EXAMPLE OF A $T_1$ TOPOLOGICAL SPACE WITHOUT A NOETHERIAN BASE

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Abstract. A Noetherian base $\mathcal{B}$ of a topological space $X$ is a base for the topology of $X$ which has the following property: If $B_1 \subseteq B_2 \subseteq \cdots$ is a nondecreasing sequence of elements of $\mathcal{B}$, then $\{B_n\}_{n \in \mathbb{N}}$ is finite. In this article we give an example of a $T_1$ topological space without a Noetherian base.

I. Introduction.

Definition 1.1. A collection $\mathcal{C}$ of subsets of a set $X$ is Noetherian if $\mathcal{C}$ does not contain a strictly increasing infinite chain.

There are large classes of topological spaces which have a Noetherian base (see [3]), for example if $X$ is a normed linear space, the collection of open balls of radius $1/n$ ($n \in \mathbb{N}$) constitutes a Noetherian base of $X$. On the other hand, $\mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ is a non $T_1$-space with no Noetherian base.

An important unsolved problem is the following:

Does Con($\text{ZFC}$) imply that Con ($\text{ZFC} + \text{there exists a } T_2\text{-space without a Noetherian base}$)?

However, the following is known:

Theorem 1.2[1 and 4]. Let $\alpha$ be an ordinal. The space $\alpha$ has a Noetherian base if and only if $\alpha + 1$ does not contain a strongly inaccessible cardinal.

In the section that follows we give an example, in ZFC, of a $T_1$-space that has no Noetherian base.

II. A $T_1$ topological space with no Noetherian base.

Definition 2.1. A topological space $X$ is Noetherianly refinable or in abbreviated notation, $N$-refinable, if each open covering has a Noetherian open refinement.

It is easy to see that if $X$ has a Noetherian base then it is $N$-refinable and that $X$ is $N$-refinable if and only if each open cover has a refinement which is an antichain of open sets.

Lemma 2.2 [2]. Let $\alpha$ be an uncountable regular cardinal. Let $E \subseteq \alpha$ be a stationary subset of $\alpha$ and let $\phi : E \rightarrow \alpha$ be a regressive function. Then, there is $\xi < \alpha$ such that $|\phi^{-1}(\xi)| = \alpha$. 

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For each \( \lambda \leq \omega_1 \), let \( \mathcal{B}_\lambda = \{ A \subset \lambda : |\lambda - A| < \aleph_0 \} \). If \( B_1, B_2 \in \mathcal{B} = \bigcup_{\lambda \leq \omega_1} \mathcal{B}_\lambda \), then \( B_1 \cap B_2 \in \mathcal{B} \). Therefore \( \mathcal{B} \) is a base for a topology \( \tau \) in \( \omega_1 \).

**Remark 2.3.** \( A \in \tau - \mathcal{B} \) if and only if \( A = \lambda - C \), where \( \lambda < \omega_1 \) and \( C \) is a cofinal subset in \( \lambda \) of order type \( \omega \) (o.t. \( C = \omega \)).

**Theorem 2.4.** \( (\omega_1, \tau) \) is a \( T_1 \)-space which is not \( N \)-refinable (and therefore, \( (\omega_1, \tau) \) does not have a Noetherian base).

**Proof.** Let us suppose that \( \mathcal{A} \subset \tau \) is a refinement of \( \mathcal{E} = \{ \lambda + 1 : \lambda \in \omega_1 \} \). Let \( \lambda_0 = 0 \) and let \( A_0 \in \mathcal{A} \) be such that \( \lambda_0 \in A_0 \). Then, there is \( \lambda_1 \in \omega_1 \) such that \( A_0 = \lambda_1 - C_1 \), where \( C_1 \) is either finite or is an infinite cofinal subset of \( \lambda_1 \) of order type \( \omega \) (see 2.3). Let \( A_0' = A_0 \cup \{ \eta \in C_1 : \eta > \lambda_0 \} \). Let \( A_1 \in \mathcal{A} \) be such that \( \lambda_1 \in A_1 \). There is \( \lambda_2 \in \omega_1 \) such that \( A_1 = \lambda_2 - C_2 \), where \( C_2 \) is finite or is an infinite cofinal subset of \( \lambda_2 \) of order type \( \omega \). Let \( A_1' = A_1 \cup \{ \eta \in C_2 : \eta > \lambda_1 \} \).

Let us suppose that for some \( \gamma < \omega_1 \), we have chosen the collections: \( \{ A_\beta \}_\beta \subset \mathcal{A} \) and \( \{ A'_\beta \}_\beta \subset \mathcal{A} \), such that \( \lambda_\beta \in A_\beta = \lambda_{\beta + 1} - C_{\beta + 1} \), where \( C_{\beta + 1} \) is either finite or is an infinite cofinal subset of \( \lambda_{\beta + 1} \) of order type \( \omega \). Moreover, for each \( \beta < \gamma \), \( A'_\beta = A_\beta \cup \{ \eta \in C_{\beta + 1} : \eta > \lambda_\beta \} \).

We construct, inductively, \( \lambda_\gamma \in \omega_1 \), \( A_\gamma \in \mathcal{A} \) and \( A'_\gamma \):

If \( \gamma \) is a nonlimit ordinal and \( \gamma - 1 \) is the immediate predecessor of \( \gamma \), then there exist \( \lambda_\gamma < \omega_1 \) such that \( A_{\gamma - 1} = \lambda_\gamma - C_\gamma \). If \( \gamma \) is a limit ordinal, let \( \lambda_\gamma = \sup \{ \lambda_\beta : \beta < \gamma \} \). In both cases, let \( A_\gamma \in \mathcal{A} \) such that \( \lambda_\gamma \in A_\gamma \). There is \( \lambda_{\gamma + 1} \in \omega_1 \) such that \( A_\gamma = \lambda_{\gamma + 1} - C_{\gamma + 1} \), where \( C_{\gamma + 1} \) is either finite or is an infinite cofinal subset of \( \lambda_{\gamma + 1} \) of order type \( \omega \) (see 2.3). Let \( A'_\gamma = A_\gamma \cup \{ \eta \in C_{\gamma + 1} : \eta > \lambda_\gamma \} \).

By the inductive construction, \( \{ \lambda_\beta \}_\beta \subset \omega_1 \) is cofinal in \( \omega_1 \).

Let \( \mathcal{A}' = \{ A'_\beta : \beta < \omega_1 \} \). It is easy to see that each \( A'_\beta \) is an open set. In fact, \( A'_\beta \in \mathcal{B} \) for each \( \beta < \omega_1 \).

We claim that:

1. If \( \mathcal{A} \) is an antichain, then \( \mathcal{A}' \) is also an antichain.

In fact, let \( A'_\gamma, A'_\beta \in \mathcal{A}' \) where \( \gamma < \beta \). \( A'_\gamma = A_\gamma \cup \{ \eta \in C_{\gamma + 1} : \eta > \lambda_\gamma \} \) and \( A'_\beta = A_\beta \cup \{ \eta \in C_{\beta + 1} : \eta > \lambda_\beta \} \). \( A'_\gamma \) does not contain \( A'_\beta \) since \( \lambda_\beta \in A'_\beta - A'_\gamma \). On the other hand, if \( \eta_0 \in A'_\gamma - A'_\beta \), then \( \eta_0 \in A'_\beta - A'_\gamma \). Therefore \( \mathcal{A}' \) is an antichain.

2. Let \( E' = \bigcup_{\gamma \in \omega_1} A'_\gamma \) and let \( G = \omega_1 - E' \). Then, the set \( G \) is empty or has order type \( \leq \omega \). Furthermore \( E = \{ \alpha \in E' : \alpha \) is a limit ordinal \( \} \) is a stationary subset of \( \omega_1 \).

In fact, let us suppose that \( G \) is a subset of \( \omega_1 \) such that o.t. \( G > \omega \). Let \( \eta_0 \in G \) be such that o.t. \( \{ \eta \in G : \eta < \eta_0 \} > \omega \). Since \( \{ \lambda_\gamma \}_\gamma \subset \omega_1 \) is a cofinal subset in \( \omega_1 \), then, there is \( \lambda_\xi \) such that \( \eta_0 < \lambda_\xi \). But \( \lambda_\xi \in A'_\xi = \lambda_{\xi + 1} - C'_{\xi + 1} \), where \( C'_{\xi + 1} = \{ \eta \in C_{\xi + 1} : \eta < \lambda_\xi \} \) and o.t. \( C'_{\xi + 1} \leq \omega \). Therefore \( A'_\xi \cap G \neq \emptyset \). This contradiction proves that o.t. \( G \leq \omega \). As an immediate consequence the set \( E = \{ \alpha \in E' : \alpha \) is a limit ordinal \( \} \) is a stationary subset of \( \omega_1 \).

For each \( \eta \in E \), let \( g(\eta) \) be the smallest \( \gamma \) such that \( \gamma \in A'_\gamma = \lambda_{\gamma + 1} - C'_{\gamma + 1} \). If \( T_\eta = \{ \xi < \omega_1 : \lambda_\xi \leq \eta \} \), then \( g(\eta) \) is the supremum of \( T_\eta \) and therefore \( \lambda_{g(\eta)} \leq \eta \). Since \( \eta \) is a limit ordinal, \( \lambda_{g(\eta)} \leq \eta \) and \( \eta \in A'_{g(\eta)} = A_\gamma(\eta) - C'_{g(\eta) + 1} \) (where \( C'_{g(\eta) + 1} \subset \lambda_{g(\eta)} \) is a finite set) there is \( a_\eta < \eta \) such that \( C'_{g(\eta) + 1} \subset a_\eta \). The function \( \phi(\eta) = a_\eta \) is a regressive function. Since \( E \) is a stationary subset in \( \omega_1 \), there is \( \xi < \omega_1 \) such that \( |\phi^{-1}(\eta)| = \omega_1 \) (Lemma 2.2). Let \( M = \phi^{-1}(\xi) \). Since \( |M| = \omega_1 \) and
If $|\xi| = \omega$, there exist an infinite subset $K$ of $M$ and a finite subset $C \subseteq \xi$, such that $A'_{\xi(k)} = \lambda_{\xi(k)} + 1 - C$ for each $k \in K$. Therefore $\{A'_{\xi(k)} : k \in K\}$ is an infinite strictly increasing chain of elements of $\mathcal{A}'$. It follows from (1) that $\mathcal{A}$ is not an antichain, that is, $(\omega_1, \tau)$ is not $\mathcal{N}$-refinable.

**Bibliography**


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