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# ON SOME GENERALIZATIONS OF COMPACTNESS IN SPACES $C_p(X, 2)$ AND $C_p(X, \mathbb{Z})$

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ABSTRACT. We discuss topological properties of a space $X$ which imply that the spaces $C_p(X, 2)$ and $C_p(X, \mathbb{Z})$ have properties similar to compactness, such as $\sigma$-compactness and $\sigma$-countable compactness. In particular, for a zero-dimensional space $X$, we prove: (1) $X$ is normal and $C_p(X, 2)$ is $\sigma$-compact iff $X$ is an Eberlein-Grothendieck space and the set of non-isolated points in $X$ is Eberlein compact, and (2) $C_p(X, \mathbb{Z})$ is $\sigma$-compact iff $X$ is an Eberlein compact space.

## 1. Introduction

All spaces are assumed to be Tychonoff unless otherwise stated. Given two spaces X and Y, we denote by $C(X, Y)$ the set of all continuous functions from $X$ to $Y$, and $C_p(X, Y)$ is the set $C(X, Y)$ equipped with the topology of pointwise convergence (that is, the topology inherited by $C(X, Y)$ as a subspace of the space $Y^X$ of all functions from $X$ to $Y$ with the Tychonoff product topology). The space $C_p(X, \mathbb{R})$ is denoted as $C_p(X)$, and $C_p^*(X)$ stands for the subset of all bounded elements in $C_p(X)$. For points $x_0, \ldots, x_n$ in $X$ and subsets $A_1, \ldots, A_n$ of $Y$, we will denote by $[x_0, \ldots, x_n; A_0, \ldots, A_n]$ the subset of $Y^X$ of those functions $f$ such that $f(x_i) \in A_i$ for every $i \in \{0, \ldots, n\}$.

The symbols $\mathbb{R}$, $I$, $\omega$, $\mathbb{Z}$ and 2 stand for the real line, interval $[0, 1]$, the natural numbers, the discrete group of the integer numbers and the discrete group $\{0, 1\}$, respectively. The letters $t$, $n$, $m$, $k$ will denote natural numbers; and if $t$ is a natural number, we will use the same symbol $t$ to denote the discrete space of cardinality $t$. For topological spaces $X$ and $Y$, the symbol $X \cong Y$ means that $X$ and $Y$ are homeomorphic. The space $\beta(\omega)$ is the Stone-Čech compactification of the natural numbers, and $\omega^*$ is equal to $\beta(\omega) \setminus \omega$. If $\mathcal{P}$ is a topological property, then a space $X$ is $\sigma$-$\mathcal{P}$ if $X$ is the countable union of subspaces satisfying $\mathcal{P}$. A space $X$ is a *P-space* if the intersection of a countable family of open subsets of $X$ is still an open set. A subspace $Y$ of $X$ is *bounded* in $X$ if for every $f \in C(X)$, $f \restriction Y$ is a bounded function, or equivalently, if every sequence of open sets in $X$, which meets $Y$, has an accumulation point in $X$. A subspace $Y$ of a space $X$ is *$C^*$-embedded* in $X$ if for every $f \in C^*(Y)$ there is $g \in C^*(X)$ such that $g \restriction Y = f$. A space $X$ is *$\omega$-discrete* if every subset $Y$ of $X$ of cardinality $\leq \aleph_0$ is discrete; and $X$ is *b-discrete* if every subset $Y$ of $X$ of cardinality $\leq \aleph_0$ is discrete and $C^*$-embedded in $X$.

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