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p-Fréchet–Urysohn property of function spaces

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Abstract

In this paper, we study the p-Fréchet–Urysohn property of function spaces, for $p \in \beta(\omega)\setminus\omega$. We prove that $C_\pi(X)$ is p-Fréchet–Urysohn if and only if X has (γ_p), where (γ_p) is the natural p-version of property (γ) (this is a generalization of a result due to Gerlits and Nagy). We note the following implications: X is second countable $\Rightarrow X$ has (γ_p) for some $p \in \beta(\omega)\setminus\omega \Rightarrow X^n$ is Lindelöf for all $1 \leqslant n < \omega$. We deal with the question when is $C_\pi(\mathbb{R})$ a p-Fréchet–Urysohn space. It is shown that there is $p \in \beta(\omega)\setminus\omega$ such that $C_\pi(\mathbb{R})$ is p-Fréchet–Urysohn; if p is semiselective, then every subset X of $\mathbb{R}$ satisfying (γ_p) has measure zero and if p is selective, then X is a strong measure zero set; and we can find $p \in \beta(\omega)\setminus\omega$ such that $C_\pi(\mathbb{R})$ is p-Fréchet–Urysohn and is not strongly p-Fréchet–Urysohn. Finally, we prove that $\mathbb{R}^\omega$ does not have (γ_p) whenever p is a P-point of $\beta(\omega)\setminus\omega$.

Key words: Function space; ω-cover; FU(p)-space; γ_p-property; Rudin–Keisler order; Rapid; Semiselective; Selective; P-point; Q-point

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1. Introduction and preliminaries

In this paper we consider only completely regular Hausdorff spaces. For $A \subseteq X$, $\mathrm{Cl}(A)$ stands for the closure of A in X and, for $x \in X$, $\mathcal{N}(x)$ is the set of neighborhoods of x in X. For a space X we define $C_\pi(X)$ to be the set of all continuous real valued functions on X endowed with the topology of pointwise

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convergence. If $f: X \to Y$ is a continuous function then $\bar{f}: \beta X \to \beta Y$ denotes the Stone–Čech extension of f. The remainder of βX is $X^* = \beta X \setminus X$. For $p \in \omega^*$, $\xi(p)$ denotes the subspace $\omega \cup \{p\}$ of $\beta(\omega)$. For p, $q \in \omega^*$ we define $p \leqslant_{\mathrm{RK}} q$, the Rudin–Keisler ordering, if there is a function $f: \omega \to \omega$ such that $\bar{f}(q) = p$. If p, $q \in \omega^*$, then $p \approx_{\mathrm{RK}} q$ means that $p \leqslant_{\mathrm{RK}} q$ and $q \leqslant_{\mathrm{RK}} p$ (equivalently, there is a permutation σ of ω such that $\bar{\sigma}(p) = q$). The *type* of $p \in \omega^*$ is $T(p) = \{q \in \omega^*: p \approx_{\mathrm{RK}} q\}$. If $p \in \omega^*$, then p is a *P-point* if for every partition $\{A_n: n < \omega\}$ of ω with $A_n \notin p$, for each $n < \omega$, there is $A \in p$ such that $|A \cap A_n| < \omega$ for all $n < \omega$; p is a *Q-point* if for every partition $\{A_n: n < \omega\} \subseteq [\omega]^{<\omega}$ of ω there is $A \in p$ such that $|A \cap A_n| \leqslant 1$ for each $n < \omega$; p is *rapid* if for every function $f: \omega \to \omega$ there is $A \in p$ such that $|A \cap f(n)| \leqslant n$ for each $n < \omega$; p is *semiselective* if p is a P-point and rapid; and p is *selective* if p is a P-point and a Q-point. Observe that every Q-point is rapid and so every selective ultrafilter is semiselective. Kunen (see [5] or [8,9.6]) showed that the selective ultrafilters on ω^* are precisely the RK-minimal points of ω^*.

A collection $\mathscr{G}$ of subsets of X is an *ω-cover* of X if for every finite subset F of X there is $G \in \mathscr{G}$ such that $F \subseteq G$.

The authors of [16] introduced the following property for a space X:

(γ) if $\mathscr{G}$ is an open ω-cover of X, then there is a sequence $(G_n)_{n<\omega}$ in $\mathscr{G}$ such that $X = \underline{\mathrm{Lim}}\, G_n$, where

$$\underline{\mathrm{Lim}}\, G_n = \{x \in X: (\exists m < \omega)(\forall n > m)[x \in G_n]\}.$$

It is shown in [15,16] that a space X has $(\gamma) \Leftrightarrow C_\pi(X)$ is sequential $\Leftrightarrow C_\pi(X)$ is a Fréchet–Urysohn space: the second equivalence was also proved in [23].

In [3], Bernstein introduced for $p \in \omega^*$ the *p-limit* notion of a sequence of points in a space X: $x = p\text{-}\lim x_n$ if for each $V \in \mathscr{N}(x)$ we have that $\{n < \omega: x_n \in V\} \in p$. Bernstein's concept suggests the following generalization of sequential and Fréchet–Urysohn spaces:

Definition 1.1. Let $p \in \omega^*$ and X a space.

(1) (Kombarov [17]) X is *p-sequential* if for every nonclosed subset A of X there is a sequence $(x_n)_{n<\omega}$ in A and $x \notin A$ such that $x = p\text{-}\lim x_n$.

(2) (Comfort–Savchenko) X is an *FU(p)-space* if for every $A \subseteq X$ and $x \in \mathrm{Cl}(A)$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x = p\text{-}\lim x_n$.

It is then natural to define the p-version of property (γ):

Definition 1.2. Let $p \in \omega^*$ and let X be a space.

(1) If $(G_n)_{n<\omega}$ is a sequence of nonempty subsets of X, then

$$\underline{\mathrm{Lim}}_p\ G_n = \{x \in X: \{n < \omega: x \in G_n\} \in p\}.$$

(2) X has (γ_p) if for every open ω-cover $\mathscr{G}$ of X there is a sequence $(G_n)_{n<\omega}$ in $\mathscr{G}$ such that $X = \underline{\mathrm{Lim}}_p G_n$.

1.3. Observe that if $(G_n)_{n<\omega}$ is a sequence of subsets of a space X and $p \in \omega^*$, then $\underline{\mathrm{Lim}}_p G_n = \bigcup_{A \in p} \bigcap_{n \in A} G_n$.

The following lemma will be useful.

Lemma 1.4. *Let $p \in \omega^*$ and X a space. Then $X = \underline{\mathrm{Lim}}_p G_n$ if and only if $X = \bigcup_{n \in A} G_n$ for every $A \in p$.*

Proof. ($\Rightarrow$) Let $A \in p$ and $x \in X$. By assumption, $\{n < \omega \colon x \in G_n\} \in p$. Hence, we may pick $m \in A$ such that $x \in G_m$. Thus, $X = \bigcup_{n \in A} G_n$.

($\Leftarrow$) If there is $x \in X$ such that $B = \{n < \omega \colon x \in G_n\} \notin p$, then $x \notin \bigcup_{n \in A} G_n$, where $A = \omega \setminus B$, which is a contradiction. $\square$

The next theorem establishes the connection between the Rudin–Keisler order on ω^* and properties (γ_p).

Theorem 1.5. *Let p, $q \in \omega^*$. If $p \leqslant_{\mathrm{RK}} q$, then every space with (γ_p) has (γ_q).*

Proof. Let X be a space with (γ_p) and let $f : \omega \to \omega$ be onto such that $\bar{f}(q) = p$. Let $\mathscr{G}$ be an open ω-cover of X. Then there is a sequence $(G_n)_{n<\omega}$ in $\mathscr{G}$ such that $X = \underline{\mathrm{Lim}}_p G_n$. For each $n < \omega$, set $F_n = G_{f(n)}$. If $x \in X$, then $A = \{n < \omega \colon x \in G_n\} \in p$ and so $f^{-1}(A) = \{m < \omega \colon x \in G_{f(m)} = F_m\} \in q$. Thus, $X = \underline{\mathrm{Lim}}_q F_n$. $\square$

We do not know whether the converse to Theorem 1.5 holds:

Question 1.6. Let p, $q \in \omega^*$. If every space with (γ_p) has (γ_q), must we have that $p \leqslant_{\mathrm{RK}} q$?

It should be remarked that if $X = \underline{\mathrm{Lim}}_p G_n$ then $\mathscr{G} = \{G_n \colon n < \omega\}$ is an ω-cover of X. Thus, in a space satisfying (γ_p) every open ω-cover has a countable ω-subcover. This last property is denoted by (ε) in [16], where it is shown that a space X has $(\varepsilon) \Leftrightarrow X^n$ is Lindelöf for each $1 \leqslant n < \omega \Leftrightarrow$ the tightness of $C_\pi(X)$ is countable (the last equivalence was proved by Arkangel'skii [2, 4.1.2] and Pytkeev [22]). It is evident that every closed subspace of a space with (ε) has (ε) too; more general:

Lemma 1.7. *If X has (ε), then every F_σ-subset of X has (ε) too.*

Proof. Let $F = \bigcup_{n<\omega} F_n$ such that F_n is a closed subset of X for each $n < \omega$, and let $1 \leqslant m < \omega$. Then we have that

$$F^m = \bigcup \{F_{n_1} \times \cdots \times F_{n_m} \colon n_j < \omega \text{ for } 1 \leqslant j \leqslant m\},$$

and each $F_{n_1} \times \cdots \times F_{n_m}$ is a closed subset of X^m. Since X^m is Lindelöf, we have that each $F_{n_1} \times \cdots \times F_{n_m}$ is Lindelöf and so F^m is Lindelöf. $\square$

The following questions appear to be natural.

(I) Is it true that a space X has (γ_p) if and only if $C_\pi(X)$ is an FU(p)-space?

(II) If $C_\pi(X)$ is p-sequential, must $C_\pi(X)$ be an FU(p)-space?

In Section 2, we answer question (I) in the affirmative (Theorem 2.10). We do not know the response to the second one. Also, in Section 2, we show that if X has (ε) and $w(X) \leqslant 2^\omega$, then there is $p \in \omega^*$ such that $C_\pi(X)$ is an FU(p)-space (Theorem 2.3). In particular, $C_\pi(\mathbb{R})$ is an FU(p)-space for some $p \in \omega^*$. However, we prove that if p is semiselective, then $C_\pi(\mathbb{R})$ is not an FU(p)-space (Corollary 3.9).

The authors are grateful to the referee for valuable remarks and for improving an earlier version of Lemma 3.14.

2. Property (γ_p) and function spaces

In this section, we shall show that a space X has (γ_p) if and only if $C_\pi(X)$ is an FU(p)-space, for $p \in \omega^*$. First, we give some basic results concerning properties (γ_p).

Theorem 2.1. *Let $p \in \omega^*$.*

(a) *Property (γ_p) is preserved under continuous functions.*

(b) *If F is a F_σ-subset of a space X with (γ_p), then F has (γ_p).*

(c) *If X has (γ_p), then X^n has (γ_p) for each $1 \leqslant n < \omega$.*

Proof. (a) Let X be a space with (γ_p) and let $f: X \to Y$ be a continuous function from X onto a space Y. Let $\mathscr{G}$ be an open ω-cover of Y. We have that $\mathscr{F} = \{f^{-1}(G)\colon G \in \mathscr{G}\}$ is an open ω-cover of X. Since X has (γ_p) there is a sequence $(F_n)_{n<\omega}$ in $\mathscr{F}$ such that $X = \underline{\mathrm{Lim}}_{\,p} F_n$. For each $n < \omega$, we have that $F_n = f^{-1}(G_n)$ for some $G_n \in \mathscr{G}$. It is then evident that $Y = \underline{\mathrm{Lim}}_{\,p} G_n$.

(b) Assume that X has (γ_p) and $F = \bigcup_{n<\omega} F_n$, where F_n is a closed subset of X and $F_n \subseteq F_{n+1}$, for $n < \omega$. Let $\mathscr{G}$ be an open ω-cover of F. By Lemma 1.7, $\mathscr{G}$ has a countable ω-subcover $\mathscr{G}' = \{G_n\colon n < \omega\}$. For each $n < \omega$ we choose an open subset K_n of X such that $G_n = K_n \cap F$ and set $H_n = K_n \cup (X \setminus F_n)$. We note that $\mathscr{H} = \{H_n\colon n < \omega\}$ is an open ω-cover of X. By hypothesis, there is a sequence $(H_{n_k})_{k<\omega}$ in $\mathscr{H}$ such that $X = \underline{\mathrm{Lim}}_{\,p} H_{n_k}$. Let us check that $F = \underline{\mathrm{Lim}}_{\,p} G_{n_k}$. If $x \in F$, then we have that $\{k < \omega\colon x \in H_{n_k}\} \in p$ and $x \in F_m$ for some $m < \omega$. Hence,

$$\{k < \omega\colon m \leqslant n_k,\ x \in H_{n_k}\} \subseteq \{k < \omega\colon x \in G_{n_k}\} \in p.$$

(c) Assume that X has (γ_p). It suffices to show that $X \times X$ has (γ_p). Indeed, fix an open base $\mathscr{B}$ of X closed under finite unions. Let $\mathscr{G}$ be an open ω-cover of $X \times X$. Set $\mathscr{B}' = \{B \times B\colon B \in \mathscr{B},\ B \times B \subseteq G \text{ for some } G \in \mathscr{G}\}$. We claim that $\mathscr{B}'$ is an ω-cover of $X \times X$. Let $\{(x_0, y_0), \ldots, (x_r, y_r)\} \subseteq X \times X$. By assumption, there is $G \in \mathscr{G}$ such that

$$\{(x_i, y_j)\colon i, j < r\} \cup \{(x_i, x_j)\colon i, j < r\} \cup \{(y_i, y_j)\colon i, j < r\}$$
$$\cup \{(y_i, x_j)\colon i, j < r\} \subseteq G.$$

We may find A_i, $B_i \in \mathscr{B}$ for $i \leqslant r$ so that $(x_i, y_j) \in A_i \times B_j \subseteq G$, $(y_i, x_j) \in B_i \times A_j \subseteq G$, $(x_i, x_j) \in A_i \times A_j \subseteq G$ and $(y_i, y_j) \in B_i \times B_j \subseteq G$, for each $i,j \leqslant r$. Define $B = (\bigcup_{i \leqslant r} A_i) \cup (\bigcup_{i \leqslant r} B_i)$. Then, $B \in \mathscr{B}$ and $B \times B = (\bigcup_{i,j \leqslant r} A_i \times B_j) \cup (\bigcup_{i,j \leqslant r} B_i \times A_j) \cup (\bigcup_{i,j \leqslant r} A_i \times A_j) \cup (\bigcup_{i,j \leqslant r} B_i \times B_j) \subseteq G$. Thus, $\mathscr{B}'$ is an open ω-cover of $X \times X$ and hence $\mathscr{D} = \{B \in \mathscr{B}\colon B \times B \in \mathscr{B}'\}$ is also an ω-cover of X. We then have that there is a sequence $(B_n)_{n<\omega}$ in $\mathscr{D}$ such that $X = \underline{\mathrm{Lim}}_p B_n$. For each $n < \omega$, choose $G_n \in \mathscr{G}$ so that $B_n \times B_n \subseteq G_n$. If $(x, y) \in X \times X$, then $\{n < \omega\colon x \in B_n\} \cap \{m < \omega\colon y \in B_m\} \subseteq \{k < \omega\colon (x, y) \in B_k \times B_k\} \in p$. Therefore, $X \times X = \underline{\mathrm{Lim}}_p (B_n \times B_n) = \underline{\mathrm{Lim}}_p G_n$. □

Lemma 2.2. *Let $p \in \omega^*$, X a space with (ε) and $\mathscr{B}$ a base of X closed under finite unions. We have that X has (γ_p) if and only if for every countable open ω-cover $\{B_n\colon n < \omega\} \subseteq \mathscr{B}$ of X there is $q \in \omega^*$ such that $q \leqslant_{\mathrm{RK}} p$ and $X = \underline{\mathrm{Lim}}_q B_n$.*

Proof. ($\Rightarrow$) Let $\{B_n\colon n < \omega\}$ be an open ω-cover of X. By hypothesis there is a sequence $(B_{n_k})_{k<\omega}$ of $\{B_n\colon n < \omega\}$ such that $X = \underline{\mathrm{Lim}}_p B_{n_k}$. Define $f\colon \omega \to \omega$ by $f(k) = n_k$ for each $k < \omega$, and $q = \bar{f}(p)$. If $x \in X$, then $\{k < \omega\colon x \in B_{n_k}\} \in p$ and since $q = \bar{f}(p)$, $\{f(k) < \omega\colon x \in B_{n_k}\} = \{f(k) < \omega\colon x \in B_{f(k)}\} \in q$. Hence, $\{n < \omega\colon x \in B_n\} \in q$. Thus, $X = \underline{\mathrm{Lim}}_q B_n$.

($\Leftarrow$) Let $\mathscr{G}$ be an open ω-cover of X and consider the set $\mathscr{D} = \{B \in \mathscr{B}\colon B \subseteq G$ for some $G \in \mathscr{G}\}$. Notice that $\mathscr{D}$ is also an open ω-cover of X. Since X has (ε), we may assume that $\mathscr{D} = \{B_n\colon n < \omega\}$. By assumption there is $q \in \omega^*$ and a function $f\colon \omega \to \omega$ such that $\bar{f}(p) = q$ and $X = \underline{\mathrm{Lim}}_q B_n$. For $x \in X$, we have that $A = \{n < \omega\colon x \in B_n\} \in q$ and so $f^{-1}(A) = \{k < \omega\colon x \in B_{f(k)}\} \in p$. Thus, $X = \underline{\mathrm{Lim}}_p B_{f(k)}$. For each $k < \omega$ choose $G_k \in \mathscr{G}$ so that $B_{f(k)} \subseteq G_k$. Therefore, $(G_k)_{k<\omega}$ is a sequence in $\mathscr{G}$ and $X = \underline{\mathrm{Lim}}_p G_k$. □

Theorem 2.3. *If X has (ε) and $w(X) \leqslant 2^\omega$, then there is $p \in \omega^*$ such that X has (γ_p).*

Proof. Let $\mathscr{B}$ be a base of X closed under finite unions and $|\mathscr{B}| \leqslant 2^\omega$. Let $\mathscr{D}$ be the set of all countable open ω-covers of X with elements in $\mathscr{B}$. It is clear that $|\mathscr{D}| \leqslant 2^\omega$. Now, each $b \in \mathscr{D}$ will be enumerated as follows: if b is infinite, then $\{B_n\colon n < \omega\}$ will be a faithful enumeration of b and if b is finite, then $\{B_n\colon n < \omega\}$ will be an enumeration of b so that each element of b appears infinite many times in the enumeration. Fix $b = \{B_n\colon n < \omega\} \in \mathscr{D}$. For each $x \in X$ put $S(x) = \{n < \omega\colon x \in B_n\}$ and $\mathscr{F}_b = \{S(x)\colon x \in X\}$. Since b is an ω-cover of X and $\{B_n\colon n < \omega\}$ is a nice enumeration of b, $\mathscr{F}_b$ is a filter subbase in ω which can be extended to a free filter on ω. Hence, for each $b \in \mathscr{D}$, we may choose $q_b \in \omega^*$ such that $\mathscr{F}_b \subseteq q_b$. We then have that $X = \underline{\mathrm{Lim}}_{q_b} B_n$, for each $b \in \mathscr{D}$. Since $|\{q_b\colon b \in \mathscr{D}\}| \leqslant |\mathscr{D}| \leqslant 2^\omega$, by 10.9 of [8] or 6.4 of [5], there is $p \in \omega^*$ such that $q_b \leqslant_{\mathrm{RK}} p$ for each $b \in \mathscr{D}$. The conclusion follows from Lemma 2.2. □

Corollary 2.4. *If X is a countable space, then there is $p \in \omega^*$ such that $C_\pi(X)$ has (γ_p).*

Proof. If X is countable, then $C_\pi(X) \subseteq \mathbb{R}^\omega$ and so $C_\pi(X)$ is second countable. By Theorem 2.3, there is $p \in \omega^*$ such that $C_\pi(X)$ has (γ_p). □

We turn now to the principal result of this section (Theorem 2.10). We prove this theorem by using arguments that are similar to those applied in the proof of Theorem 2 ($(\gamma) \Leftrightarrow$ (iv)) in [16]. For the sake of completeness, we present the proof with all the details. The following lemmas constitute the essential modifications to prove our theorem.

For a space X, $\varepsilon > 0$ and $f \in C_\pi(X)$, the set $\{x \in X\colon |f(x)| < \varepsilon\}$ is denoted by $\operatorname{coz}_\varepsilon f$. If F is a finite subset of X and V is an open subset of $\mathbb{R}$, then we put $[F, V] = \{f \in C_\pi(X)\colon f(F) \subseteq V\}$. A subbase of $C_\pi(X)$ is the set $\{[F, V]\colon F \subseteq X$ is finite and $V \subseteq \mathbb{R}$ is open$\}$.

Lemma 2.5. *Let $\emptyset \neq \Phi \subseteq C_\pi(X)$, $f \in \operatorname{Cl} \Phi$ and $\varepsilon > 0$. Then $\mathscr{G}(\Phi, f, \varepsilon) = \{\operatorname{coz}_\varepsilon(g - f)\colon g \in \Phi\}$ is an open ω-cover of X.*

Proof. Fix $x_0, \ldots, x_n \in X$. Since $f \in \operatorname{Cl} \Phi$ there is $h \in W \cap \Phi$, where $W = [x_0, (f(x_0) - \varepsilon, f(x_0) + \varepsilon)] \cap \cdots \cap [x_n, (f(x_n) - \varepsilon, f(x_n) + \varepsilon)]$. Then, $|f(x_j) - h(x_j)| < \varepsilon$ for each $j \leqslant n$; that is, $x_0, \ldots, x_n \in \operatorname{coz}_\varepsilon(h - f)$. □

Lemma 2.6. *Let $f \in C_\pi(X)$, $\varepsilon > 0$ and $\mathscr{H}$ an open ω-cover of X with $X \notin \mathscr{H}$. If $\Phi(\mathscr{H}, f, \varepsilon) = \{g \in C_\pi(X)\colon \operatorname{coz}_\varepsilon(g - f) \subseteq H$ for some $H \in \mathscr{H}\}$, then $f \in \operatorname{Cl} \Phi(\mathscr{H}, f, \varepsilon) \setminus \Phi(\mathscr{H}, f, \varepsilon)$ and $\mathscr{G}(\Phi(\mathscr{H}, f, \varepsilon), f, \delta)$ refines $\mathscr{H}$ for each $\delta > 0$.*

Proof. Let $F = \{x_0, \ldots, x_n\} \subseteq X$ and

$$W = [x_0, (f(x_0) - \rho, f(x_0) + \rho)] \cap \cdots \cap [x_n, (f(x_n) - \rho, f(x_n) + \rho)].$$

Since $\mathscr{H}$ is an ω-cover there is $H \in \mathscr{H}$ such that $F \subseteq H$ and since X is Tychonoff there is a continuous function $g\colon X \to [0, \varepsilon]$ such that $g(x_j) = 0$ for $j \leqslant n$, and $g(x) = \varepsilon$ for all $x \in X \setminus H$. Then, $f - g \in W \cap \Phi(\mathscr{H}, f, \varepsilon)$. Thus, $f \in \operatorname{Cl} \Phi(\mathscr{H}, f, \varepsilon)$ and since $X \notin \mathscr{H}$ then $f \notin \Phi(\mathscr{H}, f, \varepsilon)$. □

Lemma 2.7. *Let $p \in \omega^*$, X a space, $f \in C_\pi(X)$ and $(f_n)_{n<\omega}$ a sequence in $C_\pi(X)$. Then $f = p - \lim f_n$ if and only if $\underline{\operatorname{Lim}}_p \operatorname{coz}_\varepsilon(f_n - f) = X$ for each $\varepsilon > 0$.*

Proof. ($\Rightarrow$) Let $\varepsilon > 0$ and $x \in X$. By assumption we have that $\{n < \omega\colon f_n \in [x, (f(x) - \varepsilon/2, f(x) + \varepsilon/2)]\} \in p$ and so $\{n < \omega\colon |f_n(x) - f(x)| < \varepsilon\} = \{n < \omega\colon x \in \operatorname{coz}_\varepsilon(f_n - f)\} \in p$. Thus, $X = \underline{\operatorname{Lim}}_p \operatorname{coz}_\varepsilon(f_n - f)$.

($\Leftarrow$) Let $\{x_0, \ldots, x_k\} \subseteq X$, $\varepsilon > 0$ and $W = \bigcap_{j \leqslant k} [x_j, (f(x_j) - \varepsilon, f(x_j) + \varepsilon)]$. For each $j \leqslant k$ we have that $\{n < \omega\colon x_j \in \operatorname{coz}_\varepsilon(f_n - f)\} \in p$ and so

$$\bigcap_{j \leqslant k} \{n < \omega\colon |f_n(x_j) - f(x_j)| < \varepsilon\} \subseteq \{n < \omega\colon f_n \in W\} \in p.$$

Therefore, $f = p$-lim f_n. □

The following lemma generalizes Lemma 2.7 and plays a very important role in the proof of Theorem 2.10 below.

Lemma 2.8. *Let $p \in \omega^*$, X a space, $f \in C_\pi(X)$, $(f_n)_{n<\omega}$ a sequence in $C_\pi(X)$ and $(\varepsilon_n)_{n<\omega}$ a sequence of positive real numbers. If for every $\varepsilon > 0$ we have that $\{n<\omega: \varepsilon_n < \varepsilon\} \in p$ and $X = \underline{\mathrm{Lim}}_p \mathrm{coz}_{\varepsilon_n}(f_n - f)$, then $f = p$-$\lim f_n$.*

Proof. Let $W = \bigcap_{j \leqslant k}[x_j, (f(x_j) - \varepsilon, f(x_j) + \varepsilon)]$, where $\{x_0, \dots, x_k\} \subseteq X$ and $\varepsilon > 0$. Since $\underline{\mathrm{Lim}}_p \mathrm{coz}_{\varepsilon_n}(f_n - f) = X$, we have that $\{n<\omega: x_0, \dots, x_k \in \mathrm{coz}_{\varepsilon_n}(f_n - f)\} \in p$. This implies that

$$\{n<\omega: |f_n(x_j) - f(x_j)| < \varepsilon_n < \varepsilon\} \subseteq \{n<\omega: f_n \in W\} \in p$$

for every $j \leqslant k$. That is, $f = p$-$\lim f_n$. □

The next corollary is a direct application of Lemma 2.8.

Corollary 2.9. *Let $p \in \omega^*$, f, $(f_n)_{n<\omega}$ and $(\varepsilon_n)_{n<\omega}$ as in Lemma 2.8. If $\varepsilon_0 > \cdots > \varepsilon_n > \cdots > 0$ and $\varepsilon_n \to 0$, then $f = p$-$\lim f_n$.*

Theorem 2.10. *Let $p \in \omega^*$. Then $C_\pi(X)$ is an FU(p)-space if and only if X has (γ_p).*

Proof. ($\Rightarrow$) Assume that $C_\pi(X)$ is an FU(p)-space and let $\mathscr{G}$ be an open ω-cover of X such that $X \notin \mathscr{G}$. If $\Phi = \Phi(\mathscr{G}, 0, 1)$, then $0 \in \mathrm{Cl}\ \Phi \setminus \Phi$ (see Lemma 2.6). Since $C_\pi(X)$ is an FU(p)-space there is a sequence $(f_n)_{n<\omega}$ in Φ such that $0 = p$-$\lim f_n$. For each $n<\omega$ choose $G_n \in \mathscr{G}$ such that $\mathrm{coz}_1 f_n \subseteq G_n$. We verify that $X = \underline{\mathrm{Lim}}_p G_n$. Indeed, fix $x \in X$. Since $0 = p$-$\lim f_n$, $\{n<\omega: |f_n(x)| < 1\} \in p$. Hence,

$$\{n<\omega: x \in \mathrm{coz}_1 f_n\} \subseteq \{n<\omega: x \in G_n\} \in p.$$

($\Leftarrow$) Let $\Phi \subseteq C_\pi(X)$ such that $0 \in \mathrm{Cl}\ \Phi \setminus \Phi$. Without loss of generality we may assume that X is infinite. Let $\{x_n: n<\omega\}$ be an infinite subset of X such that $x_n \neq x_m$ for $n<m<\omega$. By Lemma 2.5, for each $n<\omega$, $\mathscr{G}_n = \mathscr{G}(\Phi, 0, \varepsilon_n)$ is an open ω-cover of X, where $\varepsilon_n = 1/2^n$. Define, for each $n<\omega$, $\mathscr{U}_n = \{G \setminus \{x_n\}: G \in \mathscr{G}_n\}$ and $\mathscr{U} = \bigcup_{n<\omega} \mathscr{U}_n$. It is not hard to prove that $\mathscr{U}$ is an open ω-cover of X. Since X has (γ_p) there is a sequence $(U_k)_{k<\omega}$ in $\mathscr{U}$ such that $X = \underline{\mathrm{Lim}}_p U_k$. For each $k<\omega$ there is $f_k \in \Phi$, $n_k < \omega$ and $G_k \in \mathscr{G}_{n_k}$ such that $U_k = G_k \setminus \{x_{n_k}\}$ and $G_k = \mathrm{coz}_{\varepsilon_{nk}} f_k$. Thus, $X = \underline{\mathrm{Lim}}_p(\mathrm{coz}_{\varepsilon_{nk}} f_k)$. Suppose that there is $\varepsilon > 0$ such that $\{k<\omega: \varepsilon < \varepsilon_{n_k}\} \in p$. Then there is $m<\omega$ such that $A = \{k<\omega: n_k = m\} \in p$ and so $X = \bigcup_{k \in A} U_k$ and $U_k \in \mathscr{U}_m$ for each $k \in A$, which is a contradiction since $x_m \notin U$ for all $G \in \mathscr{U}_m$. Thus, $\{k<\omega: \varepsilon_{n_k} < \varepsilon\} \in p$ for all $\varepsilon > 0$. The conclusion follows from Corollary 2.9. □

Malykhin and Shakhmatov [18] have shown that if we add a single Cohen real to a countable model of MA + $\neg$CH, then in the generic extension there exist two spaces X and Y such that $C_\pi(X)$ and $C_\pi(Y)$ are Fréchet–Urysohn (hence, FU(p)-spaces for all $p \in \omega^*$) and $C_\pi(X) \times C_\pi(Y)$ has uncountable tightness; in particular, it is not an FU(p)-space for each $p \in \omega^*$. Van Douwen remarked (see [6, p. 1222]) that the results of [21] imply the existence of spaces X and Y of weight not bigger than 2^ω such that X and Y have (ε) and $X \times Y$ is not Lindelöf. In virtue of Theorems 2.3 and 2.10, we may find $p \in \omega^*$ so that $C_\pi(X)$ and $C_\pi(Y)$ are FU(p)-spaces and, by [2, 4.1.2], the tightness of $C_\pi(X \times Y)$ is uncountable; that is, $C_\pi(X \times Y)$ cannot be a FU(p)-space.

As a consequence of Theorem 2.10 we have the following result similar to Theorem 1 in [24]:

Corollary 2.11. *Let $p \in \omega^*$. If $C_\pi(X)$ is an* FU(p)*-space, then $C_\pi(X)^\omega$ is so.*

Proof. Assume that $C_\pi(X)$ is an FU(p)-space. By Theorem 2.10, we have that X has (γ_p) and, by Theorem 2.1(c), $X \times X$ has (γ_p). We may assume that X is infinite. Choose a countably infinite discrete subset D of X. Since $X \times D$ is an F_σ-subset of $X \times X$, we obtain that $X \times D \cong X \times \omega$ has (γ_p). From Theorem 2.10 it follows that $C_\pi(X \times \omega) \cong C_\pi(X)^\omega$ is an FU(p)-space. □

A generalization of Theorem 2.3 can be achieved by using Theorem 2.10 and Theorem 3.12 from [14] as follows:

Theorem 2.12. *If $d(X) = \omega = L(X^n)$ for all $1 \leqslant n < \omega$, then there is $p \in \omega^*$ such that X has (γ_p).*

Proof. If $d(X) = L(X^n) = \omega$ for all $1 \leqslant n < \omega$, then $|C_\pi(X)| \leqslant 2^\omega$ and, by a theorem of Arhangel'skii and Pytkeev [2,4.1.2], $t(C_\pi(X)) = \omega$. Applying 3.12 from [14], there is $p \in \omega^*$ for which $C_\pi(X)$ is an FU(p)-space. By virtue of Theorem 2.10, X has (γ_p). □

Now, we consider the following generalization of (γ') in [16]:

(γ'_p) if $(\mathcal{G}_n)_{n<\omega}$ is a sequence of open ω-covers of X, then for each $n < \omega$ there is $G_n \in \mathcal{G}_n$ such that $X = \underline{\mathrm{Lim}}_p G_n$.

It is shown in [16] that X has $(\gamma) \Leftrightarrow X$ has $(\gamma') \Leftrightarrow C_\pi(X)$ is a strictly Fréchet–Urysohn space (X is a strictly Fréchet–Urysohn space if $x \in \mathrm{Cl}(A_n)$ for $n < \omega$ implies that $x_n \to x$, where $x_n \in A_n$ for $n < \omega$). By slightly modifing the proof of Theorem 2.10, we have:

Theorem 2.13. *Let $p \in \omega^*$ and X a space. Then X has (γ'_p) if and only if $C_\pi(X)$ is a strictly* FU(p)*-space.*

Also, Gerlits and Nagy [16] proved that a subset of $\mathbb{R}$ with (γ) has the Rothberger property C'' (hence, it is a strong measure zero set): a space X has C'' provided that for each sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X there is a sequence $(G_n)_{n<\omega}$, with $G_n \in \mathcal{G}_n$ and $X = \bigcup_{n<\omega} G_n$. Daniels [9,p.100] gave a more direct proof of this result. The argument used in Daniels' proof also shows:

Theorem 2.14. *Let $p \in \omega^*$. If $X \subseteq \mathbb{R}$ has (γ'_p), then X has C'' and hence X has strong measure zero.*

We remark, by Theorems 2.3, 2.10, 2.13 and 2.14, that there is $p \in \omega^*$ such that $C_\pi(\mathbb{R})$ is an FU(p)-space and is not a strictly FU(p)-space. On the other hand, the authors of [16] established the equivalence of the Fréchet–Urysohn property and the strictly Fréchet–Urysohn property on the spaces of continuous functions; more general, Nyikos [20] proved that every Fréchet–Urysohn topological group is strictly Fréchet–Urysohn.

3. The reals $\mathbb{R}$ and property (γ_p)

In [7, Q. 484], the following question is posed.

(III) Does $\xi(p)$ embed as a closed subspace into a p-sequential group?

It is well known that X is a closed subspace of the topological group $C_\pi(C_\pi(X))$, for each space X. This makes natural to ask:

Question 3.1. For each $p \in \omega^*$ does $C_\pi(\xi(p))$ have (γ_p)?

We will answer this question in the negative fashion when p is a semiselective ultrafilter on ω. Unfortunately, we do not know any example of a point $p \in \omega^*$ for which $C_\pi(\xi(p))$ has (γ_p) yet. Nevertheless, by Corollary 2.4, for each $p \in \omega^*$ there is $q \in \omega^*$ such that $C_\pi(\xi(p))$ has (γ_q).

3.2. Let $p \in \omega^*$. It is known that $C_\pi(\xi(p))$ is linearly isomorphic to the product $\mathbb{R} \times C^0_\pi(\xi(p))$, where $C^0_\pi(\xi(p)) = \{(x_n)_{n<\omega} \in \mathbb{R}^\omega\colon (\forall \varepsilon > 0)(\exists A \in p)[\,|x_n| < \varepsilon$ for all $n \in A]\}$. We can identify p with the closed subset $\{\chi_A\colon A \in p\}$ of $C^0_\pi(\xi(p))$, where $\chi_A(n) = 0$ if $n \in A$ and $\chi_A(n) = 1$ if $n \notin A$ for $n < \omega$. Also, the topology of p as a subspace of 2^ω coincides with the topology on p inherited from $C^0_\pi(\xi(p))$.

Lemma 3.3. *Let p, $q \in \omega^*$. If $p \subseteq 2^\omega$ satisfies (γ_q) then $p \leqslant_{\mathrm{RK}} q$.*

Proof. Assume that $p = \{\chi_A\colon A \in p\} \subseteq 2^\omega$ satisfies (γ_q). For $n < \omega$, set $G_n = \{\chi_A \in p\colon n \in A\}$ and notice that G_n is an open subset of p. If $A_0, \ldots, A_k \in p$ and $m \in \bigcap_{j=0}^{k} A_j$, then $\{\chi_{A_0}, \ldots, \chi_{A_k}\} \subseteq G_m$. Hence, $\mathscr{G} = \{G_n\colon n < \omega\}$ is an open ω-cover of p. Then, there is a sequence $(G_{n_j})_{j<\omega}$ in $\mathscr{G}$ such that $p = \underline{\mathrm{Lim}}_{\,q} G_{n_j}$. Define $f\colon \omega \to \omega$ by $f(j) = n_j$ for each $j < \omega$. We claim that $\bar{f}(q) = p$, In fact, if $A \in p$, then $f^{-1}(A) = \{j < \omega\colon f(j) = n_j \in A\} = \{j < \omega\colon \chi_a \in G_{n_j}\} \in q$. Thus, $\bar{f}(q) = p$; that is, $p \leqslant_{\mathrm{RK}} q$. $\square$

In 3.2, we pointed out that p as a subspace of 2^ω can be embedded as a closed subspace of $C_\pi(\xi(p))$ for each $p \in \omega^*$. This observation and Lemma 3.3 imply:

Corollary 3.4. *Let p, $q \in \omega^*$. If $C_\pi(\xi(p))$ has (γ_q) then $p \leqslant_{\mathrm{RK}} q$.*

Corollary 3.5. *For every $p \in \omega^*$ there is a subset X of $\mathbb{R}$ such that X does not have (γ_p).*

Proof. Let $p \in \omega^*$. Choose $q \in \omega^*$ such that $p <_{\mathrm{RK}} q$. We have that $q \subseteq 2^\omega \subseteq \mathbb{R}$, where $2^\omega = \{\sum_{n=1}^\infty i_n/3^n\colon i_n = 0$ or $i_n = 2$, for $1 \leqslant n < \omega\}$ and $q = \{\sum_{n=1}^\infty i_n/3^n\colon \{n < \omega\colon i_n = 0\} \in q\}$. If q has (γ_p), by Lemma 3.3 we obtain that $q \leqslant_{\mathrm{RK}} p$, which is a contradiction. Thus, $X = q$ does not have (γ_p). $\square$

Galvin and Miller [13] pointed out that the combinatorial principle $P(c)$ is equivalent to the statement: every subspace X of $\mathbb{R}$ of cardinality less that the continuum has (γ). In this equivalence, we may replace (γ) by (γ_p) for any $p \in \omega^*$.

For each space X we can embed $\mathbb{R}$ as a closed subspace of $C_\pi(X)$: we identify each $r \in \mathbb{R}$ with the constant function of $C_\pi(X)$ of value r. From Theorem 2.1(b) it follows that $\mathbb{R}$ has (γ_p) whenever $C_\pi(\xi(p))$ has (γ_p) for $p \in \omega^*$. Thus, Question 3.1 can be reduced to the following:

(IV) Does $\mathbb{R}$ have (γ_p) for each $p \in \omega^*$?

Next, we shall show that $\mathbb{R}$ does not have (γ_p) whenever every RK-predecessor of p is rapid (this is the case when p is semiselective and when $p = q^\omega$ for some selective ultrafilter q). It is interesting to note that (by Theorem 2.3) there is $p \in \omega^*$ such that $\mathbb{R}$ has (γ_p), and that $\mathbb{R}$ cannot have (γ) since every space with (γ) is zero-dimensional [16, p.157].

Theorem 3.6. *If all* RK-*predecessors of $p \in \omega^*$ are rapid, then every subset of $\mathbb{R}$ with (γ_p) has measure zero. In particular, $\mathbb{R}$ does not have (γ_p).*

Proof. Let X be a subset of $\mathbb{R}$ with (γ_p). Without loss of generality we may assume that $X \subseteq [0, \infty)$. Fix $\varepsilon > 0$ and, for each $n < \omega$, we let

$$\begin{aligned}\mathscr{G}'_n = &\left\{\left[0, \varepsilon/n^2 2^{n+1}\right)\right\}\\ &\cup \left\{\left(k\varepsilon/n^2 2^{n+2}, (k+2)\varepsilon/n^2 2^{n+2}\right)\colon 1 \leqslant k \leqslant n^3 2^{n+2} - 3\right\}\\ &\cup \left\{\left(\varepsilon n - \varepsilon/n^2 2^{n+1}, \varepsilon n\right]\right\},\end{aligned}$$

and

$$\mathscr{G} = \left\{\bigcup_{j=1}^k I_j\colon I_j \in \mathscr{G}'_n \text{ for } 1 \leqslant j \leqslant k \leqslant n \text{ and } I_i \cap I_j = \emptyset \text{ for } i \neq j\right\}.$$

Observe that if $G \in \mathscr{G}_n$ and if μ is the Lebesgue measure on $\mathbb{R}$, then $\mu(G) \leqslant \varepsilon/n2^{n+1}$, for $n < \omega$. It is not hard to show that $\mathscr{G} = \bigcup_{n<\omega} \mathscr{G}_n$ is an open ω-cover of X. Now, enumerate faithfully $\mathscr{G}$ by $\{G_n\colon n < \omega\}$. In virtue of Lemma 2.2 there is $q \in \omega^*$ such that $q \leqslant_{\mathrm{RK}} p$ and $X \subseteq \underline{\mathrm{Lim}}_q G_n$. Define $f\colon \omega \to \omega$ by $f(k) = \max\{n < \omega\colon G_n \in \mathscr{G}_k\}$ for each $k < \omega$. Since q is rapid, there is $A \in q$ such that $|A \cap f(k)| \leqslant k$ for all $k < \omega$. For $k < \omega$ we set $A_k = \{n < \omega\colon G_n \in \mathscr{G}_k\}$ which is a finite subset of ω. If $n \in A_k$, then $G_n \in \mathscr{G}_k$ and $n \leqslant f(k)$, for $n,k < \omega$. Thus, $|A \cap A_k| \leqslant |A \cap$

$f(k)| \leqslant k$ for all $k < \omega$. For each $k < \omega$, put $A \cap A_k = \{n(k, 1), \ldots, n(k, r_k)\}$, where $r_k \leqslant k$. We have that $A = \bigcup_{k<\omega} A_k$ and so $X \subseteq \bigcup_{k<\omega} \bigcup_{j=1}^{r_k} G_{n(k,j)}$. Since

$$\mu\left(\bigcup_{j=1}^{r_k} G_{n(k,j)}\right) \leqslant \sum_{j=1}^{r_k} \mu(G_{n(k,j)}) \leqslant r_k \varepsilon / k 2^{k+1} \leqslant \varepsilon / 2^{k+1}$$

for each $k < \omega$, we have that $\mu(X) \leqslant \sum_{j=1}^{\infty} \varepsilon/2^{k+1} = \varepsilon/2 < \varepsilon$. Therefore, $\mu(X) = 0$, as required. □

For a selective $p \in \omega^*$ we have a stronger result:

Theorem 3.7. *If $p \in \omega^*$ is selective and X has (γ_p), then X has property C''. Hence, every subset of $\mathbb{R}$ with (γ_p) has strong measure zero..*

Proof. Let $(\mathscr{G}_n)_{n<\omega}$ be a sequence of open covers of X. Let $\{A_k: k < \omega\}$ be a partition of ω in infinite subsets. For each $k < \omega$ we define $\mathscr{D}_k = \{\bigcup_{j \leqslant m_k} G_j$: $\exists m_k \leqslant k$ and for each $j \leqslant m_k$, $\exists t_j \in A_k$ with $G_j \in \mathscr{G}_{t_j}\}$. Observe that $\bigcup_{k<\omega} \mathscr{D}_k$ is an open ω-cover of X for $k < \omega$. We may assume that X is infinite. Then, choose an infinite subset $\{x_k: k < \omega\}$ of X. Now, for $k < \omega$, let us set $\mathscr{F}_k = \{G \setminus \{x_k\}$: $G \in \mathscr{D}_k\}$. It is clear that $\mathscr{F}_k$ does not cover X for each $k < \omega$, and $\mathscr{F} = \bigcup_{k<\omega} \mathscr{F}_k$ is an open ω-cover of X. By assumption there is a sequence $(H_n)_{n<\omega}$ in $\mathscr{F}$ such that $X = \underline{\mathrm{Lim}}_p H_n$. For $k < \omega$ let $B_k = \{n < \omega$: $H_n \in \mathscr{F}_k\}$. Since p is selective and $B_k \notin p$ for each $k < \omega$, there is $B \in p$ such that $|B \cap B_k| \leqslant 1$ for all $k < \omega$. For $k < \omega$ we set $B \cap B_k = \{n_k\}$. We have that for each $k < \omega$, $H_{n_k} = \bigcup_{j \leqslant m_k} G_j^{n_k}$, where $G_j^{n_k} \in \mathscr{G}_{t(j,n_k)}$ and $t(j, n_k) \in A_{n_k}$ for $j \leqslant m_k \leqslant n_k$. Thus, $X = \bigcup_{k<\omega} H_{n_k} = \bigcup_{k<\omega} \bigcup_{j \leqslant m_k} G_j^{n_k}$ and $t(i, n_{k_0}) \neq t(j, n_{k_1})$ whenever $k_0 \neq k_1$, $i \leqslant m_{k_0}$ and $j \leqslant m_{k_1}$, as required. □

The following lemma is known and it is a consequence of the facts that every RK-predecessor of a P-point if a P-point and every RK-predecessor of a rapid ultrafilter via a finite-to-one function is rapid. Here, we include a proof.

Lemma 3.8. *If $p \in \omega^*$ is semiselective, then every* RK*-predecessor of p is semiselective.*

Proof. Assume that $p \in \omega^*$ is semiselective and let $q \leqslant_{\mathrm{RK}} p$. It is not hard to prove that q is a P-point as well. So, we only need to show that q is rapid. Indeed, let $f: \omega \to \omega$ be a function and let $g: \omega \to \omega$ be onto such that $\bar{g}(p) = q$. Since p is a P-point and $g^{-1}(n) \notin p$ for $n < \omega$, there is $A \in p$ such that $|A \cap g^{-1}(n)| < \omega$. Hence, without loss of generality we may suppose that $g^{-1}(n)$ is finite for all $n < \omega$. Now, define $h: \omega \to \omega$ by $h(n) = \max\{k < \omega$: $q(k) \leqslant f(n)\}$ for each $n < \omega$. Since p is rapid there is $B \in p$ such that $|B \cap h(n)| \leqslant n$ for all $n < \omega$. For $n < \omega$, if $k \in B$ and $g(k) \leqslant f(n)$, then $k \leqslant h(n)$ and $k \in B \cap h(n)$; hence, $|g(B) \cap f(n)| \leqslant |B \cap h(n)| n$ for each $n < \omega$, and $g(B) \in q$. □

As an immediate consequence of Theorem 3.6 and Lemma 3.8 we have:

Corollary 3.9. *If $p \in \omega^*$ is semiselective, then $\mathbb{R}$ does not have (γ_p).*

Next, we will show in Lemmas 3.12 and 3.15 that $\mathbb{R}$ does not have (γ_{p^θ}) for all $\theta \leqslant \omega$, whenever p is a selective ultrafilter on ω. We need the following definition and lemmas.

Definition 3.10. For p, $q \in \omega^*$, the *tensor* product of p and q is

$$p \otimes q = \{A \subseteq \omega \times \omega \colon \{n < \omega \colon \{m < \omega \colon (n, m) \in A\} \in q\} \in p\}.$$

Observe that $p \otimes q$ is an ultrafilter on $\omega \times \omega$, and it can be viewed as an ultrafilter on ω via a fixed bijection between ω and $\omega \times \omega$. This product $\otimes$ is not an associative operation on ω^*. Nevertheless, $\otimes$ induces a semigroup structure on the set of types of ω^* by setting $T(p) \otimes T(q) = T(p \otimes q)$ for p, $q \in \omega^*$. Thus, if $p \in \omega^*$ then p^n stands for any point in $T(p)^n$ for $1 \leqslant n < \omega$. For each $p \in \omega^*$ Booth [4] defined $T(p)^\omega$ as follows: choose an embedding $e : \omega \to \omega^*$ such that $e(n) \approx_{\mathrm{RK}} p^n$ for $1 \leqslant n < \omega$, then $T(p)^\omega = T(\bar{e}(p))$. As above, p^ω stands for a point in $T(p)^\omega$ for $p \in \omega^*$.

We omit the proof of the following two lemmas.

Lemma 3.11 (Miller [19]). *Let p, $q \in \omega^*$. Then $p \otimes q$ is rapid if and only if q is rapid.*

Lemma 3.12 [14]. *If $p \in \omega^*$ is selective and $1 \leqslant n < \omega$, then the set of all* RK*-predecessors of p^n is $P_{\mathrm{RK}}(p^n) = \bigcup_{1 \leqslant k \leqslant n} T(p)^k$.*

The next result is a direct application of Theorem 3.6 and Lemmas 3.11 and 3.12.

Corollary 3.13. *If $p \in \omega^*$ is selective, then $\mathbb{R}$ does not have (γ_{p^n}) for each $1 \leqslant n < \omega$.*

Lemma 3.14. *Let $p \in \omega^*$ and let $\{p_n \colon n < \omega\}$ be a set of rapid ultrafilters on ω. If $q = p$-$\lim p_n$, then q is rapid.*

Proof. Assume that $q = p$-$\lim p_n$. Let $\{B_n \colon n < \omega\}$ be a set of pairwise disjoint finite subsets of ω. For each $n < \omega$ choose $A_n \in p_n$ such that

(1) $A_n \cap B_j = \emptyset$ for every $j \leqslant n$;

(2) $|A_n \cap B_m| \leqslant m$ for every $m < \omega$.

We set $A = \bigcup_{n<\omega} A_n$. So $A \in q$ and, for each $m < \omega$, $|A \cap B_m| \leqslant |\bigcup_{j<m}(A_j \cap B_m)| \leqslant \Sigma_{j<m} |A_j \cap B_m| \leqslant m^2$. It then follows from Theorem 3 of [19] that q is rapid. □

Lemma 3.15. *If $p \in \omega^*$ is selective, then every* RK*-predecessor of p^ω is rapid.*

Proof. Let $q \in \omega^*$ such that $q \leqslant_{\mathrm{RK}} p^\omega$. By definition, there are an embedding $e : \omega \to \omega^*$ and a function $f : \omega \to \omega$ such that $\bar{f}(p^\omega) = q$, $e(n) \approx_{\mathrm{RK}} p^n$ for $1 \leqslant n <$

ω, and $\bar{e}(p)=p^{\omega}$. Put $h=\bar{f}\circ e:\omega\to\beta(\omega)$. If $\{n<\omega: h(n)\in\omega\}\in p$, then $q=\bar{h}(p)$ $\leqslant_{\mathrm{RK}}$ p and since p is RK-minimal, $p\approx_{\mathrm{RK}} q$. Then we may assume that $h(n)\in\omega^*$ for each $n<\omega$. We have that $h(n)\leqslant_{\mathrm{RK}} e(n)\approx_{\mathrm{RK}} p^n$ for $1\leqslant n<\omega$. By Lemma 3.12, there is $k_n\leqslant n$ such that $h(n)=p^{k_n}$ for each $1\leqslant n<\omega$. In virtue of Lemma 9.4 in [5], we may suppose that h is an embedding; that is, h is one-to-one and $\{h(n): n<\omega\}$ is discrete in ω^*. By Lemma 3.11, we have that $h(n)$ is rapid for all $1\leqslant n<\omega$. Applying Lemma 3.14 we obtain that $q=p\text{-lim } h(n)=\bar{h}(p)$ is rapid. □

Corollary 3.16. *If $p\in\omega^*$ is selective, then $\mathbb{R}$ does not have $(\gamma_{p^{\omega}})$.*

Booth [4] showed that p^{ω} has 2^{ω} RK-type predecessors for $p\in\omega^*$. Hence, if p is selective, then p^{ω} is not a P-point, it has 2^{ω} RK-type predecessors and $\mathbb{R}$ does not have $(\gamma_{p^{\omega}})$ (by Corollary 3.16). On the other hand, by Lemma 2.3, we can find $p\in\omega^*$ for which p has exactly 2^{ω} type-RK predecessors and $\mathbb{R}$ has (γ_p). These observations suggest the next problem.

Problem 3.17. Classify those $p\in\omega^*$ for which $\mathbb{R}$ does not have (γ_p).

In the next theorem, we show that if $\mathbb{R}$ has (γ_p), then some other spaces do, and vice versa.

Theorem 3.18. *For $p\in\omega^*$ the following are equivalent.*

(1) *$\mathbb{R}$ has (γ_p);*
(2) *$[0, 1]$ has (γ_p);*
(3) *the Cantor space 2^{ω} has (γ_p);*
(4) *$[0, 1]^{\omega}$ has (γ_p);*
(5) *$\Sigma=\bigcup_{n<\omega}[1/n, 1-1/n]^{\omega}$ has (γ_p);*
(6) *$C_{\pi}^*(\omega)$ has (γ_p);*
(7) *every compact metric space has (γ_p);*
(8) *every zero-dimensional, second countable compact space has (γ_p);*
(9) *every σ-compact metric space has (γ_p);*
(10) *every locally compact, separable metric space has (γ_p).*

Proof. We proof the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5); (5) ⇔ (6); (5) ⇒ (9); (7) ⇒ (8); (8) ⇒ (3); (9) ⇒ (10); (10) ⇒ (1); (10) ⇒ (7) and (5) ⇒ (9). In fact, (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are immediate consequences of Theorem 2.1; (5) ⇔ (6) follows from the fact that $C_{\pi}^*(\omega)$ is homeomorphic to the space Σ; (7) ⇒ (8), (9) ⇒ (10), (10) ⇒ (1) and (8) ⇒ (3) are trivial; and (5) ⇒ (9) holds since every σ-compact metric space is homeomorphic to a closed subspace of Σ (see [27, Lemma 5.1]). □

We have shown (in Theorem 2.1(c)) that if X has (γ_p) for $p\in\omega^*$, then X^n has (γ_p) for each $1\leqslant n<\omega$. We do not know whether $\mathbb{R}^{\omega}$ has (γ_p) whenever $\mathbb{R}$ has (γ_p) yet. The following two theorems could be useful.

Theorem 3.19. *If $p \in \omega^*$ is a P-point, then $\mathbb{R}^\omega$ does not have (γ_p).*

Proof. Let $p \in \omega^*$ be a P-point and assume that $\mathbb{R}^\omega$ has (γ_p). For each $n < \omega$ let $\mathscr{G}'_n$ be a countable cover of $\mathbb{R}$ consisting of open intervals of length $1/2^n$. For each $n < \omega$ set

$$\mathscr{G}_n = \left\{ \bigcup_{1 \leqslant j \leqslant n} I_j \colon I_j \in \mathscr{G}'_n,\ I_i \cap I_j = \emptyset \text{ for } i \neq j \right\}.$$

Now, for each $n < \omega$, define $\mathscr{F}_n = \{\Pi_n^{-1}(G) \colon G \in \mathscr{G}_n\}$, where $\Pi_n : \mathbb{R}^\omega \to \mathbb{R}$ is the projection map on the nth coordinate. It is not hard to prove that $\mathscr{F} = \bigcup_{n<\omega} \mathscr{F}_n$ is an open ω-cover of $\mathbb{R}^\omega$. By assumption there is a sequence $(F_n)_{n<\omega}$ in $\mathscr{F}$ such that $\mathbb{R}^\omega = \underline{\operatorname{Lim}}_p F_n$. For each $k < \omega$ put $A_k = \{n < \omega \colon F_n \in \mathscr{F}_k\}$. Then, $\{A_k \colon k < \omega\}$ is a partition of ω. We verify that $A_k \notin p$ for each $k < \omega$. Indeed, assume that $A_k \in p$ for some $k < \omega$. For each $m < \omega$ we have that $S(m) = \{n < \omega \colon m \in F_n\} \in p$ (here, we identify m with the constant function on $\mathbb{R}^\omega$ of value m). Hence, we may take $n \in A_k \cap (\bigcap_{i \leqslant k+1} S(i))$. If $F_n = \Pi_k^{-1}(G)$ for some $G \in \mathscr{G}_k$, then $i \in G$ for each $i \leqslant k+1$, which is a contradiction. Thus, $A_k \notin p$ for all $k < \omega$. Since p is a P-point, there is $A \in p$ such that $|A \cap A_k| < \omega$ for each $k < \omega$, and $\mathbb{R}^\omega = \bigcup_{n \in A} F_n$ (by Lemma 1.4). For each $k < \omega$ enumerate $A \cap A_k = \{n(k, 0), \ldots, n(k, r_k)\}$. We then have that $\mathbb{R}^\omega = \bigcup_{k<\omega} \bigcup_{i \leqslant r_k} F_{n(k,i)}$. For each $k < \omega$ and each $i \leqslant r_k$, choose $G_{n(k,i)} \in \mathscr{G}_k$ so that $F_{n(k,i)} = \Pi_k^{-1}(G_{n(k,i)})$. Notice that $\mathbb{R} \neq \bigcup_{i \leqslant r_k} G_{n(k,i)}$. Hence, for each $k < \omega$ we pick $x_k \in \mathbb{R} \setminus \bigcup_{i \leqslant r_k} G_{n(k,i)}$. If $x = (x_k)_{k<\omega}$, then $x \notin F_{n(k,i)}$ for each $k < \omega$ and each $i \leqslant r_k$. But this is impossible. Therefore, $\mathbb{R}^\omega$ does not have (γ_p). □

Theorem 3.20. *Let $p \in \omega^*$. If all the spaces of any of the following classes have (γ_p), then the other classes of spaces do.*

(1) $\mathbb{R}^\omega$;
(2) ω^ω;
(3) *the hedgehog* $J(\omega)^\omega$;
(4) *the Hilbert space* $\mathbb{H}$;
(5) $\mathbb{H}^\omega$;
(6) $\sigma_\omega = \{x \in \mathbb{H} \colon x_i = 0$ *for all but finitely many* $i\}^\omega$;
(7) *every* $C_\pi(X)$ *for a metric countable space* X;
(8) *every* $C_\pi^*(X)$ *for a separable metric space* X;
(9) *every* $C_\pi(X)$ *for a countable space* X *such that* $C_\pi(X)$ *is an* $F_{\sigma\delta}$*-subset of* $\mathbb{R}^\omega$;
(10) *every* $C_\pi^*(X)$ *for a countable space* X *such that* $C_\pi(X)$ *is an* $F_{\sigma\delta}$*-subset of* $\mathbb{R}^\omega$;
(11) *every* $F_{\sigma\delta}$*-subset of* $[0, 1]^\omega$;
(12) *every separable, completely metrizable space*;
(13) *every zero-dimensional, separable completely metrizable space*;
(14) *every* G_δ*-subset of a separable completely metrizable space*;
(15) *every* G_δ*-subset of* $\mathbb{R}$.

Proof. The trivial implications are (12) ⇒ (13), (12) ⇒ (14), (14) ⇒ (15) and (15) ⇒ (2); (1) ⇒ (2) follows from Theorem 2.1(b); and (1) ⇔ (3) ⇔ (4) ⇔ (5) are consequences of the facts that $\mathbb{H}$ and $J(\omega)^\omega$ are homeomorphic [25,26] and that $\mathbb{R}^\omega$ and $\mathbb{H}$ are homeomorphic [1].

Since σ_ω, $C_\pi(X)$ and $C_\pi^*(X)$ are homeomorphic whenever X is countable and $C_\pi(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^\omega$ (see [10,11]), we obtain that (6) ⇔ (7) ⇔ (8) ⇔ (9) ⇔ (10).

(11) ⇒ (1) is a direct application of 5.2 in [27].

We have that $\mathbb{R}^\omega$ is homeomorphic to $(0, 1)^\omega$, which is an $F_{\sigma\delta}$-subset of $[0, 1]^\omega$; hence, (11) ⇒ (1).

Since Σ is a continuous image of the disjoint union of countably many copies of $[0, 1]^\omega$, say $\oplus_{n<\omega}([0, 1]^\omega)_n \cong \omega \times [0, 1]^\omega$, $\Sigma^\omega \cong \sigma_\omega$ is a continuous image of the closed subspace $(\omega \times [0, 1]^\omega)^\omega \cong \omega^\omega \times [0, 1]^\omega$ of $\mathbb{R}^\omega$ and so (1) ⇒ (6).

We know that every zero-dimensional, separable completely metrizable space is homeomorphic to a closed subspace of ω^ω (see [12, 7.3.H]) and then (2) ⇒ (13).

From the facts that $\omega^\omega \times 2^\omega$ satisfies the conditions of (13), σ_ω is a continuous image of $\omega^\omega \times [0, 1]^\omega$ and $\omega^\omega \times [0, 1]^\omega$ is a continuous image of $\omega^\omega \times 2^\omega$, it follows that (13) ⇒ (6). □

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