: $Y^\omega = \bigcup_{\alpha<\lambda} K_\alpha$. We may and shall assume that each K_α is of the type $\prod_{n<\omega} K_{\alpha,n}$, where $K_{\alpha,n}$ is a finite subset of Y. Fix $y \in X$, non isolated in (X,τ). If $(b_n)_{n<\omega}$ is a sequence in Y converging to y with $b_n \neq b_m$ if $n \neq m$, then there is $\beta < \lambda$ such that

$$y \in \mathrm{cl}(\bigcup_{n<\omega}(K_{\beta,n} \setminus \{y\}). \tag{1}$$

As a subspace of (X,τ), $\bigcup_{n<\omega}(K_{\beta,n} \setminus \{y\})$ is sequential and countable, so there is a sequential base $\mathcal{P}_{\beta,y}$ of $\bigcup_{n<\omega}(K_{\beta,n} \setminus \{y\})$ such that

$$fn(\mathcal{P}_{\beta,y}) \leq \theta. \tag{2}$$

Now, we shall consider

$$\mathcal{P} = \bigcup\{\mathcal{P}_{\beta,y} : \beta \text{ and } y \text{ satisfy } (1)\} \cup \{\{z\} : z \notin X'\}, \tag{3}$$

and for each $y \in X'$ we take $\mathcal{P}_y = \{(P_n)_{n<\omega} : (P_n)_{n<\omega} \in \mathcal{P}_{\beta,y}\}$, and for $y \notin X'$, let $\mathcal{P}_y = \{(P_n)_{n<\omega}\}$ where $P_n = \{y\}$ for every $n < \omega$.

Claim: $\mathcal{P}$, with the assignment $y \to \mathcal{P}_y$, constitutes a sequential base of (X,τ).

Conditions (1) and (2)($\Rightarrow$) in Definition 5.1.(a) are evident for this $\mathcal{P}$ and this assignment $y \to \mathcal{P}_y$. Now, let $V \subset X$ be such that for every $x \in V$ and $(P_n)_{n<\omega} \in \mathcal{P}_x$ there exists m such that $P_m \subset V$. We have to prove that V is open. Assume the contrary. Thus, $X \setminus V$ is not closed in (X,τ), so there is a sequence of points in $X \setminus V$, $(b_n)_{n<\omega}$, converging to some $y \in V$. Fix $\beta < \lambda$ such that $(b_n)_{n<\omega} \in \prod_{n<\omega} K_{\beta,n}$. Since $\mathcal{P}_{\beta,y}$ is a sequential base, $V \cap (\bigcup_{n<\omega} K_{\beta,n} \cup \{y\})$ is open in the subspace $\bigcup_{n<\omega} K_{\beta,n} \cup \{y\}$. Hence, there is $\Omega \in \tau$ such that

$$V \cap (\bigcup_{n<\omega} K_{\beta,n} \cup \{y\}) = \Omega \cap (\bigcup_{n<\omega} K_{\beta,n} \cup \{y\}). \tag{4}$$

Since $y \in V$, $y \in \Omega$. Then, there is m_0 such that $b_m \in \Omega$ for every $m \geq m_0$. But this means that $b_m \in V$ for every $m \geq m_0$, which contradicts the choice of $(b_n)_{n<\omega}$. So, V must be open, and $\mathcal{P}$ with the assignment $y \to \mathcal{P}_y$ is a sequential base of (X,τ).

Moreover, $|\mathcal{P}_y| \leq \theta \cdot \lambda$ for every $y \in X$. $\square$

6.8. Corollary. *Let X be a non-discrete Fréchet-Urysohn space. Then,*

$$ec(X) \cdot \mathfrak{d} \leq \check{C}(C_p(X,I) \leq \theta \cdot kcov(ec(X)^\omega).$$

7. $\check{C}(C_p(X))$ FOR COUNTABLE SPACES X

Recall that a separable completely metrizable space is called a *Polish space*; every subspace X of a Polish space which is a continuous image of a Polish space is called *analytic* (that is equivalent to saying that X is the continuous image of the space of irrational numbers with the topology inherited by the real line). A subspace Y of a Polish space X is called *co-analytic* if $X \setminus Y$ is analytic. A separable metrizable space is co-analytic if it is homeomorphic to a co-analytic set in a Polish space.

7.1. Proposition. *If X is a Polish space and $F \subset X$ is co-analytic, then $\check{C}(F) \leq \mathfrak{d}$.*

Proof. We have that $X \setminus F$ is analytic, so it is a continuous image of $\mathbb{P}$. Then $\check{C}(F) = kcov(X \setminus F) \leq kcov(\mathbb{P}) = \mathfrak{d}$. $\square$

The following result is Corollary 21.21 in [Ke]

7.2. Theorem. *Let X be a separable metrizable co-analytic space. Then X is Polish if and only if it contains no closed subset homeomorphic to $\mathbb{Q}$.*

7.3. Proposition. *Let X be a Polish space and let $F \subset X$ be co-analytic. If F is not Polish, then $\check{C}(F) = \mathfrak{d}$.*

Proof. By Proposition 7.1, $\check{C}(F) \leq \mathfrak{d}$. Now, since F is not Polish it contains a closed copy of $\mathbb{Q}$. Hence, $\mathfrak{d} = \check{C}(\mathbb{Q}) \leq \check{C}(F)$. $\square$

If X is a countable space, then $C_p(X, I)$ is a separable metrizable subspace of the Polish space I^ω. Moreover, if X is not discrete, $C_p(X, I)$ is not completely metrizable; so, in particular, if $C_p(X, I)$ is co-analytic it must contain a closed copy of $\mathbb{Q}$. Following this line of thoughts we get the following:

7.4. Theorem. *If X is a countable non-discrete EG-space, then $\check{C}(C_p(X, I)) = \mathfrak{d}$.*

Proof. Theorem 4.2 gives us $\check{C}(C_p(X, I)) \leq \mathfrak{d}$. On the other hand, by Proposition 4.5, $C_p(X, I)$ is co-analytic. Since X is not discrete, $C_p(X, I)$ is not a Polish space. Hence, $C_p(X, I)$ contains a closed copy of $\mathbb{Q}$ (Theorem 7.2). This implies that $\mathfrak{d} \leq \check{C}(C_p(X, I))$. $\square$

Of course, there are countable *EG*-spaces which are not metrizable. One way to construct such spaces is as follows: Let A be a dense countable subset of $C_p([0,1])$. A is a countable *EG*-space, and A is not metrizable.

On the other hand, it is possible to give an example of a countable space which is not a *EG*-space. The Fréchet fan $V(\aleph_0)$ is a classic example.

7.5. Lemma. *Let $a \in X$ and let $\{V_\alpha : \alpha < \kappa\}$ be a fundamental system of neighborhoods of a in X. Let $H_a = \bigcap_{n<\omega} \bigcup_{E \in \mathcal{E}_n} \bigcup_{\alpha<\kappa} (I^{X \setminus V_\alpha} \times E^{V_\alpha})$ (see Notations 5.2). Then, H_a is the set of all functions from X into I continuous at a.*

Proof. Let $g : X \to I$ be continuous at a. For each $n < \omega$ there is $E \in \mathcal{E}_n$ such that $g(a) \in E$ and, since g is continuous at a, there is $\beta < \kappa$ such that $g[V_\beta] \subset E$, hence $g \in I^{X \setminus V_\beta} \times E^{V_\beta}$ and $g \in H_a$.

On the other hand, let $h \in I^X$ be a non continuous function at a. Hence, there is $n < \omega$ with the property

$$(*) \qquad h[V_\alpha] \not\subset (h(a) - (1/2^{n+1}), h(a) + (1/2^{n+1}))$$

for all $\alpha < \kappa$. If $E \in \mathcal{E}_n$, the length of E is $\leq 1/2^{n+1}$. Because of (*), $h \notin I^{X\setminus V_\alpha} \times E^{V_\alpha}$ for all $E \in \mathcal{E}_n$, which implies $h \notin H_a$. $\square$

For $a \in X$, denote by $\widetilde{\Delta}(X,a)$ the cardinal number $\min\{|V| : V$ is open and $a \in V\}$, and let $\widetilde{\Delta}(X) = sup\{\widetilde{\Delta}(X,a) : a \in X\}$.

7.6. Corollary. *For every non-discrete space X,*

$$ec(X) \leq \check{C}(C_p(X,I)) \leq ec(X) \cdot kcov(\chi(X)^{\widetilde{\Delta}(X)}).$$

Proof. Let Y be an essential subspace of X of cardinality $ec(X)$ ($X \setminus Y$ is discrete and clopen). Let $\kappa = \chi(Y)$, and for each $y \in Y$, $\{V_{y,\alpha} : \alpha < \kappa\}$ be a fundamental system of neighborhoods of y in Y. Because of Lemma 7.5,

$$C_p(Y,I) = \bigcap_{y\in Y} \bigcap_{n<\omega} \bigcup_{E\in\mathcal{E}_n} \bigcup_{\alpha<\kappa} (I^{X\setminus V_{y,\alpha}} \times E^{V_{y,\alpha}}).$$

On the other hand,

$$I^{X\setminus V_{y,\alpha}} \times E^{V_{y,\alpha}} = \bigcap_{b\in V_{y,\alpha}} (I^{X\setminus\{b\}} \times E^{\{b\}}).$$

So,

$$\bigcup_{E\in\mathcal{E}_n} \bigcup_{\alpha<\kappa} (I^{X\setminus V_{y,\alpha}} \times E^{V_{y,\alpha}}) = \bigcup_{E\in\mathcal{E}_n} \bigcup_{\alpha<\kappa} \bigcap_{b\in V_{y,\alpha}} (I^{X\setminus\{b\}} \times E^{\{b\}}).$$

Let us call this last set G_y. Since $I^{X\setminus\{b\}} \times E^{\{b\}}$ is an open set in I^Y, $|\mathcal{E}_n| < \aleph_0$ and $\kappa \leq \chi(Y)$, G_y is a $G_{\widetilde{\Delta}(Y),\chi(Y)}$-set. Then $\check{C}(G_y) \leq kcov(\chi(Y)^{\widetilde{\Delta}(Y)})$ (Corollary 3.5 in [OT]). So $\check{C}(C_p(X,I)) = \check{C}(C_p(Y,I)) \leq ec(X) \cdot kcov(\chi(Y)^{\widetilde{\Delta}(Y)})$ (Corollary 1.11 in [OT]). But $\chi(Y) = \chi(X)$ and $\widetilde{\Delta}(Y) = \widetilde{\Delta}(X)$; so, we have finished the proof of this corollary. $\square$

7.7. Corollary. *For every countable space X, $\check{C}(C_p(X,I)) \leq kcov(\chi(X)^\omega)$.*

In [LMP], Lutzer, van Mill and Pol defined for each subset S of the Cantor set 2^ω a topological space Σ_S as follows: let $T_n = 2^n$ be the set of functions from $\{0,1,...,n-1\}$ into $\{0,1\}$. Let $T = \bigcup_{n\geq1} T_n$ and partially order T by function extension. A *branch* of T is a maximal linearly ordered subset of T, i.e., a linearly ordered subset $B \subset T$ having $|B \cap T_n| = 1$ for each $n \geq 1$. Given $x \in 2^\omega$, the set $B_x = \{(x(0)),(x(0),x(1)),(x(0),x(1),x(2)),...\}$ is a branch of T. Conversely, each branch B of T has the form $B = B_x$ for a unique $x \in 2^\omega$. For each subset $S \subset 2^\omega$, the collection $\{T \setminus (B_{x_1} \cup ... \cup B_{x_n} \cup F) : n \geq 1, x_i \in S$ and $F \in [T]^{<\omega}\}$ is a filter base. Let p_S be the filter generated by that filter base. Let $\Sigma_S = T \cup \{p_S\}$. Topologize Σ_S by isolating each point of T and by using the family $\{P \cup \{p_S\} : P \in p_S\}$ as a neighborhood base at p_S. All spaces Σ_S are Fréchet (the sequence $(1),(0,1),(0,0,1),(0,0,0,1),...$ converges to p_S). So:

7.8. Proposition. *For every $S \subset 2^\omega$,*

$$\mathfrak{d} \leq \check{C}(C_p(\Sigma_S)) = \check{C}(C_p(\Sigma_S,I)).$$

7.9. Proposition. *If $S \subset 2^\omega$ is a co-analytic subset of 2^ω, then $\mathfrak{d} = \check{C}(C_p(\Sigma_S)) = \check{C}(C_p(\Sigma_S, I))$.*

Proof. Since S is co-analytic in 2^ω, then $C_p(\Sigma_S, I)$ is co-analytic in I^{Σ_S} ([LMP], Theorem 3.1); so, $I^{\Sigma_S} \setminus C_p(\Sigma_S, I)$ is analytic. This means that

$$kcov(I^{\Sigma_S} \setminus C_p(\Sigma_S, I) \leq \mathfrak{d}.$$

Therefore, $\check{C}(C_p(\Sigma_S, I)) \leq \mathfrak{d}$. By Proposition 7.8, we must have $\check{C}(C_p(\Sigma_S, I)) = \mathfrak{d}$. $\square$

For each filter $\mathcal{F}$ on ω, we define the space $X_\mathcal{F} = \omega \cup \{\mathcal{F}\}$ with the following topology: each point in ω is isolated and a basic system of neighborhoods for $\mathcal{F}$ are the sets of the form $\{\mathcal{F}\} \cup F$ where $F \in \mathcal{F}$.

In [C], Jean Calbrix considers, in addition to spaces of type Σ_S, countable spaces constructed as follows: Let X be a non-discrete metrizable separable space which contains a countable dense set ω where each of its points is isolated. Let $A = X \setminus \omega$. We identify A to a point p and we obtain the quotient space $X_\mathcal{F} = \omega \cup \{p\}$, where $\mathcal{F}$ is the filter in ω generated by the trace in ω of the basic system of neighborhoods of p in $X_\mathcal{F}$. Calbrix called this kind of filters as filters of type $\mathcal{A}$. He proved also that every countable k-space with only a non-isolated point is Fréchet-Urysohn ([C], Lemma 2.1), and he mentions that for every filter $\mathcal{F}$ of type $\mathcal{A}$, $X_\mathcal{F}$ is a sequential space. Thus for this kind of spaces, we have:

7.10. Corollary. *For an $\mathcal{A}$-filter $\mathcal{F}$, $\mathfrak{d} \leq \check{C}(C_p(X_\mathcal{F}, I)) \leq |X| \cdot \mathfrak{d}$. In particular, if $|X| \leq \mathfrak{d}$, $\check{C}(C_p(X_\mathcal{F}, I)) = \mathfrak{d}$.*

Proof. This is a consequence of Corollary 5.8. $\square$

The final remarks of this section are due to a colleague. We truly thank him for his comments.

It is consistent with ZFC that there is a countable space X such that $\mathfrak{d} < \check{C}(C_p(X, I))$. Indeed, let S be a Bernstein set in the Cantor set 2^ω (that is, both S and $2^\omega \setminus S$ do not contain perfect subsets of 2^ω). Since all compact subsets of S and $2^\omega \setminus S$ are countable and $|S| = |2^\omega \setminus S| = 2^\omega$, we have that $kcov(S) = \check{C}(S) = 2^\omega$. Now, consider the corresponding space Σ_S defined in some paragraphs above. Since Σ_S is countable, $kcov(\omega^{|\Sigma_S|}) = \mathfrak{d}$. From the fact that $C_(\Sigma_S)$ contains a closed copy of S ([LMP]), it follows that $kcov(C_p(\Sigma_S)) = \check{C}(C_p(\Sigma_S)) = 2^\omega$.

This example also shows that the strict inequality in Corollary 2.7 and the relation $\check{C}(C_p(X, I)) > kcov(ec(X)^{t_\mathbb{R}})$ (see Theorem 3.2) can consistently happen.

Since for every metrizable and separable space X, $|X| \leq 2^\omega$, if $\mathfrak{d} = 2^\omega$, then equalities in Corollary 7.10 hold. On the other hand, assume that $\mathfrak{d} < 2^\omega$ and let X be an uncountable compact metrizable space containing a countable dense set ω consisting of isolated points. Then $X_\mathcal{F}$ is homeomorphic to a convergent sequence. Hence, $\check{C}(C_p(X_\mathcal{F}, I)) = \mathfrak{d} < 2^\omega = |X|$.

8. Relations between $\check{C}(C_p(X))$ and other cardinal topological functions

For each space X we define $\mathrm{nov}(X) = \min\{|\mathcal{C}| : \mathcal{C}$ is a cover of X constituted by nowhere dense subsets of $X\}$, and, if X is realcompact, $Exp(X) = \min\{\kappa : X$ can be embedded as a closed subset of $\mathbb{R}^\kappa\}$. We also consider $\mathfrak{N} = \min\{\tau : nov(I^\tau) = \tau\}$

8.1. Lemma. *([vD], Lemma 8.19) Let X be a separable metrizable space (or more generally, a space that has a perfectly normal compactification) that is not locally compact. Then $Exp(X) = kcov(bX \setminus X)$ for each compactification bX of X.*

8.2. Corollary. *Let X be a non-discrete countable space. Then, $\check{C}(C_p(X, I)) = Exp(C_p(X, I))$.*

We always have that $\omega_1 \leq \mathfrak{N} \leq nov(\mathbb{R}) \leq \mathfrak{d}$ ([M]), and Martin's Axiom implies $\mathfrak{N} = \mathfrak{c}$. The following result can be proved similarly to Theorem 5.3 in [OT].

8.3. Proposition. *If X is a non-discrete countable space, then $\check{C}(C_p(X, I)) \geq nov(\mathbb{R})$.*

8.4. Lemma. *For every cardinal number $\tau \geq \omega$, $nov(\{0,1\}^\tau) = nov([0,1]^\tau)$.*

Proof. It is well known that there is an onto continuous irreducible map $G : \{0,1\}^\tau \to [0,1]^\tau$. If $S \subset \{0,1\}^\tau$ is closed and nowhere dense, then $G[S]$ is closed and nowhere dense. And for each closed and nowhere dense subset $S \subset [0,1]^\tau$, $G^{-1}[S]$ is nowhere dense. Therefore, $nov(\{0,1\}^\tau) = nov([0,1]^\tau)$. □

In [OT] it is asked if $nov(\mathbb{R})$ is equal to $\mathfrak{N} = \min\{\tau : nov(I^\tau) = \tau\}$ ([OT], Question 5.2). It was proved in [M] that it is consistent with ZFC that $nov(\mathbb{R}) = \aleph_2$ and $nov(\{0,1\}^{\omega_1}) = \omega_1$. This proves, using Lemma 8.4, that the answer to Question 5.2 in [OT] is consistently negative.

It is easy to prove that $\check{C}(C_p(X, I)) \geq \check{C}(C_p(X, \{0,1\})) \geq nov(I^\tau)$ for every non-discrete 0-dimensional space X, where $\tau = ec(X)$.

In [OT] it is asked what the Čech number of the Σ-product of ω_1 copies of $[0,1]$ is; and it is remarked that the relation $\omega_1 \leq \check{C}(\Sigma I^{\omega_1}) \leq \mathfrak{d}$ holds. Observe that ΣI^{ω_1} and $I^{\omega_1} \setminus \Sigma I^{\omega_1}$ are dense subsets of I^{ω_1}, so $nov(I^{\omega_1}) \leq \check{C}(\Sigma I^{\omega_1})$. On the other hand, $\Sigma(0,1)^{\omega_1} \cong C_p(Y)$, where Y is the one point Lindelöfication of the discrete space of cardinality ω_1. Moreover, $\Sigma(0,1)^{\omega_1} = \bigcap_{\lambda<\omega_1} A_\lambda$ where $A_\lambda = \{f \in \Sigma I^{\omega_1} : f(\lambda) \in (0,1)\}$. Since A_λ is an open subset of ΣI^{ω_1}, $\Sigma(0,1)^{\omega_1}$ is a G_{ω_1}-set of ΣI^{ω_1}. This means that $\check{C}(\Sigma(0,1)^{\omega_1}) \leq \omega_1 \cdot \check{C}(\Sigma I^{\omega_1})$ (see Proposition 1.12 in [OT]). On the other hand, ΣI^{ω_1} is a closed subset of $\Sigma \mathbb{R}^{\omega_1}$. So, we have:

8.5. Proposition. $\omega_1 \leq nov(I^{\omega_1}) \leq \check{C}(\Sigma I^{\omega_1}) = \check{C}(\Sigma \mathbb{R}^{\omega_1}) \leq kcov((\omega_1)^\omega) = \mathfrak{d}$.

By the way, $ec(\Sigma I^{\omega_1}) = \mathfrak{c}$, then, using Corollary 4.8 in [OT] we know that $\check{C}(C_p(\Sigma I^{\omega_1}, I)) \geq \mathfrak{c}$. Moreover, Kombarov proved in [Ko] that the tightness of ΣI^{ω_1} is $\aleph_0$, so, by Theorem 3.2, $\mathfrak{c} \geq \check{C}(C_p(\Sigma I^{\omega_1}, I))$. In summary:

8.6. Proposition. $\check{C}(C_p(\Sigma I^{\omega_1}, I)) = \mathfrak{c}$.

9. The Cech number of $C_p(X)$ and additional axioms consistent with ZFC

Let X be a nondiscrete space. We know that $\check{C}(C_p(X)) \geq \omega_1$; so, CH implies that $\check{C}(C_p(X)) \geq \mathfrak{c}$. We get the same conclusion if we assume MA: Let $\omega \leq \kappa < \mathfrak{c}$. $MA(\kappa)$ implies that if $(\Omega_\alpha)_{\alpha<\kappa}$ is a family of open dense subsets of $\mathbb{R}^X$, then $\bigcap_{\alpha<\kappa} \Omega_\alpha$ is dense in $\mathbb{R}^X$.

9.1. Proposition. *If we assume $MA(\kappa)$ and X is not discrete, then $\check{C}(C_p(X)) > \kappa$.*

Proof. Assume that $\check{C}(C_p(X)) \leq \kappa$. We may assume that $ec(X) = |X|$. Then, $|X|$ must be $\leq \kappa$. We have $C_p(X) = \bigcap_{\alpha<\kappa} \Omega_\alpha$ and Ω_α is an open subset of $\mathbb{R}^X$

Fix $f_0 : X \to \mathbb{R}$ non continuous (it exists because X is not discrete). $f_0 + C_p(X)$ and $C_p(X)$ are dense in $\mathbb{R}^X$, $W_\alpha = f_0 + \Omega_\alpha$ is open and dense in $\mathbb{R}^X$ for each $\alpha < \kappa$; hence $(f_0 + C_p(X)) \cap C_p(X)$ is not empty, which implies that f_0 would be continuous; a contradiction. □

9.2. Corollary. *$MA(\kappa)$ implies $\kappa < \mathfrak{d}$.*

Proof. If $MA(\kappa)$ and $\kappa = \mathfrak{d}$, then for every non-discrete space X, $\check{C}(C_p(X)) > \mathfrak{d}$ (Proposition 9.1). But this is not the case, for example, $\check{C}(C_p([0,\omega])) = \mathfrak{d}$. □

Martin Axiom is equivalent to $MA(\kappa)$ for every $\kappa < \mathfrak{c}$. (Also MA implies $\mathfrak{d} = \mathfrak{c}$, see Theorem 5.1 in [vD]) Then, by Proposition 9.1, we get:

9.3. Corollary. *If we assume MA and X is not discrete, then $\check{C}(C_p(X,I)) \geq \mathfrak{c} = \mathfrak{d}$.*

Proof. Let Y be a clopen subset of X such that $|Y| = ec(X)$ and $X \setminus Y$ is discrete. We have that $\check{C}(C_p(X,I)) = \check{C}(C_p(Y,I)) = \check{C}(C_p(Y))$. Since Y is not discrete, Proposition 9.1 guaranties that $\check{C}(C_p(Y)) \geq \mathfrak{d} = \mathfrak{c}$. □

We will finish this section proving that it is consistent with ZFC the existence of a countable space X for which $\omega_1 = \check{C}(C_p(X,I)) < \mathfrak{d}$.

9.4. Proposition. *Let $q \in \omega^*$ and consider the set $X = \omega \cup \{q\}$ as a subspace of $\beta\omega$. Then, $\check{C}(C_p(X,I)) \leq \chi(q, \beta(\omega))$.*

Proof. Let $g : X \to I$ non continuous; then it is not continuous at q and there is $\omega > m \geq 1$ such that

$$g^{-1}[(g(q) - 1/m, g(q) + 1/m)] \notin q.$$

Since q is an ultrafilter, then $U = X \setminus g^{-1}[(g(q) - 1/m, g(q) + 1/m)] \in q$. So, $t \in U$ implies $|g(t) - g(q)| \geq 1/m$.

Let $\mathcal{B}$ be a local base of q in $\beta\omega$ of cardinality $\chi(q, \beta(\omega))$. We have that

$$I^X \setminus C_p(X,I) = \bigcup_{1\leq m} \bigcup_{U\in\mathcal{B}} \{f \in I^X : |f(t) - f(q)| \geq 1/m \ \forall\ t \in U\}.$$

Claim. $F_{U,m} = \{f \in I^X : |f(t) - f(q)| \geq 1/m \ \forall\ t \in U\}$ is closed in I^X.

Indeed, assume $h \in cl(F_{U,m}) \setminus F_{U.m}$; then, there is $b \in U$ such that $|h(b) - h(q)| < 1/m$. Choose $r_1, r_2 > 0$ such that $r_1 + r_2 + |h(b) - h(q)| < 1/m$, and consider $V = \prod_{x\in X} V_x$ the open neighborhood of h where $V_x = I$ if $x \notin \{q, b\}$, $V_b = (h(b) - r_1, h(b) + r_1)$ and $V_q = (h(q) - r_2, h(q) + r_2)$. Choose $f \in F_{U,m} \cap V$; it follows that $|f(b) - h(b)| < r_1$ and $|f(q) - h(q)| < r_2$. Thus, we have

$$1/m > |h(b) - h(q)| + r_1 + r_2 > |f(b) - f(q)| \geq 1/m,$$

which is a contradiction. □

The following example answers question 5.1 in [OT] in the affirmative.

9.5. Example. *There is a model M of ZFC containing a countable space X with the property $\check{C}(C_p(X,I)) = \omega_1 < \mathfrak{d}$.* (We thank Professor Frank Tall for his concern to this problem and for leading us to [BS]).

Proof. In [BS] it was proved that there is a model M of ZFC in which there is a free ultrafilter q with $\chi(q, \beta(\omega)) = \aleph_1$ and $\mathfrak{d} = \aleph_2$. So, in M, $\check{C}(C_p(\omega \cup \{q\}, I)) = \aleph_1 < \mathfrak{d}$. $\square$

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Universidade de São Paulo, Caixa Postal 66281, CEP 05311-970, São Paulo, Brasil
E-mail address: alas@ime.usp.br

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, México D.F. 04510, México.
E-mail address: atamariz@servidor.unam.mx