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THE α-BOUNDIFICATION OF α

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ABSTRACT. A space X is $< \alpha$-bounded if for all $A \subseteq X$ with $|A| < \alpha$, $\mathrm{cl}_X A$ is compact. Let $B(\alpha)$ be the smallest $< \alpha$-bounded subspace of $\beta(\alpha)$ containing α. It is shown that the following properties are equivalent: (a) α is a singular cardinal; (b) $B(\alpha)$ is not locally compact; (c) $B(\alpha)$ is α-pseudocompact; (d) $B(\alpha)$ is initially α-compact. Define $B^0(\alpha) = \alpha$ and $B^n(\alpha) = \{\mathrm{cl}_{\beta(\alpha)} A : A \subseteq B^{n-1}(\alpha), |A| < \alpha\}$ for $0 < n < \omega$. We also prove that $B^2(\alpha) \neq B^3(\alpha)$ when $\omega = \mathrm{cf}(\alpha) < \alpha$. Finally, we calculate the cardinality of $B(\alpha)$ and prove that, for every singular cardinal α, $|B(\alpha)| = |B(\alpha)|^\alpha = |N(\alpha)|^{\mathrm{cf}(\alpha)}$ where $N(\alpha) = \{p \in \beta(\alpha) : \text{there is } A \in p \text{ with } |A| < \alpha\}$.

0. INTRODUCTION

In [15] O'Callaghan proved the following properties of the α-boundification $B(\alpha)$ of the discrete space of cardinality α (for definitions see 1.3 and 1.4).

0.1. (a) If α is a regular cardinal, then $B(\alpha)$ is the set of nonuniform ultrafilters on α.

(b) α is a singular cardinal if and only if $B(\alpha)$ contains a uniform ultrafilter.

(c) If we assume one of the following statements:

(i) GCH,

(ii) α is a strong limit cardinal,

(iii) α is a regular cardinal,

then $B(\alpha) \neq \beta(\alpha)$. Moreover, if (i) or (ii) holds, then $|B(\alpha)| \leq 2^\alpha$.

From 0.1 it follows that if α is regular, then $B(\alpha) = N(\alpha) = B^\xi(\alpha)$ for each $0 < \xi < \alpha^+$. Hence, $B(\alpha)$ is known when α is a regular cardinal. Thus, the following question, due to Comfort, appears natural.

0.2. Is $B^2(\alpha) \neq B^3(\alpha)$ whenever α is a singular cardinal?

In this paper we are principally concerned with singular cardinals. It is shown, in §2, that $B(\alpha) \neq \beta(\alpha)$ for every cardinal α (Corollary 2.4), and we will

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also obtain some topological properties of $B(\alpha)$. In §3 we answer Comfort's question 0.2 in the affirmative when $\omega = \mathrm{cf}(\alpha) < \alpha$. In these two sections Kunen's α-good ultrafilters will play an important role. In the last section, the cardinality of $B(\alpha)$ is calculated and we will prove that $|B(\alpha)| = |B(\alpha)|^\alpha = |N(\alpha)|^{\mathrm{cf}(\alpha)}$ for every singular cardinal.

1. Preliminaries

Throughout this paper, all spaces are assumed to be completely regular and Hausdorff. If X is a space and $B \subseteq X$, $\mathrm{cl}_X B$ denotes the closure of B in X. For $x \in X$, $\mathscr{N}(x)$ is the set of neighborhoods of x in X. $\mathscr{P}(X)$ is the set of all subsets of a set X. The Greek letters stand for ordinal numbers; in particular, α, κ, θ denote infinite cardinal numbers; γ, ν, μ denote arbitrary cardinals; and δ, ξ, λ, η denote ordinal numbers. For a cardinal α, we let α^+ stand for the smallest cardinal greater than α. For κ, γ cardinals we set $[\kappa]^\gamma = \{M \subseteq \kappa : |M| = \gamma\}$ and $[\kappa]^{<\gamma} = \{M \subseteq \kappa : |M| < \gamma\}$.

We do not distinguish notationally between a cardinal number α and the discrete space whose underlying set is that cardinal. For a space X, βX stands for the Stone-Čech compactification of X. If $f: X \to Y$ is a continuous function, we let $\overline{f}: \beta X \to \beta Y$ stand for the Stone extension of f. The remainder of βX is $X^* = \beta X \backslash X$; in particular, $\alpha^* = \beta(\alpha)\backslash\alpha$. For $A \subseteq \alpha$ we have that (see [2, Chapter 2])

$$\widehat{A} = \{p \in \beta(\alpha) : A \in p\} = \mathrm{cl}_{\beta(\alpha)} A \quad \text{and} \quad A^* = \widehat{A}\backslash A .$$

We shall use the terminology and notation of Comfort and Negrepontis [2].

The notion of α-bounded space was introduced in [7] and modified by Comfort as follows.

1.1. **Definition.** A space X is $<\alpha$*-bounded* if for every $A \subseteq X$ of cardinality less than α, $\mathrm{cl}_X A$ is a compact set.

It is evident that every space is $<\omega$-bounded, and if X is $<\alpha$-bounded, then X is $<\gamma$-bounded for every $\gamma \leq \alpha$. The basic properties of $<\alpha$-bounded spaces are summarized in the following proposition (see, e.g., [7, 8]).

1.2. **Proposition.** *Let α be a cardinal number. Then*

(a) *every compact space is $<\alpha$-bounded*;
(b) *every closed subset of a $<\alpha$-bounded space is $<\alpha$-bounded*;
(c) *the product of a set of $<\alpha$-bounded spaces is $<\alpha$-bounded*;
(d) *the intersection of a set of $<\alpha$-bounded spaces is $<\alpha$-bounded*;
(e) *the continuous image of a $<\alpha$-bounded space is $<\alpha$-bounded.*

Notice that (d) is a particular case of Lemma 2 of [8], and it is a consequence of (b) and (c).

1.3. For a $<\alpha$-bounded space Z and $X \subseteq Z$, we set

$$B_\alpha(X, Z) = \bigcap\{Y : X \subseteq Y \subseteq Z \text{ and } Y \text{ is } <\alpha\text{-bounded}\}.$$

It follows from 1.2(d) that $B_\alpha(X, Z)$ is the smallest $<\alpha$-bounded space containing X and is contained in Z. If $Z = \beta X$, then $B_\alpha(X, Z)$ will be denoted by $B_\alpha(X)$. In this case $B_\alpha(X)$ has the following extension property: For each

$<\alpha$-bounded space Y and each continuous function $f: X \to Y$, there exists a continuous function $\hat{f}: B_\alpha(X) \to Y$ such that $\hat{f}|_X = f$ [5, 8]. $B_\alpha(X)$ is called the *α-boundification* of X.

1.4. *Notation.* For a space X, we define

$B^0_\alpha(X) = X$;

$B^{\xi+1}_\alpha(X) = \bigcup\{\mathrm{cl}_{\beta X} A : A \subseteq B^\xi_\alpha(X)$ and $|A| < \alpha\}$ for each ordinal $\xi < \alpha^+$; and

$B^\xi_\alpha(X) = \bigcup_{\lambda<\xi} B^\xi_\alpha(X)$ if ξ is a limit ordinal.

Notice that $B_\alpha(X) = \bigcup_{\xi<\alpha^+} B^\xi_\alpha(X)$.

Let $p \in \beta(\alpha)$. The norm of p is $\|p\| = \min\{|A| : A \in p\}$, and p is *κ-uniform* if $\|p\| \geq \kappa$. If p is α-uniform, then p is said to be *uniform.* We set $U(\alpha) = \{p \in \beta(\alpha) : \|p\| = \alpha\}$; $N(\alpha) = \{p \in \beta(\alpha) : \|p\| < \alpha\}$; and $W_\kappa(\alpha) = \{p \in \beta(\alpha) : \|p\| = \kappa\}$ $(\kappa < \alpha)$. We write $B(\alpha)$ and $B^\xi(\alpha)$ instead of $B_\alpha(\alpha)$ and $B^\xi_\alpha(\alpha)$, for $\xi < \alpha^+$, respectively. Note that $N(\alpha) = B^1(\alpha)$.

2. Some topological properties of $B(\alpha)$

In this section, we observe that $B(\alpha) \neq \beta(\alpha)$ for every cardinal α (Corollary 2.4), and we prove that some topological conditions are equivalent to the singularity of α (Theorem 2.10). We will first give some definitions and preliminary results.

2.1. **Definition.** Let X be a space, $B \subseteq X$, and $p \in \mathrm{cl}_X B$.

(a) $a(p, B) = \min\{|M| : M \subseteq B$ and $p \in \mathrm{cl}_X M\}$,

(b) p is said to be a *weak P_α-point* of X if $a(p, X\backslash\{p\}) \geq \alpha$. A *weak P-point* is a weak P_{ω_1}-point.

2.2. **Definition.** (a) Let γ be a cardinal. A function $h: [\gamma]^{<\omega} \to \mathscr{P}(X)$ is called *monotone* if $h(A) \subseteq h(B)$ for $A, B \in [\gamma]^{<\omega}$ and $B \subseteq A$, and f is said to be *multiplicative* if $h(A \cup B) = h(A) \cup h(B)$ for $A, B \in [\gamma]^{<\omega}$.

(b) $p \in X$ is *α-good* if for each $\kappa < \alpha$ and each monotone function $f: [\kappa]^{<\omega} \to \mathscr{N}(p)$ there is a multiplicative function $g: [\kappa]^{<\omega} \to \mathscr{N}(p)$ which refines f (i.e., $g(A) \subseteq f(A)$ for all $A \in [\kappa]^{<\omega}$).

2.3. **Proposition.** *Let X be a space.*

(a) *If X is $<\alpha$-bounded and p is a weak P_α-point of X, then $X\backslash\{p\}$ is $<\alpha$-bounded.*

(b) (*Kunen*) *If X is a compact* 0*-dimensional space and $p \in X$ is α^{++}-good, then p is a weak P_{α^+}-point* (*for a proof see* [4, 4.8]).

(c) [14] *There are 2^{2^α} countably incomplete uniform ultrafilters in $\beta(\alpha)$ which are α^+-good.*

As an immediate consequence of Proposition 2.3 we have:

2.4. **Corollary.** (a) *For every limit cardinal α, there are 2^{2^α} countably incomplete weak P_α-points in $U(\alpha)$.*

(b) *$|\beta(\alpha)\backslash B(\alpha)| = 2^{2^\alpha}$ for every α. In particular, we have that $B(\alpha) \neq \beta(\alpha)$ for every cardinal α* (*see* 0.1(c)).

In the next theorem, we are going to give some topological properties of $B(\alpha)$ when α is a singular cardinal. The concepts and results that follow are needed.

2.5. **Definition** (Saks-Woods). Let $M \subseteq \beta(\alpha)$. A space X is said to be *M-compact* if for every function $f: \alpha \to X$ we have that $\overline{f}(p) \in X$ for each $p \in M$.

The definition of p-compactness for a point $p \in \beta(\alpha)$ was given initially by Bernstein [1]. For other results on spaces required to be p-compact simultaneously for various p, see Woods [18] and Saks [17].

In [10] the topological properties which are productive, closed hereditary, and surjective are characterized in terms of ultrafilters as follows.

2.6. **Proposition.** *Let P be a topological property which is productive, closed hereditary, and surjective. A space X of cardinality α has P if and only if X is $P(\alpha)$-compact, where $P(\alpha)$ is the maximal P-reflection of α. In particular, a space X is $< \alpha$-bounded iff X is $B(\alpha)$-compact.*

2.7. **Definition.** Let X be a space and $\omega \leq \alpha$.

(a) X is said to be a *α-pseudocompact* if every continuous image of X in $\mathbb{R}^\alpha$ is compact.

(b) X is *initially α-compact* if every open cover $\mathscr{U}$ of X, with $|\mathscr{U}| \leq \alpha$, has a finite subcover.

The following lemma is due to Retta [16].

2.8. **Lemma.** *Let X be a space. Then X is α-pseudocompact if and only if every cozero cover of X of cardinality $\leq \alpha$ has a finite subcover.*

2.9. **Lemma** [6]. *If α is singular, then every $< \alpha$-bounded space is initially α-compact.*

Now we will prove the main result of this section.

2.10. **Theorem.** *The following conditions are equivalent.*

(a) *α is a singular cardinal.*
(b) *$B(\alpha)$ is not locally compact.*
(c) *$B(\alpha)$ is α-pseudocompact.*
(d) *Every $< \alpha$-bounded space is α-pseudocompact.*
(e) *$B(\alpha)$ is initially α-compact.*
(f) *Every $< \alpha$-bounded space is initially α-compact.*

Proof. (d) $\Rightarrow$ (c) and (f) $\Rightarrow$ (e) are trivial, and (a) $\Rightarrow$ (f), (a) $\Rightarrow$ (d), and (e) $\Rightarrow$ (c) are direct consequences of Lemmas 2.8 and 2.9. In order to complete the proof we will show (a) $\Leftrightarrow$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose that α is singular and $B(\alpha)$ is locally compact. Then $B(\alpha) \cap U(\alpha)$ is a nonempty open subset of $U(\alpha)$. Fix an arbitrary $p \in U(\alpha)$. We will show that $p \in B(\alpha)$. Indeed, since the type $T(p) = \{q \in \beta(\alpha) :$ there is a permutation h of α such that $\overline{h}(p) = q\}$ of p is a dense subset of $U(\alpha)$ (see [2] for a proof), there is $q \in T(p) \cap B(\alpha) \cap U(\alpha)$. Choose a permutation f of α such that $\overline{f}(q) = p$. According to Proposition 2.6, we have that $B(\alpha)$ is $B(\alpha)$-compact and so $\overline{f}(q) = p \in B(\alpha)$; thus, $p \in B(\alpha)$. But this implies that $\beta(\alpha) = B(\alpha)$, a contradiction to Corollary 2.4(b).

(b) $\Rightarrow$ (a) Suppose that α is a regular cardinal. From 0.1(a) it follows that $B(\alpha) = N(\alpha)$. Since $N(\alpha)$ is open in $\beta(\alpha)$, we have that $B(\alpha)$ is locally compact.

(c) $\Rightarrow$ (a) If α is a regular cardinal and $\mathscr{C} = \{\mathrm{cl}_{\beta(\alpha)}\kappa : \kappa < \alpha\}$, then $\mathscr{C}$ is a cozero cover of $N(\alpha) = B(\alpha)$ of cardinality α without a finite subcover. Now Retta's result (Lemma 2.8) implies that $B(\alpha)$ is not α-pseudocompact.

3. The sets $B^2(\alpha)$ and $B^3(\alpha)$

Our main goal here is to give an answer to Comfort's question 0.2 in the affirmative when $\omega = \mathrm{cf}(\alpha) < \alpha$ (see 3.5, 3.7, and 3.8). Because of 0.1, we will only be concerned with singular cardinals. Thus, throughout this section, α will denote a singular cardinal.

3.1. **Definition** (Keisler [11]). For $p \in X$, let

$$G(p) = \min\{\gamma : \gamma \text{ is a cardinal number and } p \text{ is not } \gamma^+\text{-good}\}.$$

$G(p)$ is called the *degree of goodness of* p.

3.2. We point out that $B^2(\alpha)\backslash B^1(\alpha)$ is dense in $U(\alpha)$. Indeed, let $\{\kappa_\xi : \xi < \mathrm{cf}(\alpha)\}$ be a strictly increasing sequence of cardinals converging to α ($\kappa_\xi \nearrow \alpha$), and let $A \in [\alpha]^\alpha$. Choose $p_\xi \in W_{\kappa_\xi}(\alpha) \cap \widehat{A}$ for each $\xi < \mathrm{cf}(\alpha)$. If $p \in \beta(\alpha)$ is a complete accumulation point of $\{p_\xi : \xi < \mathrm{cf}(\alpha)\}$, then $p \in (B^2(\alpha)\backslash B^1(\alpha)) \cap \widehat{A}$.

We will prove in 3.5 that, for each $\kappa < \alpha$, the subset of $B^2(\alpha)\backslash B^1(\alpha)$ of all ultrafilters of degree of goodness equal to κ^+ is dense in $U(\alpha)$. For our purpose we need the following two lemmas (they are Theorems 10.5 and 10.6 of [2], respectively).

3.3. **Lemma** (Keisler [12]). *Let $\omega \leq \gamma \leq \kappa$, let r be a function from κ onto γ, and let $e: \beta(\gamma) \to \beta(\kappa)$ be a continuous function such that $\overline{r} \circ e$ is the identity function on $\beta(\gamma)$. If $q \in \beta(\gamma)$ is countably incomplete and $p = e(q) \in \beta(\kappa)$, then p and q have the same degree of goodness.*

Let κ be a cardinal number. A family $\mathscr{F}$ of subsets of κ is said to have *the uniform finite intersection property* if $\mathscr{F} \neq \varnothing$ and $|\bigcap_{k \leq n} A_k| = \kappa$ whenever $n < \omega$ and $A_k \in \mathscr{F}$ for every $k \leq n$.

3.4. **Lemma.** *Let $\omega \leq \gamma \leq \kappa$. Every family of subsets of κ with the uniform finite intersection property and of cardinality at most κ is contained in 2^{2^κ} distinct uniform ultrafilters each of which is countably incomplete and has degree of goodness equal to γ^+.*

By an *α-partition* of α we mean a collection $\mathscr{F}$ of subsets of α such that: (a) $\alpha = \bigcup \mathscr{F}$; (b) $|A| = \alpha$ for every $A \in \mathscr{F}$; and (c) $A \cap B = \varnothing$ whenever A and B are distinct elements of $\mathscr{F}$.

3.5. **Theorem.** *For every $\kappa < \alpha$, the set*

$$\{p \in \beta(\alpha) : p \in B^2(\alpha) \cap U(\alpha) \text{ and } G(p) = \kappa^+\}$$

is dense in $U(\alpha)$.

Proof. Let $A \in [\alpha]^\alpha$, $\{A_\xi : \xi < \kappa\}$ be an α-partition of A, and $\{\alpha_\eta : \eta < \mathrm{cf}(\alpha)\}$ be a strictly increasing sequence of cardinals converging to α. For every $(\xi, \eta) \in \kappa \times \mathrm{cf}(\alpha)$, pick $p(\xi, \eta) \in \widehat{A_\xi} \cap W_{\alpha_\eta}(\alpha)$. For every $\xi < \kappa$ we choose a complete accumulation point p_ξ of $\{p(\xi, \eta) : \eta < \mathrm{cf}(\alpha)\}$. It is not difficult to see that $p_\xi \in \widehat{A_\xi} \cap U(\alpha) \cap B^2(\alpha)$ for each $\xi < \kappa$. Let $f: \kappa \to \beta(\alpha)$ be defined by $f(\xi) = p_\xi$ for $\xi < \kappa$. According to Lemma 3.4, we can take a countably incomplete ultrafilter $q \in \beta(\kappa)$ with $G(q) = \kappa^+$. Then $\overline{f}(q) \in B^2(\alpha) \cap U(\alpha) \cap \widehat{A}$ and, by Lemma 3.3, $G(\overline{f}(q)) = \kappa^+$.

The following theorem answers question 0.2 in the affirmative when $\omega = \mathrm{cf}(\alpha) < \alpha$ (see Corollary 3.8). We need the following lemma; its proof is standard in showing that regular Lindelöf spaces are normal.

3.6. **Lemma.** *Let X be a normal space. Let $E = \bigcup_{n<\omega} E_n$ and $D = \bigcup_{n<\omega} D_n$ be subsets of X such that $\mathrm{cl}_X(E_n) \cap \mathrm{cl}_X(D) = \mathrm{cl}_X(D_n) \cap \mathrm{cl}_X(E) = \varnothing$ for every $n < \omega$. Then there are two disjoint cozero sets $S, T \subseteq X$ satisfying $E \subseteq S$ and $D \subseteq T$.*

3.7. **Theorem.** *Assume that $\mathrm{cf}(\alpha) = \omega$. For each $n < \omega$, let $p_n \in U(\alpha)$ with $G(p_n) = \kappa_n^+$ where $\kappa_n \nearrow \alpha$. If p is an accumulation point of $D = \{p_n : n < \omega\}$, then $a(p, N(\alpha)) = \alpha$.*

Proof. Let $A \subseteq N(\alpha)$ be of cardinality $\gamma < \alpha$, and let $A_n = \{x \in A : \kappa_n < \|x\| \leq \kappa_{n+1}\}$ for $n < \omega$. By 2.3(b), there is $N < \omega$ such that $p_n \notin \mathrm{cl}_{\beta(\alpha)} A$ for every $n > N$. Hence, without loss of generality, we may suppose that $D \cap \mathrm{cl}_{\beta(\alpha)} A = \varnothing$. If $p \in U(\alpha)$, $M \subseteq W_\kappa(\alpha)$ with $\kappa < \alpha$, and $p \in \mathrm{cl}_{\beta(\alpha)} M$, then $|M| = \alpha$; hence, $\mathrm{cl}_{\beta(\alpha)} D \cap \mathrm{cl}_{\beta(\alpha)} A_n = \varnothing$ for all $n < \omega$. By Lemma 3.6, we can find two disjoint cozero sets S and T of $\beta(\alpha)$ such that $A \subseteq S$, $D \subseteq T$. Since α^* is an F-space [2, 14.9], $\mathrm{cl}_{\beta(\alpha)} D \cap \mathrm{cl}_{\beta(\alpha)} A = \varnothing$.

The next corollary is an immediate consequence of 3.5 and 3.7.

3.8. **Corollary.** *If $\mathrm{cf}(\alpha) = \omega$, then $B^3(\alpha) - B^2(\alpha) \neq \varnothing$.*

4. The cardinality of $B(\alpha)$

We have mentioned (0.1(c)) that $|B(\alpha)| \leq 2^\alpha$ when α satisfies some additional properties. In this section we improve this result by calculating $|B(\alpha)|$ for every α (Theorems 4.9, 4.13, and 4.18). We will also establish the relations among $|B(\alpha)|$, $|\beta(\alpha)|$, and $|N(\alpha)|$.

The following concept is basic in this section. For other properties of ultraproducts not considered here and historical notes see [2, Chapter 12].

4.1. **Definition.** Let $p \in \beta(\alpha)$, and let κ be a cardinal. We define the binary relation $\equiv$ on κ^α by

$$f \equiv g \quad \text{if } \{\xi < \alpha : f(\xi) = g(\xi)\} \in p.$$

It is easy to see that $\equiv$ is an equivalence relation on κ^α. We let κ^α/p be the set of $\equiv$-equivalence classes. κ^α/p is called the *ultraproduct of κ^α modulo p*.

The next theorem follows from Lemma 2 of [13] (see [2, 12.22]).

4.2. **Theorem.** *Let $p \in \beta(\alpha)$ be countably incomplete with $G(p) = \alpha^+$. If κ is an infinite cardinal, then $|\kappa^\alpha/p| = \kappa^\alpha$.*

The proof of the following lemma is straightforward.

4.3. **Lemma** [6]. *Let $\omega \leq \kappa \leq \alpha$ be cardinals, $p \in U(\kappa)$, and $\{A_\xi : \xi < \kappa\}$ be a partition of α. If $f, g : \kappa \to \alpha^*$ are functions such that $f(\xi), g(\xi) \in \widehat{A}_\xi$ for every $\xi < \kappa$, then $\overline{f}(p) = \overline{g}(p)$ if and only if $\{\xi < \kappa : f(\xi) = g(\xi)\} \in p$.*

In the following two results we calculate the cardinality of $W_\kappa(\alpha)$ and $N(\alpha)$ that will allow us to estimate $|B(\alpha)|$.

4.4. **Lemma.** *For $\omega \leq \kappa \leq \alpha$ we have that $|W_\kappa(\alpha)| = \alpha^\kappa \cdot 2^{2^\kappa}$.*

Proof. It is evident that $2^{2^\kappa} \leq |W_\kappa(\alpha)| \leq 2^{2^\kappa} \cdot \alpha^\kappa$, so we only need to show the inequality $\alpha^\kappa \leq |W_\kappa(\alpha)|$. Let $\{A_\xi : \xi < \kappa\}$ be an α-partition of α. For each $\xi < \kappa$, let $\{p_{\xi,\zeta} : \zeta < \alpha\}$ be a strongly discrete subset of ultrafilters contained in $W_\kappa(\alpha) \cap \widehat{A}_\xi$. Fix $q \in U(\kappa)$ countably incomplete κ^+-good and, for each $f \in \alpha^\kappa$, we define $\phi_f : \kappa \to \alpha^*$ by $\phi_f(\xi) = p_{\xi, f(\xi)}$ for $\xi < \kappa$. Let $p_f = \overline{\phi}_f(q)$. Clearly, for every $f \in \alpha^\kappa$, $p_f \in W_\kappa(\alpha)$. Since $\{\xi < \kappa : \phi_f(\xi) = \phi_g(\xi)\} = \{\xi < \kappa : f(\xi) = g(\xi)\}$, and by Lemma 4.3, we have that $p_f = \overline{\phi}_f(p) = \overline{\phi}_g(p) = p_g$ if and only if $f \equiv g$. Using Theorem 4.2, we have that $\alpha^\kappa = |\alpha^\kappa / q| \leq |W_\kappa(\alpha)|$.

The next result appears in [3] without proof.

4.5. **Corollary.** *For every α, the following equality holds:*

$$|N(\alpha)| = \alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma}.$$

Proof. We have that $N(\alpha) = \bigcup\{W_\gamma(\alpha) : \gamma < \alpha\}$. By virtue of 4.4, it follows that $|N(\alpha)| = \sum_{\gamma<\alpha} 2^{2^\gamma} \cdot \alpha^\gamma = \alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma}$.

4.6. Observe that $|N(\alpha)| = \alpha^{<\alpha}$ if α is a strong limit cardinal; otherwise, $|N(\alpha)| = \sum_{\gamma<\alpha} 2^{2^\gamma} = \sup_{\gamma<\alpha} 2^{2^\gamma}$.

The following two lemmas are needed in order to calculate the cardinality of $B(\alpha)$. For a proof of Lemma 4.7 see [9, Lemma 6.5 and exercise 6.14].

4.7. **Lemma.** *If α is a strong limit singular original, then*

$$\alpha^{<\alpha} = \alpha^{\mathrm{cf}(\alpha)} = 2^\alpha.$$

4.8. **Lemma.** *Let α be a nonstrong limit cardinal such that, for some cardinal $\theta < \alpha$, $\sum_{\gamma<\alpha} 2^{2^\gamma} = 2^{2^\theta}$. Then*

(a) $\mathrm{cf}(|N(\alpha)|) \geq \alpha^+$, *and*
(b) $|N(\alpha)| = |N(\alpha)|^\gamma$ *for every* $\gamma < \alpha$.

Proof. Set $\theta < \alpha$ such that $|N(\alpha)| = 2^{2^\theta}$ and $2^\theta \geq \alpha$.

(a) $\mathrm{cf}(|N(\alpha)|) = \mathrm{cf}(2^{2^\theta}) > 2^\theta \geq \alpha$.
(b) If γ is a cardinal less than α, then

$$|N(\alpha)|^\gamma = 2^{(2^\theta)\cdot\gamma} = 2^{2^\theta} = |N(\alpha)|.$$

4.9. **Theorem.** *Let $\omega \leq \alpha$. Then $|B(\alpha)| = |N(\alpha)|$ whenever α satisfies one of the following properties:*

(a) *α is a regular cardinal.*
(b) *α is a singular cardinal which is not a strong limit and $\sup_{\gamma<\alpha} 2^{2^\gamma} = 2^{2^\theta}$ for some $\theta < \alpha$. In this case, $|B(\alpha)| = 2^{2^\theta}$.*
(c) *α is a singular strong limit. In this case we have $|B(\alpha)| = 2^\alpha$.*

Proof. When α is regular, the conclusion is a consequence of 0.1(a). Let α be a singular cardinal. Suppose that $|B^{\xi}(\alpha)| = |N(\alpha)|$ for every $\xi < \eta < \alpha^+$. If η is a limit ordinal, then

$$|N(\alpha)| \leq |B^{\eta}(\alpha)| = \left|\bigcup_{\xi<\eta} B^{\xi}(\alpha)\right| \leq \sum_{\xi<\eta} |B^{\xi}(\alpha)| = |\eta| \cdot |N(\alpha)| = |N(\alpha)|.$$

If $\eta = \xi + 1$, then

$$|B^{\eta}(\alpha)| \leq \sum\{|B^{\xi}(\alpha)|^{\gamma} \cdot 2^{2^{\gamma}} : \gamma < \alpha\} = \sum\{|N(\alpha)|^{\gamma} \cdot 2^{2^{\gamma}} : \gamma < \alpha\}.$$

Hence, if α satisfies (b) (resp. (c)), then we obtain the equality $|B^{\eta}(\alpha)| = |N(\alpha)|$ because of Lemma 4.8 (resp. Lemma 4.7). Therefore, in these two cases, $|N(\alpha)| \leq |B(\alpha)| \leq \alpha^+ \cdot |N(\alpha)| = |N(\alpha)|$.

Note that if α is a strong limit cardinal, then $|B(\alpha)| = |N(\alpha)| = \alpha^{<\alpha}$.

In 4.13 we shall have $|B(\alpha)|$ for those cardinals not considered in the previous theorem. We need the following definition and lemma.

4.10. **Definition.** Let $\omega \leq \kappa \leq \alpha$. A collection $\mathscr{G}$ of subsets of α is *κ-almost disjoint* if $|G| \geq \kappa$ for $G \in \mathscr{G}$ and $|G_0 \cap G_1| < \kappa$ for $G_0, G_1 \in \mathscr{G}$ and $G_0 \neq G_1$.

A proof of the following lemma can be found in [2, 12.2].

4.11. **Lemma.** *Let κ, γ be two cardinal numbers with $\omega \leq \kappa$ and $2 \leq \gamma$. Then there is a κ-almost disjoint family $\mathscr{G} \subseteq \mathscr{P}(\gamma^{<\kappa})$ on $\gamma^{<\kappa}$ of cardinality γ^{κ}.*

4.12. We will denote by L the set of cardinals that do not satisfy any properties considered in 4.9; that is, $L = \{\alpha : \alpha$ is a singular nonstrong limit cardinal such that $\sup_{\gamma<\alpha} 2^{2^{\gamma}} > 2^{2^{\nu}}$ for every $\nu < \alpha\}$. Observe that (see 4.6) if α is not a strong limit and $\{2^{2^{\gamma}}\}_{\gamma<\alpha}$ is not eventually constant (in particular, if $\alpha \in L$), then $\mathrm{cf}(\alpha) = \mathrm{cf}(|N(\alpha)|)$.

4.13. **Theorem.** *If $\alpha \in L$, then $|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} = 2^{\kappa}$ where $\kappa = 2^{<\alpha}$. Moreover, if $\omega \leq \gamma < \mathrm{cf}(\alpha)$, then $|N(\alpha)| = |N(\alpha)|^{\gamma} < |B(\alpha)|$.*

Proof. Let μ be a cardinal less than $|N(\alpha)|$. We choose $\gamma < \alpha$ such that $2^{\gamma} \geq \alpha$ and $\mu < 2^{2^{\gamma}}$. If $\nu < \alpha$, then $\mu^{\nu} \leq (2^{2^{\gamma}})^{\nu} = 2^{(2^{\gamma}) \cdot \nu} = 2^{2^{\gamma}} < |N(\alpha)|$; therefore (see Theorem 19 in [9]),

($*$) for every $\mathrm{cf}(\alpha) \leq \nu < \alpha$ we obtain $|N(\alpha)|^{\nu} = |N(\alpha)|^{\mathrm{cf}(\alpha)}$, and
($**$) $|N(\alpha)|^{<\mathrm{cf}(\alpha)} = |N(\alpha)|$.

By using inductively the equality in ($*$) we obtain that $|B^{\xi}(\alpha)| \leq |N(\alpha)|^{\mathrm{cf}(\alpha)}$ for every $\xi < \alpha^+$. Hence,

$$|B(\alpha)| \leq |N(\alpha)|^{\mathrm{cf}(\alpha)} \cdot \alpha^+ = |N(\alpha)|^{\mathrm{cf}(\alpha)}.$$

We are now going to prove that $|N(\alpha)|^{\mathrm{cf}(\alpha)} \leq |B^2(\alpha) \backslash B^1(\alpha)|$. Let $\mathscr{G} = \{G_f : f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}\}$ be a $\mathrm{cf}(\alpha)$-almost disjoint family on $|N(\alpha)|$ of cardinality $|N(\alpha)|^{\mathrm{cf}(\alpha)}$ (see Lemma 4.11 and ($**$)); let $G_f = \{\lambda_{f,\xi} : \xi < \mathrm{cf}(\alpha)\}$ be a faithful indexing of G_f for each $f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$; and let $\mathscr{A} = \{A_\delta : \delta < \mathrm{cf}(\alpha)\}$ be an α-partition of α and $\alpha_\delta \nearrow \alpha$.

Since $|\widehat{A_\delta} \cap B^1(\alpha)| = |N(\alpha)|$ for each $\delta < \mathrm{cf}(\alpha)$, we can take $B_\delta = \{p_{\delta,\xi} : \xi < |N(\alpha)|\} \subseteq \widehat{A_\delta} \cap B^1(\alpha)$ such that $\|p_{\delta,\xi}\| = \alpha_\delta$ for $\xi < \mathrm{cf}(\alpha)$ and $p_{\delta,\xi} \neq p_{\delta,\zeta}$ for

$\xi < \zeta < \mathrm{cf}(\alpha)$. For each $f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$, we consider the function $\phi_f\colon \mathrm{cf}(\alpha) \to B^1(\alpha)$ defined by $\phi_f(\xi) = p_{\xi, \lambda_{f,\xi}}$ for $\xi < \mathrm{cf}(\alpha)$. Fix $q \in U(\mathrm{cf}(\alpha))$. Then $\overline{\phi}_f(q) \in \mathrm{cl}_{\beta(\alpha)} \phi_f(\mathrm{cf}(\alpha)) \subseteq B^2(\alpha)$. It suffices to prove that the relation $f \to \overline{\phi}_f(q)$ from $|N(\alpha)|^{\mathrm{cf}(\alpha)}$ to $B^2(\alpha)$ is one-to-one. Indeed, let $f, g \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$ such that $f \neq g$. It is evident that $\phi_f(\xi) = \phi_g(\xi)$ iff $\lambda_{f,\xi} = \lambda_{g,\xi}$ and so $|\{\xi < \mathrm{cf}(\alpha) : \phi_f(\xi) = \phi_g(\xi)\}| = |\{\xi < \mathrm{cf}(\alpha) : \lambda_{f,\xi} = \lambda_{g,\xi}\}| \le |G_f \cap G_g| < \mathrm{cf}(\alpha)$. Hence, $\{\xi < \mathrm{cf}(\alpha) : \phi_f(\xi) = \phi_g(\xi)\} \notin q$. From Lemma 4.3, it follows that $\overline{\phi}_f(q) \neq \overline{\phi}_g(q)$. Reasoning as in 3.2, we can prove that $\overline{\phi}_f(q) \in B^2(\alpha)\backslash B^1(\alpha)$ for each $f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$; therefore, $|N(\alpha)|^{\mathrm{cf}(\alpha)} \le |B^2(\alpha)\backslash B^1(\alpha)| \le |B(\alpha)|$. Thus, we have that $|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)}$.

It remains to show that $|B(\alpha)| = 2^\kappa$. Let $\theta = \sup_{\gamma<\alpha} 2^{2^\gamma} = |N(\alpha)|$. Since $\alpha \in L$, κ is a limit cardinal, $\mathrm{cf}(\alpha) = \mathrm{cf}(\theta) = \mathrm{cf}(\kappa)$, and $\theta = \sup_{\mu<\kappa} 2^\mu = 2^{<\kappa}$; therefore, $|B(\alpha)| = \theta^{\mathrm{cf}(\theta)} = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$. Because of Lemma 6.5 in [9], we conclude that $|B(\alpha)| = 2^\kappa$.

The last assertion of Theorem 4.13 is implied from the following inequality which is a consequence of $(**)$:

$$|N(\alpha)|^\gamma = |N(\alpha)| < |N(\alpha)|^{\mathrm{cf}(|N(\alpha)|)} = |N(\alpha)|^{\mathrm{cf}(\alpha)} = |B(\alpha)|.$$

We have finished the proof of Theorem 4.13. $\square$

The following result was already shown in [6]. Here we give an alternative proof (see the definition of L in 4.12).

4.14. **Corollary.** *If $\omega < \alpha$, then*

$$\alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} \le |B(\alpha)| \le \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)}.$$

Proof. If α is a strong limit, then (see 4.5, 4.7, and 4.9)

$$\alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} = |B(\alpha)| = 2^\alpha = \alpha^{\mathrm{cf}(\alpha)} = \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)}.$$

If α is not a strong limit and either α is a regular cardinal or $\sum_{\gamma<\alpha} 2^{2^\gamma} = 2^{2^\theta}$ for some $\theta < \alpha$, then

$$\alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} = \sum_{\gamma<\alpha} 2^{2^\gamma} = |B(\alpha)| \le \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)} \quad \text{(see 4.9)}.$$

Finally, when $\alpha \in L$, we have

$$|N(\alpha)| < |B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} = \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)} \quad \text{(see 4.13)}.$$

The next corollary improves Theorem 3.5 in [6].

4.15. **Corollary.** *For every singular cardinal α, we have*

(#) $$|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} = |N(\alpha)|^\alpha = |B(\alpha)|^{\mathrm{cf}(\alpha)} = |B(\alpha)|^\alpha.$$

Proof. If $\alpha \notin L$, then the result follows from 4.9. Suppose that $\alpha \in L$. In this case we have $\mathrm{cf}(|N(\alpha)|) = \mathrm{cf}(\alpha) < \alpha < |N(\alpha)|$ and $\gamma^{\mathrm{cf}(\alpha)} < |N(\alpha)|$ for every $\gamma < |N(\alpha)| = \sup_{\gamma<\alpha} 2^{2^\gamma}$. Then $|N(\alpha)|^\alpha = |N(\alpha)|^{\mathrm{cf}(\alpha)}$. Now all the equalities in (#) follow from Theorem 4.13.

4.16. **Corollary.** *Let* $\omega \leq \alpha$. *Then* $|N(\alpha)| = |B(\alpha)|$ *if and only if* $\alpha \notin L$.

It is possible to construct a model M of ZFC in which $|N(\aleph_\omega)| < |B(\aleph_\omega)|$ (see [6]). In this model, $\aleph_\omega \in L$.

4.17. **Corollary.** *Let* $\omega \leq \alpha$. *Then, for every* $1 < \xi < \alpha^+$, *we have that* $|B^\xi(\alpha)| = |B(\alpha)|$.

Proof. We have that $|N(\alpha)| \leq |B^\xi(\alpha)| \leq |B(\alpha)|$. If $\alpha \notin L$, then $|N(\alpha)| = |B(\alpha)|$ (Corollary 4.16). We have proved in 4.13 that $|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} \leq |B^2(\alpha)|$ whenever $\alpha \in L$. This completes the proof.

In the following theorem we summarize the results regarding all the possible values of $|B(\alpha)|$.

4.18. **Theorem.** *Let* $\omega \leq \alpha$, $\kappa = 2^{<\alpha}$, *and* $\theta = \sup_{\gamma<\alpha} 2^{2^\gamma}$.

(a) *If* α *is a strong limit, then*

(i) $|B(\alpha)| = \alpha$ *if and only if* α *is regular*;

(ii) $|B(\alpha)| = 2^\alpha$ *if and only if* α *is singular.*

(b) *If* α *is not a strong limit, then*

(i) $|B(\alpha)| = 2^{2^\mu}$ *for some* $\mu < \alpha$ *if and only if either* α *is a successor cardinal or* $\{2^{2^\gamma}\}_{\gamma<\alpha}$ *is eventually constant*;

(ii) $|B(\alpha)| = 2^\kappa$ *whenever* $\alpha \in L$;

(iii) $2^{2^\mu} < |B(\alpha)| = \theta = 2^{<\kappa} < 2^\kappa \leq 2^{2^\alpha}$ *for every* $\mu < \alpha$ *whenever* α *is a regular limit and* $\{2^{2^\gamma}\}_{\gamma<\alpha}$ *is not eventually constant.*

Proof. We obtain (a) as a consequence of 4.6 and 4.9(c) and 1.27 in [2]. The necessity in (b)(i) is trivial (see 4.9), and (b)(ii) is proved in 4.13. We only have to prove (b)(i)($\Rightarrow$) and (b)(iii).

(b)(i)($\Rightarrow$) In this case, α does not belong to L because $\theta^{\mathrm{cf}(\theta)} > \theta \geq 2^{2^\gamma}$ for every $\gamma < \alpha$ (if $\alpha \in L$, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\theta)$ and $|B(\alpha)| = \theta^{\mathrm{cf}(\theta)}$; see 4.13). Thus, $|B(\alpha)| = |N(\alpha)| = \theta$. So, if $|B(\alpha)| = 2^{2^\mu}$ for some $\mu < \alpha$ and $\{2^{2^\gamma}\}_{\gamma<\alpha}$ is not eventually constant, then $\mu^+ = \alpha$.

(b)(iii) Since α is a regular nonstrong limit and $\{2^{2^\gamma}\}_{\gamma<\alpha}$ is not eventually constant, $2^{2^\mu} < \theta = |B(\alpha)|$ for every $\mu < \alpha$. It is also clear that $\{2^\gamma\}_{\gamma<\alpha}$ is not eventually constant, hence, neither is $\{2^\nu\}_{\nu<\kappa}$ and so $\sup_{\nu<\kappa} 2^\nu < 2^\kappa$. The inequality $2^\kappa \leq 2^{2^\alpha}$ always holds, and $\theta = \sup_{\nu<\kappa} 2^\nu$ follows from the properties of α.

As immediate consequences of the previous theorem we have the following corollaries (in the first one we determine the conditions under which $B(\alpha)$ has the same cardinality as $\beta(\alpha)$).

4.19. **Corollary.** (a) *If* $\alpha \in L$, *then* $|B(\alpha)| = 2^{2^\alpha}$ *if and only if* $2^{2^\alpha} = 2^\kappa$ *where* $\kappa = 2^{<\alpha}$.

(b) *If* $\alpha \notin L$, *then* $|B(\alpha)| = 2^{2^\alpha}$ *if and only if* $2^{2^\alpha} = 2^{2^\mu}$ *for some* $\mu < \alpha$.

4.20. **Corollary.** *If GCH holds, then for every infinite cardinal* α *we have that* $|N(\alpha)| = |B(\alpha)| < |\beta(\alpha)|$.

4.21. **Corollary.** *If* α *is a singular cardinal, then* $|B(\alpha)| = 2^{\mu}$ *for some cardinal* μ.

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