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SOME GENERALIZATIONS OF RAPID ULTRAFILTERS IN TOPOLOGY AND ID-FAN TIGHTNESS

By

Salvador GARCÍA-FERREIRA and Angel TAMARIZ-MASCARÚA

Abstract. In this paper, we introduce the weakly k-rapid points, for $1\leqq k<\omega$, and the rapid points of topological spaces. They extend the concept of rapid ultrafilter. It is evident from the definition that every weak P-point is a rapid point and a weakly k-rapid point for $1\leqq k<\omega$. We show: (a) there is a space containing a rapid, non-weak-P-point $\Leftrightarrow$ there is a rapid ultrafilter on ω; and (b) there is a space containing a weakly k-rapid, non-weak-P-point, for some $1\leqq k<\omega\Leftrightarrow$ there is a Q-point in $\beta(\omega)\backslash\omega\Leftrightarrow$ for every $1\leqq k<\omega$, there is a space which is weakly $(k+1)$-rapid and is not weakly k-rapid. Assuming the existence of a Q-point in $\beta(\omega)\backslash\omega$, we give an example of a zero-dimensional homogeneous space without weak P-points such that all its points are rapid. Finally, the concept of Id-fan tightness is introduced as a generalization of countable strong fan tightness.

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1. Preliminaries.

By a space we mean a completely regular Hausdorff space, i.e. Tychonoff space. If X is a space and $x\in X$, then $\mathcal{N}(x)$ denotes the set of all neighborhoods of x. The closure of A in X is denoted by $\mathrm{Cl}_x(A)$ or $\mathrm{Cl}(A)$. For a set X, the set of all finite subsets of X is denoted by $[X]^{<\omega}$ and if $1\leqq m<\omega$, then $[X]^{\leq m}=\{A\subseteq X: |A|\leqq m\}$. The Stone-Čech compactification $\beta(\omega)$ of the natural numbers ω with the discrete topology can be viewed as the set of all ultrafilters on ω, and the remainder $\omega^*=\beta(\omega)\backslash\omega$ consists of all free ultrafilters on ω. For

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$p \in \omega^*$, $\xi(p)$ stands for the subspace $\{p\} \cup \omega$ of $\beta(\omega)$. All functions $f \in {}^\omega\omega$ considered throughout this paper assume only positive values.

G. Mokobodski [Mo] introduced the following class of ultrafilters to respond to a problem in measure theory.

1.1. DEFINITION. $p \in \omega^*$ is *rapid* if

$$\forall h \in {}^\omega\omega \exists A \in p \forall n < \omega (|A \cap h(n)| \leqq n).$$

Other two kinds of interesting ultrafilters on ω are:

1.2. DEFINITION. Let $p \in \omega^*$. Then

(1) p is a *Q-point* if for every partition $\{B_n : n < \omega\}$ of ω in finite subsets, there is $A \in p$ such that $|A \cap B_n| \leqq 1$ for every $n < \omega$;

(2) p is *semiselective* if $A_n \in p$ for $n < \omega$, then there is $a_n \in A_n$ for each $n < \omega$ such that $\{a_n : n < \omega\} \in p$.

In [CV], the authors say that $p \in \omega^*$ is a Q-point if $\forall \{B_n : n < \omega\} \subseteq [\omega]^{<\omega} \exists A \in p \forall n < \omega (|A \cap B_n| \leqq 1)$. But, this definition is wrong since none $p \in \omega^*$ satisfies such a condition; indeed, if $p \in \omega^*$ and $B_n = n$ for $n < \omega$, then there is not $A \in p$ such that $|A \cap B_n| \leqq 1$ for each $n < \omega$.

We know that every semiselective ultrafilter is rapid and every Q-point is rapid. The inclusions among these sorts of ultrafilters on ω are proper: It is shown in [M] that if there is a rapid ultrafilter, then there is also a rapid ultrafilter which is neither P-point and nor Q-point; (Kunen [K]) $MA \to \exists p \in \omega^*$ (p is semilective and not Q-point); and Lafflamme [L] proved that $CON(ZFC) \to CON(ZFC + \exists p \in \omega^*$ (p is Q-point and not semiselective)). The existence of these ultrafilters is independent from the axioms of ZFC. In fact, Mokobodki [Mo] proved that CH implies the existence of rapid ultrafilters on ω; Miller [M] established that $CON(ZFC) \to CON(ZFC +$there are no rapid ultrafilters); Mathias [Ma] and Taylor [T] showed that if there is a dominant family of functions in ${}^\omega\omega$ of cardinality ω_1, then there exists a Q-point in ω^* (for another sufficient condition see [CV]); and the existence of semiselective ultrafilters under MA (σ-centered) is shown in [Bo].

In the next theorem, we give four conditions which are equivalent to the rapidness of ultrafilters on ω: clauses (4) and (5) motivated the notions of rapid points and weakly k-rapid points, for $1 \leqq k < \omega$, which will be studied in section 2.

1.3. THEOREM. *For $p \in \omega^*$, the following are equivalent:*

(1) *p is rapid.*

(2) *For every sequence $(B_n)_{n<\omega}$ of finite subsets of ω,*

$$\exists A\in p\forall n<\omega(|A\cap B_n|\leqq n).$$

(3) *There is $h\in{}^{\omega}\omega$ such that for every sequence $(B_n)_{n<\omega}$ of finite subsets of ω, $\exists A\in p\forall n<\omega(|A\cap B_n|\leqq h(n))$.*

(4) *For every finite-to-one function $f\in{}^{\omega}\omega$ and every sequence $(B_n)_{n<\omega}$ of finite subsets of ω, $\exists A\in p\forall n<\omega(|A\cap B_n|\leqq f(n))$.*

(5) *For every finite-to-one function $f\in{}^{\omega}\omega$ and given $B_n\in[\omega]^{<\omega}$, for $n<\omega$, such that $B_n\cap B_m=\emptyset$ whenever $n<m<\omega$,*

$$\exists A\in p\forall n<\omega(|A\cap B_n|\leqq f(n)).$$

PROOF. The equivalences (1)$\Leftrightarrow$(2) and (2)$\Leftrightarrow$(3) are shown in [M], and the implications (1)$\Rightarrow$(5), (4)$\Rightarrow$(3) are evident.

(1)$\Rightarrow$(4). Let $f\in{}^{\omega}\omega$ be finite-to-one. Without loss of generality, we may assume that $B_n\subseteqq B_{n+1}$ for each $n<\omega$. Define $h\in{}^{\omega}\omega$ so that $h(m)=\max f^{-1}(m)$ if $f^{-1}(m)\neq\emptyset$, for $m<\omega$, and put $D_m=B_{h(m)}$ for $m<\omega$. By assumption, there is $A\in p$ such that $|A\cap D_m|=|A\cap B_{h(m)}|\leqq m$ for all $m<\omega$. If $f(n)=m$ for $n<\omega$, then we have that $n\in f^{-1}(m)$ and $|A\cap B_n|\leqq|A\cap B_{h(m)}|\leqq m=f(n)$, as desired.

(5)$\Rightarrow$(3). Let $f\in{}^{\omega}\omega$ be finite-to-one and define $h:\omega\to\omega$ by $h(n)=\sum_{i=0}^{n}f(i)$ for each $n<\omega$. We shall verify that h satisfies our conditions. In fact, let $(B_n)_{n<\omega}$ be a sequence of finite subsets of ω. For $n<\omega$, set $A_n=B_n\setminus\bigcup_{j<n}B_j$. By hypothesis, there is $A\in p$ such that $|A\cap A_n|\leqq f(n)$ for all $n<\omega$. Since $B_n\subseteqq\bigcup_{j\leqq n}A_j$ for each $n<\omega$, we have that $|A\cap B_n|\leqq\sum_{i=0}^{n}f(i)=h(n)$ for each $n<\omega$.

We remark that if a function h satisfies the condition of (3), then h must be finite-to-one. If not, then there is $m<\omega$ such that $h^{-1}(m)=\{m_j:j<\omega\}$, where $m_j<m_{j+1}$ for $j<\omega$, but there is not $A\in p$ such that $|A\cap m_j|\leqq h(m_j)=m$ for every $j<\omega$.

Our work in section 3 is based on the following definition.

1.4. DEFINITION. Let X be a space. Then

(1) [Ar_1] X has *countable tightness* if for each $x\in X$ and $A\subseteqq X$ such that $x\in\mathrm{Cl}(A)$ there is a countable subset B of A such that $x\in\mathrm{Cl}(B)$;

(2) [Ar_2] X has *countable fan tightness* if for every $x\in X$ and every sequence $(A_n)_{n<\omega}$ of subsets of X such that $x\in\bigcap_{n<\omega}\mathrm{Cl}(A_n)$, there exists $F_n\in[A_n]^{<\omega}$ such that $x\in\mathrm{Cl}(\bigcup_{n<\omega}F_n)$;

(3) [S] X has *countable strong fan tightness* if for every $x\in X$ and every

sequence $(A_n)_{n<\omega}$ of subsets of X such that $x\in\bigcap_{n<\omega}\mathrm{Cl}(A_n)$, there exists $x_n\in A_n$ such that $x\in\mathrm{Cl}(\{x_n : n<\omega\})$.

A natural generalization of countable strong fan tightness is investigated in section 3.

2. Rapid points and weakly k-rapid points.

Clauses (4) and (5) of Theorem 1.3 suggest the following definition.

2.1. DEFINITION. Let $f\in{}^{\omega}\omega$ and X a space.

(1) A point $x\in X$ is called *f-rapid* if for every sequence $(B_n)_{n<\omega}$ of finite subsets of $X\setminus\{x\}$, $\exists V\in\mathcal{N}(x)\forall n<\omega(|V\cap B_n|\leqq f(n))$. X is said to be *f-rapid* if all points of X are f-rapid.

(2) A point $x\in X$ is called *weakly f-rapid* if for every sequence $(B_n)_{n<\omega}$ of finite subsets of $X\setminus\{x\}$ such that $B_n\cap B_m=\emptyset$ whenever $n<m<\omega$, $\exists V\in\mathcal{N}(x)\forall n<\omega(|V\cap B_n|\leqq f(n))$. X is said to be *weakly f-rapid* if all points of X are weakly f-rapid.

If f is the identity function from ω to ω, then we simply say rapid (resp. weakly rapid) instead of f-rapid (resp. weakly f-rapid). The meaning of k-rapid and weakly k-rapid should be clear, for $1\leqq k<\omega$. It is evident that $p\in\omega^*$ is a Q-point iff it is weakly k-rapid in $\xi(p)$ for some $1\leqq k<\omega$.

Observe from Theorem 1.3 that $p\in\omega^*$ is a rapid ultrafilter iff p is f-rapid in $\xi(p)$ for each finite-to-one function $f\in{}^{\omega}\omega$ iff p is weakly f-rapid in $\xi(p)$ for each finite-to-one function $f\in{}^{\omega}\omega$. The next lemma shows that we cannot withdram the finite-to-one condition.

2.2. LEMMA. *For $p\in\omega^*$ and $f\in{}^{\omega}\omega$, the following are equivalent.*

(1) *p is f-rapid in $\xi(p)$.*

(2) *f is finite-to-one and p is a rapid ultrafilter.*

(3) *f is finite-to-one and p is weakly f-rapid in $\xi(p)$.*

PROOF. The implications (2)⇒(3) and (3)⇒(1) are direct consequences of Theorem 1.3.

(1)⇒(2). According to Theorem 1.3, it is enough to prove that f is finite-to-one. In fact, assume that there is $m<\omega$ such that $f^{-1}(m)=\{m_j : j<\omega\}$, where $m_j<m_{j+1}$ for $j<\omega$. Define $B_n=\{j<\omega : j\leqq n\}$ for each $n<\omega$. Then there is $A\in p$ such that $|A\cap B_n|\leqq f(n)$ for each $n<\omega$. In particular, $|A\cap B_{m_k}|\leqq f(m_k)$ $=m$ for every $k<\omega$. Since A is infinite, there must be $k<\omega$ such that $|A\cap B_{m_k}|$

$>m$, which is a contradiction.

2.3. LEMMA. *If $f:\omega\to\omega$ is not finite-to-one, then $p\in\omega^*$ is a Q-point iff p is weakly f-rapid in $\xi(p)$.*

PROOF. Only the sufficiency requires proof. Let $\{B_n : n<\omega\}\subseteq[\omega]^{<\omega}$ be a partition of ω and let $m<\omega$ such that $f^{-1}(m)=\{m_j : j<\omega\}$, where $m_j<m_{j+1}$, for $j<\omega$. Define $\{A_k : k<\omega\}$ by $A_{m_j}=\bigcup_{m_j\leqq n<m_{j+1}} B_n$, for each $j<\omega$, and $A_k=\emptyset$ otherwise. By assumption, there is $A\in p$ such that $|A\cap A_k|\leqq f(k)$ for all $k<\omega$. Hence, if $m_j\leqq n<m_{j+1}$, for some $j<\omega$, then $|A\cap B_n|\leqq|A\cap A_{m_j}|\leqq f(m_j)=m$. We may write $A=\bigcup_{1\leqq m} A_i$ so that $|A_i\cap B_n|\leqq 1$ for each $i\leqq m$ and each $n<\omega$. Since $A\in p$, there is $i\leqq m$ such that $A_i\in p$ and then $|A_i\cap B_n|\leqq 1$ for every $n<\omega$. Therefore, p is a Q-point.

We omit the proof of the next theorem since it is completely similar to that of Theorem 1.3.

2.4. THEOREM. *For a finite-to-one function $f\in{}^\omega\omega$ and $x\in X$, the following are equivalent.*

(1) *x is rapid in X.*

(2) *x is f-rapid in X.*

(3) *x is weakly f-rapid in X.*

The relationship between weakly k-rapid points, for $1\leqq k<\omega$, and rapid points is established in the next corollary.

2.5. COROLLARY. *For $1\leqq k<\omega$, every weakly k-rapid point is rapid.*

PROOF. Let $1\leqq k<\omega$ and $x\in X$. Suppose that x is weakly k-rapid in X. Let $(B_n)_{n<\omega}$ be a sequence of finite subsets of $X\setminus\{x\}$. For $n<\omega$, set $A_n=B_n\setminus\bigcup_{j<n} B_j$. By assumption, there is $V\in\mathfrak{N}(x)$ such that $|V\cap A_n|\leqq k$ for each $n<\omega$. Hence, $|V\cap B_n|\leqq\sum_{j\leqq n}|V\cap A_j|\leqq(n+1)k$, since $B_n\subseteq\bigcup_{j\leqq n} A_j$, for each $n<\omega$. Thus, x is f-rapid, where $f(n)=(n+1)k$ for every $n<\omega$. The conclusion now follows from 2.4.

Next, we shall show that if $f\in{}^\omega\omega$ is not finite-to-one, then there is $k<\omega$ such that weak f-rapidness agrees with weak k-rapidness. It will be shown in 2.11 that for every $1\leqq k<\omega$ there is a space which is weakly $(k+1)$-rapid and is not weakly k-rapid.

2.6. THEOREM. *Let $f\in{}^\omega\omega$ be non-finite-to-one and X a space. If $k=$*

$\min\{m<\omega : f^{-1}(m)$ is infinite$\}$, then $x\in X$ is weakly k-rapid iff it is weakly f-rapid.

PROOF. First, assume that $x\in X$ is weakly k-rapid. Let $(B_n)_{n<\omega}$ be a sequence in $[X\setminus\{x\}]^{<\omega}$ such that $B_i\cap B_j=\emptyset$ whenever $i<j<\omega$. Choose $r<\omega$ such that $f^{-1}(m)\subseteq r$ for each $m<k$. Then, we may find $V\in\mathcal{N}(x)$ such that $|V\cap B_n|\leqq k$, for each $n<\omega$, and $V\cap B_n=\emptyset$ for every $n<r$. Hence, if $f(n)<k$, then $|V\cap B_n|=0\leqq f(n)$. Thus, $|V\cap B_n|\leqq f(n)$ for all $n<\omega$.

Now suppose that $x\in X$ is weakly f-rapid and let $(B_n)_{n<\omega}$ be a sequence in $[X\setminus\{x\}]^{<\omega}$ such that $B_i\cap B_j=\emptyset$ whenever $i<j<\omega$. Enumerate $f^{-1}(k)$ by $\{k_n : n<\omega\}$, where $k_n<k_{n+1}$ for $n<\omega$. For every $n<\omega$, set $D_{k_n}=B_n$ and $D_m=\emptyset$ otherwise. Then, there is $V\in\mathcal{N}(x)$ such that $|V\cap D_m|\leqq f(m)$ for each $m<\omega$. Hence, $|V\cap B_n|=|V\cap D_{k_n}|\leqq f(k_n)=k$ for $n<\omega$. This shows that x is weakly k-rapid.

The weakly f-rapid points, for $f\in{}^{\omega}\omega$, satisfy the following property.

2.7. THEOREM. *If $x\in X$ is a weakly f-rapid point for $f\in{}^{\omega}\omega$, then no non-trivial sequence converges to x.*

PROOF. Assume that $\{x_n\}_{n<\omega}$ is a non-trivial sequence converging to a weakly f-rapid point x of a space X. We may assume that $x\neq x_n$ for all $n<\omega$ and $x_n\neq x_m$ for $n<m<\omega$. Define, for each $n<\omega$, $B_n=\{x_m : n+\sum_{i=0}^{n-1}f(i)\leqq m< n+1+\sum_{i=0}^{n}f(i)\}$. Notice that $|B_n|=f(n)+1$ for each $n<\omega$. By assumption there exists $V\in\mathcal{N}(x)$ such that $|V\cup B_n|\leqq f(n)$ for each $n<\omega$. So we may pick $y_n\in(X\setminus V)\cap B_n$ for each $n<\omega$; that is, $B_n\setminus V\neq\emptyset$ for each $n<\omega$. This implies that $(x_n)_{n<\omega}$ does not converge to x, which is a contradiction.

Observe from 2.7 that every non-isolated, weakly f-rapid point of a space has uncountable character.

It is evident that every weak P-point is an f-rapid point for each $f\in{}^{\omega}\omega$. For the converse, we have the following two results. Firt, we state a definition.

2.8. DEFINITION (Bernstein [B]). Let $p\in\omega^*$ and X a space. We say that $x\in X$ is the *p-limit* of a sequence $(x_n)_{n<\omega}$, we write $x=p\text{-}\lim x_n$, if for every $V\in\mathcal{N}(x)$, $\{n<\omega : x_n\in V\}\in p$.

2.9. THEOREM. *Let $f\in{}^{\omega}\omega$. There is a space X containing an f-rapid, non-weak-P-point iff f is finite-to-one and there is a rapid ultrafilter on ω.*

PROOF. Necessity. Let X be a space and $x\in X$ a f-rapid, non-weak-P-

point. Then there exists $\{x_j : j<\omega\} \subseteq X\setminus\{x\}$ such that $x\in \mathrm{Cl}_X\{x_j : j<\omega\}$. It is not hard to prove (see [GS, Lemma 2.2]) that there is $p\in\omega^*$ such that $x=p\text{-}\lim x_j$. We shall verify that p is a rapid ultrafilter on ω. Indeed, let $\{B_n : n<\omega\}\subseteq[\omega]^{<\omega}$ and define $D_n=\{x_j : j\in B_n\}$ for $n<\omega$. By assumption, we can find $V\in\mathcal{N}(x)$ such that $|V\cap D_n|\leqq f(n)$ for each $n<\omega$. Since $x=p\text{-}\lim x_j$, $A=\{j<\omega : x_j\in V\}\in p$. If $j\in A\cap B_n$, then $x_j\in V\cap D_n$. Thus, $A\in p$ and $|A\cap B_n|\leqq f(n)$ for each $n<\omega$. The conclusion now follows from Lemma 2.2.

Sufficiency. If $p\in\omega^*$ is a rapid ultrafilter and f is finite-to-one, by Lemma 2.2, then p is an f-rapid, non-weak-P-point of $\xi(p)$.

As an immediate consequence of the previous theorem we have:

2.10. COROLLARY. *If $f\in{}^\omega\omega$ is not finite-to-one, then the concepts of weak P-point and f-rapid point coincide.*

We remark that if M is a model of ZFC in which there are not rapid ultrafilters on ω (see [M]), then $M\models$ If X is a space, then $x\in X$ is a weak P-point iff x is f-rapid in X for every $f\in{}^\omega\omega$.

2.11. THEOREM. *The following statements are equivalent.*

(1) *There is a space X containing a non-weak-P-poit, weakly k-rapid for some $1\leqq k<\omega$.*

(2) *There is a Q-point $p\in\omega^*$.*

(3) *For every $1\leqq k<\omega$, there is a space which is weakly $(k+1)$-rapid and is not weakly-k-rapid.*

PROOF. To prove (1)$\Rightarrow$(2) we apply the same reasoning used in the proof of Theorem 2.9 and Lemma 2.3, and (1) is the particular case of (3) when $k=1$.

(2)$\Rightarrow$(3). Fix $1\leqq k<\omega$ and let $p\in\omega^*$. We define a topology on $\Xi(p,k)=\{p\}\cup\{(j,n) : j\leqq k,\ n<\omega\}$ as follows: $\{(j,n)\}$ is open for all $j\leqq k$ and $n<\omega$. $V\subseteq\Xi(p,k)$ is a neighborhood of p if $p\in V$ and $\{n<\omega : (j,n)\in V\}\in p$ for each $j\leqq k$. Assume that p is a Q-point. First, we show that $\Xi(p,k)$ is weakly $(k+1)$-rapid. Let $(B_m)_{m<\omega}$ be a sequence in $[\Xi(p,k)\setminus\{p\}]^{<\omega}$. For each $j\leqq k$, put $B_{j,m}=B_m\cap\{(j,n) : n<\omega\}$. Since p is a Q-point there is $A_j\in p$ such that $|A_j\cap B_{j,m}|\leqq 1$ for $m<\omega$. Then $V=\{p\}\cup_{j\leq k}\{(j,n) : n\in A_j\}\in\mathcal{N}(p)$ and it is evident that $|V\cap B_m|\leqq k+1$ for each $m<\omega$. Thus, $\Xi(p,k)$ is weakly k-rapid. Now, define $B_m=\{(j,m) : j\leqq k\}$, for each $m<\omega$, and suppose that $\Xi(p,k)$ is weakly-k-rapid. So there is $W\in\mathcal{N}(p)$ such that $|W\cap B_m|\leqq k$ for each $m<\omega$. Set $A_j=\{n<\omega : (j,n)\in W\}$ for $j\leqq k$. We have that $A=\bigcap_{j\leq k}A_j\in p$. If $m\in A$, then $(j,m)\in W\cap B_m$ for each $j\leqq k$ and so $|W\cap B_m|=k+1$, which is a contradiction.

For $1\leqq k<\omega$, it is not hard to show that if X_i is a weakly k-rapid (resp. rapid) space with more than two points, for $i\in I$, and I is infinite, then $\prod_{i\in I} X_i$ has no weakly k-rapid (resp. rapid) points. For finite products, we have that (p, p) is not weakly $(k+1)$-rapid in $\Xi(p, k)\times\Xi(p, k)$, and if x is rapid in X and y is rapid in Y, then (x, y) is rapid in $X\times Y$.

Next, we give an example, assuming the existence of a rapid ultrafilter on ω, of a rapid homogeneous space without weak P-points.

2.12. EXAMPLE. In [AF], the authors defined the homogeneous zero-dimensional space S_ω. In a similar way, for every $p\in\omega^*$, we may define the space $S_\omega(p)$ by replacing convergence sequences by p-limits in the construction (for a similar procedure see [G-F]). $S_\omega(p)$ is also a homogeneous, zero-dimensional space without weak P-points. For $p\in\omega^*$, set $S_\omega(p)=\{x\}\cup\{x_{n_1,\cdots,n_r}: n_j<\omega$ for $1\leqq j\leqq r<\omega\}$. Then, we have that $x=p\text{-lim}\, x_n$ and $x_{n_1,\cdots,n_r}=p\text{-lim}\, x_{n_1,\cdots,n_r,n}$, for every $n_1, \cdots, n_r<\omega$. To describe a neighborhood of x in $S_\omega(p)$, we put $S(A)=\{x_n : n\in A\}$ and $S(x_{n_1,\cdots,n_r}, A)=\{x_{n_1,\cdots,n_r,n} : n\in A\}$ for $x_{n_1,\cdots,n_r}\in S_\omega(p)$ and for $A\subseteq\omega$. If $\{A\}\cup\bigcup_{i\leq r<\omega}\{A_{n_1,\cdots,n_r}: n_j<\omega$ for $1\leqq j\leqq r\}$ are elements of p, then the set $\{x\}\cup S(A)\cup\bigcup_{1\leq r<\omega}(\bigcup_{n_1\in A}\bigcup_{n_2\in A_{n_1}}\cdots\bigcup_{n_r\in A_{n_1,\cdots,n_{r-1}}} S(x_{n_1,\cdots,n_r}, A_{n_1,\cdots,n_r}))$ is a basic neighborhood of x in $S_\omega(p)$. It is shown in the proof of 2.11 ((2)⇒(3)), the condition of Q-point is not essential, that the space $\Xi(p, k)$ is not weakly k-rapid for each $1\leqq k<\omega$ and for each $p\in\omega^*$. Since $\Xi(p, k)$ is homeomorphic to the subspace $\{x\}\cup\{x_{j,n}: j\leqq k,\ n<\omega\}$ of $S_\omega(p)$ for each $1\leqq k<\omega$, $S_\omega(p)$ is not weakly k-rapid for all $1\leqq k<\omega$. Now suppose that p is a rapid ultrafilter on ω. We shall show that $S_\omega(p)$ is a rapid space. It is enough to prove that x is a rapid point of $S_\omega(p)$. In fact, let $(B_m)_{m<\omega}$ be a sequence of finite subsets of $S_\omega(p)\setminus\{x\}$ and let $\sigma:\omega\to\bigcup_{1\leq r<\omega}\{(n_1, \cdots, n_r): n_j<\omega$ for $1\leqq j\leqq r\}$ be a bijection. Since p is a rapid ultrafilter, we may find $A\in p$ such that $|B_m\cap S(A)|\leqq m$ for each $m<\omega$. By induction, for each $x_{n_1,\cdots,n_r}\in S_\omega(p)$ we define $A_{n_1,\cdots,n_r}\in p$ such that

(1) $|B_m\cap S(x_{n_1,\cdots,n_r}, A_{n_1,\cdots,n_r})|\leqq m$ for each $m<\omega$; and

(2) $B_m\cap S(x_{n_1,\cdots,n_r}, A_{n_1,\cdots,n_r})=\emptyset$ for every $m\leqq\sigma^{-1}((n_1, \cdots, n_r))$.

Define

$$V=\{x\}\cup S(A)\cup\bigcup_{1\leq r<\omega}(\bigcup_{n_1\in A}\bigcup_{n_2\in A_{n_1}}\cdots\bigcup_{n_r\in A_{n_1,\cdots,n_{r-1}}} S(x_{n_1,\cdots,n_r}, A_{n_1,\cdots,n_r})).$$

For every $m<\omega$, let $z(m)=|V\cap B_m|$. Fix an arbitrary $m<\omega$ and put $V\cap B_m=\{x_{n_1^s,\cdots,n_{r_s}^s}: 1\leqq s\leqq z(m)\}$. Then $x_{n_1^s,\cdots,n_{r_s}^s}\in S(x_{n_1^s,\cdots,n_{r_s-1}^s}, A_{n_1^s,\cdots,n_{r_s}^s})\cap B_m$ for each $1\leqq s\leqq z(m)$. From (1) and (2) it follows that $\sigma^{-1}((n_1^s, \cdots, n_{r_s-1}^s))<m$, for each $1\leqq s\leqq z(m)$, and $|\{1\leqq t\leqq z(m): (n_1^s, \cdots, n_{r_s-1}^s)=(n_1^t, \cdots, n_{r_t-1}^t),\ n_{r_s}^s\neq n_{r_t}^t\}|\leqq m$, for

each $1\leqq s\leqq z(m)$. So $z(m)\leqq m^2$. Thus, $|V\cap B_m|\leqq m^2$ for every $m<\omega$. Theorem 2.4 implies that x is rapid in $S_\omega(p)$.

Finally, we state some problems.

2.13. QUESTION. Assume the existence of a Q-point $p\in\omega^*$.

(1) Is there a compact weakly k-rapid (resp. rapid) space without weak P-points, for each $1\leqq k<\omega$?

(2) It there a weakly k-rapid (resp. rapid) topological group without weak P-points, for each $1\leqq k<\omega$?

(3) For every $1\leqq k<\omega$, is there a weakly $(k+1)$-rapid homogeneous space which is not weakly k-rapid?

3. On Id-fan tightness.

We begin with a definition that generalizes countable strong fan tightness (1.4 (3)).

3.1. DEFINITION. Let $h\in{}^\omega\omega$. A space X has *h-fan tightness* if for every $x\in X$ and for every sequence $(A_n)_{n<\omega}$ of subsets of X such that $x\in\bigcap_{n<\omega}\mathrm{Cl}(A_n)$, there is $F_n\in[A_n]^{\leqq h(n)}$, for every $n<\omega$, such that $x\in\mathrm{Cl}(\bigcup_{n<\omega}F_n)$.

If $h\in{}^\omega\omega$ is the constant function of value k for $1\leqq k<\omega$, then k-fan tightness stands for h-fan tightness. Henceforth, $\mathrm{Id}:\omega\to\omega$ will denote the identity map on ω. It is evident that countable strong fan tightness $\Leftrightarrow$ 1-fan tightness $\Rightarrow$ h-fan tightness for each $h\in{}^\omega\omega\Rightarrow$ countable fan tightness $\Rightarrow$ countable tightness. There is an easy example of a space with countable tightness which does not have countable fan tightness. In fact, for $p\in\omega^*$, we define a topology on $\Xi(p,\omega)=\{p\}\cup\omega\times\omega$ as follows: the singleton $\{(n,m)\}$ is open for every (n,m) $\omega\times\omega$, and $V\in\mathfrak{N}(p)$ provided that $p\in V$ and $\{m<\omega:(n,m)\in V\}\in p$ for each $n<\omega$ (see the proof 2.11). It is not hard to show that $\Xi(p,\omega)$ has countable tightness and does not have countable fan tightness for every $h\in{}^\omega\omega$. Example 3.7 has Id-fan tightness and does not have countable strong fan tightness, and Example 3.8 has countable fan tightness and does not have h-fan tightness.

Next, we shall show that if $h\in{}^\omega\omega$, then h-fan tightness coincides with either 1-fan tightness (=countable strong fan tightness) or Id-fan tightness, First, we give some preliminary results.

3.2. LEMMA. *Let $h\in{}^\omega\omega$ and let $f\in{}^\omega\omega$ be non-bounded. Then every space with h-fan tightness has f-fan tightness.*

PROOF. Let X be a space with h-fan tightness, $x \in X$ and $(A_n)_{n<\omega}$ a sequence of subsets of X such that $x \in \bigcap_{n<\omega} \mathrm{Cl}(A_n)$. Since f is not bounded, we may choose positive integers $n_0 < n_1 < \cdots < n_k < \cdots$ such that $h(k) \leqq f(n_k)$ for each $k<\omega$. Define $B_k = A_{n_k}$ for each $k<\omega$. Then, for every $k<\omega$ there is $E_k \in [B_k]^{\leq h(k)}$ such that $x \in \mathrm{Cl}(\bigcup_{k<\omega} E_k)$. For $n<\omega$, put $F_n = E_k$ if $n = n_k$ and $F_n = \emptyset$ otherwise. Thus, we have that $\bigcup_{n<\omega} F_n = \bigcup_{k<\omega} E_k$ and $F_{n_k} = E_k \in [A_{n_k}]^{h(k) \leq f(n_k)}$ for each $k<\omega$. Therefore, $x \in \bigcup_{n<\omega} F_n$ and $F_n \in [A_n]^{\leq f(n)}$ for all $n<\omega$.

The following two corollaries are direct consequences of 3.2.

3.3. COROLLARY. *If $h, f \in {}^{\omega}\omega$ are non-bounded, then the notions of h-fan tightness and f-fan tightness are the same.*

3.4. COROLLARY. *If $h \in {}^{\omega}\omega$, then every space with h-fan tightness has Id-fan tighness.*

3.5. LEMMA. *If $h \in {}^{\omega}\omega$ is bounded then h-fan tightness agrees with countable strong fan tightness.*

PROOF. Assume that $h \in {}^{\omega}\omega$ is bounded by the integer $k<\omega$. Let X be a space with h-fan tightness, $x \in X$ and $(A_n)_{n<\omega}$ a sequence of subsets of X such that $x \in \bigcap_{n<\omega} \mathrm{Cl}(A_n)$. By assumption, for each $n<\omega$ there is $F_n \in [A_n]^{\leq k}$ such that $x \in \mathrm{Cl}(\bigcup_{n<\omega} F_n)$. We may suppose that $|F_n| = k$ for all $n<\omega$. Enumerate each F_n by $\{x_1^n, \cdots, x_k^n\}$ and set $B_j = \{x_j^n ; n<\omega\}$ for each $1 \leqq j \leqq k$. Since $x \in \mathrm{Cl}(\bigcup_{n<\omega} F_n) = \mathrm{Cl}(B_1 \cup \cdots \cup B_k) = \mathrm{Cl}(B_1) \cup \cdots \cup \mathrm{Cl}(B_k)$, there is $1 \leqq j \leqq k$ such that $x \in \mathrm{Cl}(B_j)$. Thus, $x_j^n \in A_n$ for each $n<\omega$ and $x \in \mathrm{Cl}(\{x_j^n : n<\omega\})$.

We turn now to the principal result of this section.

3.6. THEOREM. *If $h \in {}^{\omega}\omega$, then h-fan tightness coincides with either 1-fan tightness or Id-fan tightness.*

The next two examples show that Id-fan tightness is a new concept.

3.7. EXAMPLE. Let $x \notin \omega \times \omega$. We consider the following topology on $X = \{x\} \cup (\omega \setminus \{0\}) \times \omega$: the set $(\omega \setminus \{0\}) \times \omega$ has the discrete topology and a neighborhood of x consists of a finite intersection of the sets $V_f = \{x\} \cup \{(n, m) \in (\omega \setminus \{0\}) \times \omega : (n, m) \neq (n, f(n))\}$ for $f \in {}^{\omega}\omega$. Notice that X is a zero-dimensional space. We shall verify that X with this topology has Id-fan tightness and does not have strong fan tightness. Indeed, for $1 \leqq n < \omega$, we put $A_n = \{(n, m) : m<\omega\}$. In order to show that X has Id-fan tightness we note that $x \in \mathrm{Cl}(B) \setminus B$, for $B \subseteq X$, whenever for every $1 \leqq n < \omega$ there is $k_n < \omega$ such that $|B \cap A_{k_n}| > n$.

For each $1 \leqq n < \omega$, let $B_n \subseteq X$ such that $x \in \bigcap_{1 \leq n < \omega} \mathrm{Cl}(B_n)$ and $x \notin B_n$, for each $n < \omega$. Then for each $1 \leqq n < \omega$ there is $k_n < \omega$ such that $|B_n \cap A_{k_n}| > n$. For every $1 \leqq n < \omega$, choose $F_n \subseteq B_n \cap A_{k_n}$ such that $|F_n| = n$. Let $V = \bigcap_{j \leq s} V_{f_j} \in \mathcal{N}(x)$, where $f_j \in {}^\omega\omega$ for $j \leqq s < \omega$. Since $|F_{2s} \cap \{(k_{2s}, f_j(k_{2s})) : j \leqq s\}| \leqq s+1$ and $|F_{2s}| = 2s$, we obtain that $F_{2s} \cap V \neq \emptyset$ and hence $V \cap \bigcup_{1 \leq n < \omega} F_n \neq \emptyset$. Thus, $x \in \mathrm{Cl}(\bigcup_{1 \leq n < \omega} F_n)$. Suppose that X has countable strong fan tightness. Then for every $1 \leqq n < \omega$ there is $t_n < \omega$ such that $x \in \mathrm{Cl}(\{(n, t_n) : 1 \leqq n < \omega\})$. Let $f \in {}^\omega\omega$ be defined by $f(n) = t_n$ for each $1 \leqq n < \omega$. Then $V_f \cap \{(n, t_n) : 1 \leqq n < \omega\} = \emptyset$, which is a contradiction.

3.8. Example. Let $Y = \{y\} \cup (\omega \setminus \{0\}) \times \omega$, where $y \notin \omega \times \omega$. We equip $(\omega \setminus \{0\}) \times \omega$ with the discrete topology and let $\mathcal{N}(y)$ be the set of all finite intersections of the sets W_S, where $W_S = \{y\} \cup \{(n, m) : m \notin S_n, 1 \leqq n < \omega\}$ and $S = (S_n)_{1 \leq n < \omega}$ is a sequence of subsets of ω such that $|S_n| \leqq n$ for each $1 \leqq n < \omega$. We claim that Y is a zero-dimensional space which has countable fan tightness and does not have Id-fan tightness. It is evident that Y is zero-dimensional and does not have Id-fan tightness. We claim that Y does not have countable fan tightness. First, observe that $y \in \mathrm{Cl}(B) \setminus B$ if and only if for every $1 \leqq n < \omega$ there is $k_n < \omega$ such that $|B \cap A_n| > nk_n$, where $A_n = \{(n, m) : m < \omega\}$ for $1 \leqq n < \omega$. Assume that $y \in \bigcap_{1 \leq n < \omega} \mathrm{Cl}(B_n)$ and $y \notin B_n$ for each $1 \leqq n < \omega$. Then, for each $1 \leqq n < \omega$ there is $k_n < \omega$ such that $|B_n \cap A_{k_n}| > nk_n$. For each $1 \leqq n < \omega$, choose $F_n \subseteq B_n \cap A_{k_n}$ with $|F_n| > nk_n$. Let $W = \bigcap_{j \leq r} W_{s_j} \in \mathcal{N}(y)$, where $S_j = (S_n^j)_{n < \omega}$ for $j \leqq r < \omega$. Since $|F_r \cap \{(k_r, m) : m \notin S_{k_r}^j, j \leqq r\}| \leqq rk_r$ and $|F_r| > rk_r$, we have that $W \cap F_r \neq \emptyset$ and hence $W \cap (\bigcup_{1 \leq n < \omega} F_n) \neq \emptyset$. Thus, $y \in \mathrm{Cl}(\bigcup_{1 \leq n < \omega} F_n)$.

Certain ultrafilters on ω can be characterized in terms of countable fan tightness and Id-fan tightness.

3.9. Theorem. *An ultrafilter p on ω is a P-point iff $\xi(p)$ has countable fan tightness.*

3.10. Theorem. *For $p \in \omega^*$, the following statements are equivalent.*

(1) *p is semiselective*;

(2) *$\xi(p)$ has countable strong fan tightness*;

(3) *$\xi(p)$ has Id-fan tightness*;

(4) *there is $k \in {}^\omega\omega$ such that given $A_n \in p$ for $n < \omega$, there exists $F_n \in [A_n]^{\leq h(n)}$ such that $\bigcup_{n < \omega} F_n \in p$.*

Proof. The proofs of (1)$\Leftrightarrow$(2), (2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are direct from the definitions, and (4)$\Rightarrow$(3) follows from 3.2. Only the implication (3)$\Rightarrow$(1) requires proof. Assume that $\xi(p)$ has Id-fan tightness. Let $(A_n)_{n<\omega}$ be a sequence of elements of p. Without loss of generality, we may suppose that $A_{n+1}\subseteq A_n$ for $n<\omega$. Define $B_n=A_{n(n+1)/2}$ for each $n<\omega$. By hypothesis, for each $n<\omega$ there is $F_n\in[B_n]^{\leq n}$ such that $A=\bigcup_{n<\omega}F_n\in p$. By adding integers if it necessary and by induction, we may assume that $|F_n|=n$, for each $n<\omega$, and $F_n\cap F_m=\emptyset$ whenever $n<m<\omega$. Enumerate successively the F_n's by $\{a_j: j<\omega\}$. Then we have that $A=\{a_j: j<\omega\}\in p$. Fix $1<j<\omega$ and let $1\leqq n<\omega$ be such that $a_j\in F_n$. It then follows that $j\leqq n(n+1)/2$ and hence $a_j\in F_n\subseteq A_{n(n+1)/2}\subseteq A_j$, as desired.

3.11. QUESTION. Is there a topological group G such that G has Id-fan tightness (resp. countable fan tightness) and does not have countable strong fan tightness (resp. Id-fan tightness)?

For a space X we denote by $C_\pi(X)$ the function space on X with the topology of pointwise convergence. In the next theorem, we shall show that the concepts of countable strong fan tightness and Id-fan tightness coincide on the class of spaces of the form $C_\pi(X)$. Recall that X has property C'' if for every sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X there is $G_n\in\mathcal{G}_n$, for each $n<\omega$, such that $X=\bigcup_{n<\omega}G_n$. The following lemma is needed.

3.12. LEMMA. *For a space X, the following are equivalent.*

(1) *X has property C'';*

(2) *for every sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X, for each $n<\omega$ there is $\mathcal{D}_n\in[\mathcal{G}_n]^{\leq n}$ such that $X=\bigcup_{n<\omega}\bigcup\mathcal{D}_n$;*

(3) *there is $h\in{}^\omega\omega$ such that for every sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X there is $\mathcal{D}_n\in[\mathcal{G}_n]^{\leq h(n)}$, for each $n<\omega$, for which $X=\bigcup_{n<\omega}\bigcup\mathcal{D}_n$.*

PROOF. Only (3)$\Rightarrow$(1) requires proof. Let $h\in{}^\omega\omega$ satisfy the conditions of clause (3) and let $(\mathcal{G}_n)_{n<\omega}$ be a sequence of open covers of X. Without loss of generality we may suppose that h is strictly increasing. Put $\mathcal{H}_0=\mathcal{G}_0\wedge\cdots\wedge\mathcal{G}_{h(0)-1}$ and for $n<\omega$, we define $\mathcal{H}_n=\mathcal{G}_{h(n)}\wedge\cdots\wedge\mathcal{G}_{h(n+1)-1}$, where $\mathcal{G}\wedge\mathcal{H}=\{G\cap H: G\in\mathcal{G}$ and $H\in\mathcal{H}\}$ for $\mathcal{G}$ and $\mathcal{H}$ covers of X. Then for each $n<\omega$ there is $\mathcal{D}_n\in[\mathcal{H}_n]^{\leq h(n)}$ such that $X=\bigcup_{n<\omega}\bigcup\mathcal{D}_n$. We may assume that $\mathcal{D}_0=\{H_j: j<h(0)\}$ and $\mathcal{D}_n=\{H_{h(n)+j}: j<h(n+1)-h(n)\}$ for every $1\leqq n<\omega$. Now, we have that if $n<\omega$ and $j<h(n+1)-h(n)$ (resp. if $j<h(0)$), then there is $G_{h(n)+j}\in\mathcal{G}_{h(n)+j}$ (resp. $G_j\in\mathcal{G}_j$) such that $H_{h(n)+j}\subseteq G_{h(n)+j}$ (resp. $H_j\subseteq G_j$). It then follows that $X=\bigcup_{m<\omega}G_m$ and $G_m\in\mathcal{G}_m$ for each $m<\omega$.

3.13. THEOREM. *For a space X, the following are equivalent.*

(1) *$C_\pi(X)$ has countable strong fan tightness;*

(2) *each finite product of X has property C'';*

(3) *$C_\pi(X)$ has Id-fan tightness.*

PROOF. The equivalence (1)⇔(2) is shown in [S] and by a slight modification of Sakai's argument we can prove that $C_\pi(X)$ has Id-fan tightness iff each finite product of X satisfies the property of clause (2) of 3.12. Thus, (2)⇔(3) follows from 3.12.

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Salvador Garcia-Ferreíra
Instituto de Matematicas
Ciudad Universitaria (UNAM)
Mexico, D. F. 04510

Angel Tamariz-Mascarúa
Departmento de Matematicas
Facultad de Ciencias
Ciudad Universitaria (UNAM)
Mexico, D. F. 04510