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Ai-maximal independent families and irresolvable Baire spaces

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ABSTRACT

A topological space is *almost irresolvable* if it cannot be written as a countable union of subsets with empty interior. Given a cardinal κ, denote by $(\star_\kappa)$ the statement "the Cantor cube 2^{2^κ} has a dense subspace of size κ which is almost irresolvable and whose dispersion character is equal to κ." In this paper we prove:

(1) $(\star_\kappa)$ is equivalent to the existence of a dense subspace of 2^{2^κ} which is Baire submaximal and whose cardinality and dispersion character are both equal to κ. In particular, $(\star_\kappa)$ implies that κ is measurable in an inner model of ZFC.
(2) If the Continuum Hypothesis holds, $(\star_\kappa)$ fails for all κ.
(3) $(\star_\kappa)$ is equivalent to the existence of an ω_1-complete ideal I on κ containing all sets of cardinality $< \kappa$ and such that the quotient Boolean algebra $\mathcal{P}(\kappa)/I$ is isomorphic to the complete Boolean algebra that adjoins 2^κ Cohen reals.

1. Introduction

A space is *resolvable* if it contains two disjoint dense subsets. This notion was introduced by E. Hewitt in [8]. Later, R. Bolstein coined the term *almost resolvable* in [4] to designate those spaces which have a countable cover of subsets with empty interior. A natural generalization of this concept, which appeared originally in [17], is the following: call a space *almost ω-resolvable* if it possesses a cover $\{A_n : n < \omega\}$, where $\bigcup_{i<n} A_n$ has empty interior for all $n < \omega$. When a space fails to be almost resolvable, we will call it *almost irresolvable*. Similarly for *almost ω-irresolvable*.

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One of the topics studied in [7] is the existence of dense subspaces of Cantor cubes (i.e., products of the form 2^λ, where λ is a cardinal and 2 is the discrete space with two points) which are almost irresolvable or almost ω-irresolvable. The approach followed in that paper for this particular matter is to isolate the combinatorial properties that would lead to the existence of such spaces. Thus the concepts of an *ai-maximal independent family* and of a *aωi-maximal independent family* are introduced (see Definition 3.2). This translates the problem of finding topological spaces like the ones described at the beginning of the paragraph into the combinatorial problem of finding these kinds of families.

The goal of this paper is to study some consequences of the existence of ai-maximal independent families of maximum size (e.g. certain Cantor cubes have Baire submaximal dense subspaces). As a corollary we show that if this kind of family does exist, then there is an inner model of ZFC with a measurable cardinal and that under CH there are no such families.

This paper is organized as follows. Section 2 contains notation and terminology together with some elementary results that will be used several times. The main results of the article are contained in Sections 3 and 4. The last part is a selection of some of the questions we could not answer.

2. Preliminaries

Let S be a set. As usual, $\mathcal{P}(S)$ is the power set of S and $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\omega)$.

Following [13], we denote by $\mathrm{Fn}(S,2)$ the collection of all partial functions from S into 2, i.e., $p \in \mathrm{Fn}(S,2)$ iff $p \subseteq S \times 2$ is a finite function. Elements of $\mathrm{Fn}(S,2)$ are normally called *conditions*.

Let $p, q \in \mathrm{Fn}(S,2)$ be arbitrary. We say that $p \leq q$ iff $q \subseteq p$. p and q will be called *compatible* (in symbols, $p \mid q$) if $p \cup q$ is a function; otherwise they are *incompatible* (in symbols, $p \perp q$). A subset of $\mathrm{Fn}(S,2)$ in which any two different elements are incompatible will be called an *antichain*.

For a cardinal κ, $[S]^\kappa$ denotes the collection of all subsets of S which have cardinality κ. Similarly, $[S]^{<\kappa}$ is the family of all subsets of S whose cardinality is less than κ. Given a cardinal λ, we denote by $\kappa^{<\lambda}$ the cardinality of the set $[\kappa]^{<\lambda}$.

The logarithm of κ, $\log\kappa$, is defined as the least cardinal λ for which $\kappa \leq 2^\lambda$.

Let $I \subseteq \mathcal{P}(S)$. We say that I is an *ideal on S* if (1) $S \notin I$, (2) $\emptyset \in I$, (3) I is closed under finite unions, and (4) I is closed under taking subsets.

Given I, an ideal on S, and κ, an infinite cardinal, I will be called *κ-complete* if $\bigcup A \in I$ for all $A \in [I]^{<\kappa}$. Also, we will say that I is *σ-saturated* if I is ω_1-complete, $[S]^1 \subseteq I$ (i.e., I contains all singletons), and $\mathcal{P}(S)\setminus I$ contains no uncountable pairwise disjoint family. Finally, if for any $a \subseteq S$ we have that either $a \in I$ or $S \setminus a \in I$, then I is called *prime*.

Let X be a topological space. X is *crowded* if it has no isolated points. X is *submaximal* if all its dense subsets are open. When all nowhere dense subsets of X are closed, X will be called *nodec*.

Given a cardinal number κ we say that X is *κ-resolvable* if it can be expressed as a disjoint union of κ dense subsets. Otherwise, X will be called *κ-irresolvable*. X is *resolvable* (respectively *irresolvable*) if it is 2-resolvable (respectively, 2-irresolvable). Equivalently, X is resolvable if it can be expressed as the union of finitely many subsets with empty interior.

Spaces for which all non-empty open subspaces are irresolvable are called *OHI* (an acronym for *open hereditarily irresolvable*). It is a well-known fact that a space is submaximal if and only if it is nodec and OHI.

We say that a topological space is *ccc* if any family of pairwise disjoint open subsets of it is countable.

The *dispersion character* of X, $\Delta(X)$, is the least cardinality of a non-empty open subset of X.

All set-theoretic notions whose definition is not given here explicitly should be understood as in [9].

Definition 2.1. Let S be a set and let $\mathcal{C} = \{(C^0_\alpha, C^1_\alpha) : \alpha < \lambda\}$ be a non-empty family of pairs of subsets of S such that each unordered pair $\{C^0_\alpha, C^1_\alpha\}$ is a partition of S.

(1) For each non-empty $p \in \mathrm{Fn}(\lambda, 2)$ we define

$$\mathcal{C}(p) := \bigcap\{C_\alpha^{p(\alpha)} : \alpha \in \operatorname{dom} p\},$$

and $\mathcal{C}(\emptyset) = S$.

(2) We say that $\mathcal{C}$ is *independent* if $\mathcal{C}(p) \neq \emptyset$ for each $p \in \mathrm{Fn}(\lambda, 2)$.

(3) $\mathcal{C}$ will be called *uniform* if for all $p \in \mathrm{Fn}(\lambda, 2)$ we have $|\mathcal{C}(p)| = |S|$.

(4) $\mathcal{C}$ is *separating* if for each pair of distinct points $x, y \in S$ there exist $\alpha < \lambda$ and $i < 2$ such that $x \in C_\alpha^i$ and $y \in C_\alpha^{1-i}$.

Remark 2.2. To avoid trivialities we will consider only infinite independent families on infinite sets.

The proof of the following result is routine, so we omit it.

Lemma 2.3. *Let $\mathcal{C}$ be an independent family of size λ. For all $p, q \in \mathrm{Fn}(\lambda, 2)$ the following holds:*

(1) *$\mathcal{C}(p) \subseteq \mathcal{C}(q)$ iff $p \leq q$.*

(2) *$\mathcal{C}(p) \cap \mathcal{C}(q) \neq \emptyset$ iff p and q are compatible.*

As far as we know, the constructions outlined in the following three paragraphs appeared originally in [11, Observation 3.1]. They will be used constantly in this paper.

Given an independent family $\mathcal{C} = \{(C_\alpha^0, C_\alpha^1) : \alpha < \lambda\}$ on a set S, there is a topology for S which has $\{\mathcal{C}(p) : p \in \mathrm{Fn}(\lambda, 2)\}$ as a base. The topological space which results of endowing S with this topology will be denoted by $X_\mathcal{C}$. Thus, $\mathcal{C}$ is uniform iff $\Delta(X_\mathcal{C}) = |X_\mathcal{C}|$.

Another space that can be naturally associated to $\mathcal{C}$ is the following: for each $x \in S$ let $d_x : \lambda \to 2$ be defined by $d_x(\xi) = 0$ iff $x \in C_\xi^0$. Then $D_\mathcal{C}$ will denote the subspace $\{d_x : x \in S\}$ of the topological product 2^λ.

In order to establish a connection between the spaces introduced in the previous paragraphs, define $[p] := \{f \in 2^\lambda : p \subseteq f\}$ for each $p \in \mathrm{Fn}(\lambda, 2)$. Hence $\{[p] : p \in \mathrm{Fn}(\lambda, 2)\}$ is the canonical base for 2^λ. Moreover, for all $x \in X$, $d_x \in [p]$ is equivalent to $x \in \mathcal{C}(p)$. This remark has three immediate consequences: first, $D_\mathcal{C}$ is dense in 2^λ; second, the map $h : X_\mathcal{C} \to D_\mathcal{C}$ given by $h(x) = d_x$ is continuous and open, and third, the following conditions are all equivalent, (1) $\mathcal{C}$ is separating, (2) h is one-to-one, and (3) h is a homeomorphism.

Remark 2.4. If $\mathcal{C}$ is a separating independent family, then $X_\mathcal{C}$ is Tychonoff, crowded, and ccc (because any dense subset of a product of the form 2^λ is ccc).

Recall that the density of a topological space X is the least cardinality of a dense subset of X.

Lemma 2.5. *Let $\mathcal{C}$ be an independent family on a cardinal κ. If λ is the cardinality of $\mathcal{C}$, then $|\mathcal{C}(p)| \geq \log \lambda$ for each $p \in \mathrm{Fn}(\lambda, 2)$. In particular, if $|\mathcal{C}| = 2^\kappa$ and $\log(2^\kappa) = \kappa$, then $\mathcal{C}$ is uniform independent.*

Proof. Let $p \in \mathrm{Fn}(\lambda, 2)$ be arbitrary. Since $\lambda \geq \omega$, we have that $[p]$ is an open subset of 2^λ which is homeomorphic to 2^λ. It is proved in [10, 7.9 on p. 150] that the density of the Cantor cube 2^λ is equal to $\log \lambda$ and so the inequality $|D_\mathcal{C} \cap [p]| \geq \log \lambda$ follows from the fact that $D_\mathcal{C}$ is dense in 2^λ. Finally, note that for all $\alpha < \kappa$, $d_\alpha \in D_\mathcal{C} \cap [p]$ implies $\alpha \in \mathcal{C}(p)$. $\square$

3. Ai-maximal independent families

Let X be a topological space. We say that X is *almost resolvable* if X has a countable cover of subsets with empty interior. Otherwise, we will say that X is *almost irresolvable*. Clearly, all resolvable spaces are almost resolvable. Equivalently, almost irresolvable implies irresolvable.

X will be called *almost ω-resolvable* if there exists $\{Y_n : n < \omega\}$, a cover of X, such that $\bigcup_{i<n} Y_i$ has empty interior for each $n < \omega$. All spaces which lack this kind of cover will be called *almost ω-irresolvable*. Thus any space which is almost ω-resolvable is almost resolvable.

Proposition 3.1. *If X is almost irresolvable and $\Delta(X) = |X|$, then $|X|$ has uncountable cofinality.*

Proof. Without loss of generality, let us assume that the underlying set of X is the cardinal κ. Our argument will be by contrapositive so assume that $\{\alpha_n : n \in \omega\}$ is an increasing sequence of ordinals whose supremum is κ and such that $\alpha_0 = 0$. Define, for each integer n, $Y_n := [\alpha_n, \alpha_{n+1})$ to obtain a countable cover of X. Since X is almost irresolvable, int $Y_m \neq \emptyset$, for some m, and therefore $\Delta(X) < |X|$. □

Definition 3.2. Let $\mathcal{C}$ be an independent family of size λ on a set S.

(1) $\mathcal{C}$ is *ai-maximal independent* if for every partition $\{Y_n : n < \omega\}$ of S there exist $p \in \mathrm{Fn}(\lambda, 2)$ and $m < \omega$ such that $\mathcal{C}(p) \subseteq Y_m$.
(2) We say that $\mathcal{C}$ is *aωi-maximal independent* if for every partition $\{Y_n : n < \omega\}$ of S there exist $p \in \mathrm{Fn}(\lambda, 2)$ and $m < \omega$ satisfying $\mathcal{C}(p) \subseteq \bigcup_{i \le m} Y_i$.

Let $\mathcal{C}$ be an independent family of size λ on a cardinal κ. It is proved in [7, Proposition 2.6] that $\mathcal{C}$ is ai-maximal independent iff $D_{\mathcal{C}}$ is an almost irresolvable subspace of 2^λ. Similarly, [7, Proposition 2.7] states that $\mathcal{C}$ is aωi-maximal independent iff $D_{\mathcal{C}}$ is an almost ω-irresolvable subspace of 2^λ.

A complementary construction is as follows. Let $Y = \{y_\alpha : \alpha < \kappa\}$ be a dense subset of 2^λ (we are assuming that $\alpha \neq \beta$ implies $y_\alpha \neq y_\beta$). For each $\xi < \lambda$ and $i < 2$ define $B^i_\xi := \{\alpha < \kappa : y_\alpha(\xi) = i\}$. A simple argument shows that $\mathcal{B} := \{(B^0_\xi, B^1_\xi) : \xi < \lambda\}$ is a separating independent family such that $D_{\mathcal{B}} = Y$. Moreover, if Y is almost irresolvable (respectively, almost ω-irresolvable), then $\mathcal{B}$ is an ai-maximal independent (respectively, aωi-maximal independent) family of size λ on κ. In particular, we have the following.

Remark 3.3. For each dense almost ω-irresolvable subspace of 2^λ of size κ there exists an aωi-maximal independent family on κ of size λ.

Definition 3.4. Let $\mathcal{C}$ be an independent family of size 2^κ on a set S, where $\kappa := |S|$. We say that $\mathcal{C}$ is a *nice independent family on S* if the following conditions hold:

(1) $\mathcal{C}$ is separating,
(2) each element of $[X_{\mathcal{C}}]^{<\kappa}$ is closed discrete in $X_{\mathcal{C}}$,
(3) if $A \in [X_{\mathcal{C}}]^\kappa$, then either A is closed discrete in $X_{\mathcal{C}}$ or $\mathcal{C}(p) \subseteq A$ for some $p \in \mathrm{Fn}(2^\kappa, 2)$.

A nice independent family on S which is, at the same time, ai-maximal independent will be called a *nice ai-maximal independent family*. Similarly, a *nice aωi-maximal independent family* is a nice independent family which is aωi-maximal independent.

Remark 3.5. Suppose that $\mathcal{C}$ is an independent family on a set S which satisfies condition (2) above. If (3) holds, then any subset of $X_{\mathcal{C}}$ with empty interior is closed discrete, so $X_{\mathcal{C}}$ is submaximal. And vice versa, submaximality of $X_{\mathcal{C}}$ implies condition (3).

Nice independent families produce spaces with interesting properties:

Proposition 3.6. *If $\mathcal{C}$ is a nice independent family, then*

(1) *every subset of $X_{\mathcal{C}}$ is a G_δ, i.e., $X_{\mathcal{C}}$ is a Q-set space and*
(2) *$X_{\mathcal{C}}$ is not pseudocompact.*

Proof. It is proved in [3, Theorem 7.3] that every regular ccc submaximal space is a Q-set space. As we noted above, $X_{\mathcal{C}}$ is submaximal and since $\mathcal{C}$ is separating, $X_{\mathcal{C}}$ is Tychonoff and ccc (Remark 2.4). This proves (1).

Since $X_{\mathcal{C}}$ is regular, crowded, nodec, and non-empty, [3, Theorem 7.7] applies and therefore $X_{\mathcal{C}}$ is not pseudocompact. □

The following result states that we can always modify a suitable independent family to obtain a uniform nice independent family.

Theorem 3.7. *Let $\mathcal{B}$ be a uniform independent family on κ of size 2^κ. There is a uniform nice independent family $\mathcal{C}$ on κ such that if $p \in \mathrm{Fn}(2^\kappa, 2)$ and $Y \subseteq \kappa$ satisfy $\mathcal{B}(p) \subseteq Y$, then there exists $q \in \mathrm{Fn}(2^\kappa, 2)$ with $\mathcal{C}(q) \subseteq Y$.*

Proof. We will sketch the construction given in the proof of [11, Main Theorem 3.3] and argue that this construction provides us with the family $\mathcal{C}$ we need.

The first step is to enumerate $\mathcal{B} = \{(B^0_\xi, B^1_\xi) : \xi < 2^\kappa\}$ and $[\kappa]^\kappa = \{F_\xi : \xi < 2^\kappa\}$. Now partition 2^κ into two pieces, I_0 and I', such that $|I_0| = \kappa^{<\kappa}$ and $|I'| = 2^\kappa$. Also fix a partition $\{J_{b,\alpha} : b \in [\kappa]^{<\kappa} \wedge \alpha \in \kappa \setminus b\} \subseteq [I_0]^\omega$ of I_0 into countable pieces. For all $b \in [\kappa]^{<\kappa}$, $\alpha \in \kappa \setminus b$, and $\xi \in J_{b,\alpha}$ define

$$C^0_\xi := \bigl(B^0_\xi \cup b\bigr) \setminus \{\alpha\} \quad \text{and} \quad C^1_\xi := \kappa \setminus C^0_\xi = \bigl(B^1_\xi \setminus b\bigr) \cup \{\alpha\}.$$

Let $\mathbb{P} := \mathrm{Fn}(2^\kappa, 2)$ and assume that for some $\alpha < 2^\kappa$ we have constructed

(i) a sequence $\{p_\beta : \beta < \alpha\} \subseteq \mathbb{P}$,
(ii) a collection $\{K_\beta : \beta < \alpha\}$ of subsets of I' such that $|K_\beta| \in \{0, \kappa\}$ for each $\beta < \alpha$,
(iii) a bijection $f_\beta : K_\beta \to \kappa$ for each $\beta < \alpha$ with $|K_\beta| = \kappa$, and
(iv) a family $\{\{C^0_\xi, C^1_\xi\} : \xi \in K_\beta\}$ of partitions of κ for each $\beta < \alpha$

in such a way that the following holds for each $\beta < \alpha$:

(1β) If we let $I_\beta := I_0 \cup \bigcup_{\xi<\beta} K_\xi$ and

$$\mathcal{B}_\beta := \bigl\{\bigl(C^0_\xi, C^1_\xi\bigr) : \xi \in I_\beta\bigr\} \cup \bigl\{\bigl(B^0_\xi, B^1_\xi\bigr) : \xi \in 2^\kappa \setminus I_\beta\bigr\},$$

then $\mathcal{B}_\beta$ is a separating independent family of size 2^κ.
(2β) If $\mathcal{B}_\beta(q) \subseteq F_\beta$ for some $q \in \mathbb{P}$, then $K_\beta = \emptyset$ and $\mathcal{B}_\beta(p_\beta) \subseteq F_\beta$.
(3β) When $\mathcal{B}_\beta(q) \nsubseteq F_\beta$ for all $q \in \mathbb{P}$, then

(a) $p_\beta = \emptyset$,
(b) K_β is a subset of $I' \setminus \bigcup_{\xi<\beta}(K_\xi \cup \operatorname{dom} p_\xi)$ with $|K_\beta| = \kappa$, and
(c) for each $\xi \in K_\beta$:

$$C_\xi^0 := \big(B_\xi^0 \cup F_\beta\big) \setminus \big\{f_\beta(\xi)\big\} \quad \text{and} \quad C_\xi^1 := \kappa \setminus C_\xi^0 = \big(B_\xi^1 \setminus F_\beta\big) \cup \big\{f_\beta(\xi)\big\}.$$

It is shown in the proof of [11, Main Theorem 3.3] that $\mathcal{C} := \mathcal{B}_{2^\kappa}$ is a uniform independent family on κ of cardinality 2^κ which satisfies conditions (1)–(3) of Definition 3.4 and such that $X_\mathcal{C}$ is nodec.

The fact that $X_\mathcal{C}$ is submaximal is a consequence of [11, Lemma 2.7]. Another way of proving this is as follows: assume that D is a dense subset of $X_\mathcal{C}$ and let $F = X_\mathcal{C} \setminus D$. Observe that F has empty interior because D is dense. Therefore (Remark 3.5) F is closed.

Finally, suppose that $Y \subseteq \kappa$ and $p \in \mathbb{P}$ satisfy $\mathcal{B}(p) \subseteq Y$. Then $Y = F_\beta$ for some $\beta < 2^\kappa$, because $\mathcal{B}$ is uniform. It suffices to show the existence of a condition $r \in \mathbb{P}$ with $\mathcal{B}_\beta(r) \subseteq \mathcal{B}(p)$. Indeed, if this is the case, then at stage β the assumptions in (2β) hold and therefore $\mathcal{B}_\beta(p_\beta) \subseteq Y$. One easily verifies that $\mathcal{B}_\beta(p_\beta) = \mathcal{B}_\gamma(p_\beta)$ whenever $\beta \le \gamma \le 2^\kappa$. In particular, $\mathcal{C}(p_\beta) \subseteq Y$.

In order to find the condition r that we mentioned in the previous paragraph, we need the following claim.

Claim. *For each $\delta \in I_\beta \cap \operatorname{dom} p$ and any finite set $H \subseteq 2^\kappa$ with $\operatorname{dom} p \subseteq H$ there exist $\delta', \delta'' \in I_\beta \setminus H$ such that $\delta' \neq \delta''$ and*

$$C_\delta^{p(\delta)} \cap C_{\delta'}^0 \cap C_{\delta''}^1 \subseteq B_\delta^{p(\delta)}.$$

To prove the claim we will consider two cases. If $\delta \in I_0$, then $\delta \in J_{b,\alpha}$ for some $b \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus b$. Thus any pair of different points $\delta', \delta'' \in J_{b,\alpha} \setminus H$ will work. On the other hand, when $\delta \in I_\beta \setminus I_0$, there exists $\xi < \beta$ with $\delta \in K_\xi$. Set $\alpha := f_\xi(\delta)$ and notice that we only need to take $\delta' \in K_\xi \setminus H$ and $\delta'' \in J_{\emptyset,\alpha} \setminus H$.

Using finite recursion we define, for each $\delta \in I_\beta \cap \operatorname{dom} p$, a pair of ordinals $\delta', \delta'' \in I_\beta$ satisfying the conclusion of the Claim and such that

$$r := p \cup \big\{\big(\xi', 0\big) : \xi \in I_\beta \cap \operatorname{dom} p\big\} \cup \big\{\big(\xi'', 1\big) : \xi \in I_\beta \cap \operatorname{dom} p\big\}$$

is a function. Therefore $\mathcal{B}_\beta(r) \subseteq \mathcal{B}(p)$. □

Corollary 3.8. *For any cardinal κ, the existence of a uniform ai-maximal (respectively, aωi-maximal) independent family on κ of size 2^κ implies the existence of a uniform nice ai-maximal (respectively, aωi-maximal) independent family on κ.*

Proof. Suppose that $\mathcal{B}$ is a uniform ai-maximal independent family on κ with $|\mathcal{B}| = 2^\kappa$ and let $\mathcal{C}$ be the uniform nice independent family given by the previous theorem. Assume that $\{Y_n : n < \omega\}$ is a partition of κ and fix $p \in \mathrm{Fn}(2^\kappa, 2)$ and $m < \omega$ in such a way that $\mathcal{B}(p) \subseteq Y_m$. Hence there is $q \in \mathrm{Fn}(2^\kappa, 2)$ such that $\mathcal{C}(q) \subseteq Y_m$.

Similar arguments apply in the case where $\mathcal{B}$ is a uniform aωi-maximal independent family. □

Note that if X is a submaximal space and $A \subseteq X$ has void interior, then all its subsets are closed in X. Hence A is closed discrete.

A topological space X is *σ-discrete* if it can be expressed as a countable union of discrete subspaces. When X is the union of countably many closed discrete subspaces, we say that X is *strongly σ-discrete*.

Proposition 3.9. *If X is crowded submaximal, the following are equivalent.*

(1) *X is almost resolvable.*
(2) *X is almost ω-resolvable.*
(3) *X is σ-discrete.*
(4) *X is strongly σ-discrete.*

Proof. Start by noting that (4) $\to$ (3) and (2) $\to$ (1) are immediate. Now, since X is crowded, (2) is a consequence of (3). Finally, as we pointed out before, in a submaximal space all subsets with void interior are closed discrete; thus (4) follows from (1). $\square$

Definition 3.10. Let $\mathcal{C} = \{(C^0_\xi, C^1_\xi) : \xi < \lambda\}$ be an independent family on a set S.

(1) For each $p \in \mathrm{Fn}(\lambda, 2)$ we define

$$\mathcal{C} \restriction p := \{(C^0_\xi \cap \mathcal{C}(p), C^1_\xi \cap \mathcal{C}(p)) : \xi \in \lambda \setminus \mathrm{dom}\, p\}.$$

(2) We say that $\mathcal{C}$ is *globally ai-maximal independent on S* if $\mathcal{C} \restriction p$ is ai-maximal independent on $\mathcal{C}(p)$ for all $p \in \mathrm{Fn}(\lambda, 2)$.

Remark 3.11. Let $\mathcal{C}$, S, and λ be as in the definition. For each $r \in \mathrm{Fn}(\lambda, 2)$, $\mathcal{C}(r)$ is an open subspace of $X_\mathcal{C}$ which has $\{\mathcal{C}(r) \cap \mathcal{C}(p) : p \in \mathrm{Fn}(\lambda \setminus \mathrm{dom}\, r, 2)\}$ as a base for its topology. Therefore, $X_{\mathcal{C}\restriction r} = \mathcal{C}(r)$.

It is routine to verify that if $\mathcal{C}$ is a nice independent family on κ, then $\mathcal{C} \restriction r$ is a nice independent family on $\mathcal{C}(r)$, for all $r \in \mathrm{Fn}(2^\kappa, 2)$.

Now we are interested in a topological translation of globally ai-maximal independent families so we need to introduce a class of spaces: a topological space will be called *open hereditarily almost irresolvable* (OHAI, for short) if every non-empty open subspace of it is almost irresolvable.

Lemma 3.12. *Every almost irresolvable space has a non-empty open subspace which is OHAI.*

Proof. Let X be an almost irresolvable space and denote by Y be the union of all open subspaces of X which are almost resolvable. According to [17, Theorem 3.8], Y is almost resolvable. The same argument used to prove [17, Proposition 3.2] shows that $\overline{Y}$, the closure of Y in X, is almost resolvable. Thus the subspace $X \setminus \overline{Y}$ is open, non-empty, and OHAI. $\square$

We are ready to establish the topological translation we were looking for.

Proposition 3.13. *Let $\mathcal{C}$ be an independent family on a set S. Then $\mathcal{C}$ is globally ai-maximal independent iff $X_\mathcal{C}$ is OHAI.*

Proof. Let $\lambda := |\mathcal{C}|$. When $X_\mathcal{C}$ is OHAI and $p \in \mathrm{Fn}(\lambda, 2)$, $\mathcal{C}(p)$ is almost irresolvable; hence Remark 3.11 implies that $\mathcal{C} \restriction p$ is ai-maximal independent.

For the other implication assume that $X_\mathcal{C}$ is not OHAI and fix a family, $\{Y_n : n \in \omega\}$, of pairwise disjoint subsets of $X_\mathcal{C}$ whose union, Y, is a non-empty open subset of X, but each Y_n has empty interior. Then there is $p \in \mathrm{Fn}(\lambda, 2)$ with $\mathcal{C}(p) \subseteq Y$ and therefore $\{\mathcal{C}(p) \cap Y_n : n \in \omega\}$ witnesses that $\mathcal{C}(p)$ is almost resolvable. $\square$

As a consequence of the work done we obtain:

Proposition 3.14. *If $\mathcal{C}$ is an ai-maximal independent family on κ of size λ, then $\mathcal{C} \restriction r$ is globally ai-maximal independent on $\mathcal{C}(r)$, for some $r \in \mathrm{Fn}(\lambda, 2)$.*

Proof. Since $X_{\mathcal{C}}$ is almost irresolvable, Lemma 3.12 implies the existence of a condition $r \in \mathrm{Fn}(\lambda, 2)$ for which $\mathcal{C}(r)$ is OHAI. Hence, according to Remark 3.11, $X_{\mathcal{C}\restriction r}$ is OHAI, i.e., $\mathcal{C} \restriction r$ is globally ai-maximal independent (Proposition 3.13). □

Recall that a topological space *is Baire* if the intersection of any countable family of dense open subsets of it is dense.

Theorem 3.15. *If X is crowded submaximal, the following statements are equivalent.*

(1) *X is OHAI.*
(2) *X is Baire.*
(3) *If I is the collection of all subsets of X with empty interior, then I is an ω_1-complete ideal on X.*
(4) *X has no non-empty open σ-discrete subspaces.*

Proof. Let $\mathcal{D}$ be a countable family of dense open subsets of X. If $U := X \setminus \overline{\bigcap \mathcal{D}}$ were non-empty, then $\{U \setminus D : D \in \mathcal{D}\}$ would witness that U is almost resolvable. Hence, (1) → (2).

Note that in a crowded submaximal space all discrete subspaces are closed and nowhere dense. Thus, (4) follows from (2).

To prove that (4) implies (1) observe that any non-empty open subspace of X is crowded and submaximal so Proposition 3.9 applies.

To conclude our argument, let us argue that (1) and (3) are equivalent. Given that X is submaximal and crowded, I coincides with the ideal of nowhere dense subsets of X and $[X]^1 \subseteq I$. On the other hand, if I is not ω_1-complete, there is $\mathcal{A} \in [I]^\omega$ with $U := \mathrm{int} \bigcup \mathcal{A} \neq \emptyset$; therefore X is not OHAI because $\{U \cap A : A \in \mathcal{A}\}$ witnesses that U is almost resolvable. And vice versa, if V is a non-empty open subspace of X which is almost resolvable, there exists a countable family $\mathcal{F} \subseteq I$ with $V = \bigcup \mathcal{F}$, thus proving that I is not ω_1-complete. □

For a cardinal κ, we will say that a topological space X satisfies $(\dagger_\kappa)$ if X is a dense subspace of 2^{2^κ} with $\Delta(X) = |X| = \kappa$.

Theorem 3.16. *The following statements are equivalent for any cardinal κ.*

(1) *There is an almost irresolvable space which satisfies $(\dagger_\kappa)$.*
(2) *There is a Baire submaximal space which satisfies $(\dagger_\kappa)$.*
(3) *There is a Baire OHI space which satisfies $(\dagger_\kappa)$.*
(4) *There is a Baire irresolvable space which satisfies $(\dagger_\kappa)$.*
(5) *There is a Baire almost irresolvable space which satisfies $(\dagger_\kappa)$.*
(6) *There is a Baire almost ω-irresolvable space which satisfies $(\dagger_\kappa)$.*
(7) *There is an almost ω-irresolvable space which satisfies $(\dagger_\kappa)$.*

Moreover, when $\log(2^\kappa) = \kappa$, the previous statements are equivalent to

(8) *There is a Baire ω-irresolvable space which satisfies $(\dagger_\kappa)$.*

Proof. Let us prove that (2) follows from (1). If (1) holds, Corollary 3.8 guarantees the existence of a uniform nice ai-maximal independent family $\mathcal{C}$ on κ. According to Proposition 3.14, there is a condition r for which $\mathcal{C} \restriction r$ is globally ai-maximal independent on κ. Since $|\mathcal{C}(r)| = \kappa$ and $\mathcal{C} \restriction r$ is a nice independent family on $\mathcal{C}(r)$, we will assume, without loss of generality, that $\mathcal{C}$ is globally ai-maximal independent on κ. Thus $X_\mathcal{C}$ is Baire submaximal (Proposition 3.13 and Theorem 3.15) and $\Delta(X_\mathcal{C}) = \kappa$. Using the fact that $\mathcal{C}$ is separating, we have that $X_\mathcal{C}$ is homeomorphic to $D_\mathcal{C}$ (see the discussion following Lemma 2.3) and therefore $D_\mathcal{C}$ is the space needed in (2).

Implications (2) → (3) → (4) and (5) → (6) → (7) are straightforward.

An immediate consequence of [6, Theorem 3] is that any Baire irresolvable space is almost irresolvable. In particular, (4) implies (5).

Now, if (7) is true, Corollary 3.8 gives the existence of $\mathcal{C}$, a uniform nice aωi-maximal independent family on κ. By Proposition 3.9, $X_\mathcal{C}$ is the space whose existence is claimed in (1).

Since all almost irresolvable spaces are ω-irresolvable, (8) is a consequence of (5). On the other hand, the space described in (8) is Baire, crowded, and ω-irresolvable, so it possesses (according to [2, Theorem 5.9]) a dense subspace, Y, which is almost ω-irresolvable. In particular, $|Y| \leq \kappa$. Also, for each $p \in \mathrm{Fn}(2^\kappa, 2)$, the canonical basic open set $[p]$ is homeomorphic to 2^{2^κ} and thus $|Y \cap [p]| \geq \log(2^\kappa)$ (see the proof of Lemma 2.5). Hence, under the assumption $\log(2^\kappa) = \kappa$, we conclude that $\kappa \leq \Delta(Y) \leq |Y|$. Thus, Y is the space described in (7). □

As a consequence of Theorem 3.16 we obtain that the consistency strength of the existence of a space like the one described in part (7) is greater than the existence of a measurable cardinal:

Corollary 3.17. *If κ carries a uniform aωi-maximal independent family of size 2^κ, then κ is measurable in an inner model of ZFC.*

Proof. For such a κ we obtain the existence of a dense subspace Y of 2^{2^κ} which is Baire and satisfies $\Delta(Y) = |Y| = \kappa$. It is proved in [14,15], and [9, Theorem 22.33] that the existence of a space with these characteristics implies the conclusion of the corollary. □

It is natural to ask about the existence of a Tychonoff crowded almost resolvable (or almost ω-resolvable) space. Regarding this question, we get:

Proposition 3.18. *The following are equivalent in the class of all Tychonoff crowded topological spaces.*

(1) *All Baire spaces are resolvable.*
(2) *All spaces are almost ω-resolvable.*
(3) *All spaces are almost resolvable.*

Proof. Since implication (2) → (3) is immediate and [17, Corollary 4.9] is precisely (3) → (1), we only need to argue that (2) follows from (1).

Assume (1) and let (X, τ) be crowded and Tychonoff. A standard argument involving Zorn's Lemma gives σ, a $\subseteq$-maximal element of the family of all crowded Tychonoff topologies which are finer that τ (i.e., (X, σ) is a *maximal Tychonoff space* and $\tau \subseteq \sigma$). The argument used to prove [17, Theorem 4.14] shows that the existence of a Tychonoff crowded irresolvable Baire space is equivalent to the existence of a maximal Tychonoff space which is almost ω-irresolvable. Given that we are assuming (1), (X, σ) is almost ω-irresolvable and since $\sigma \subseteq \tau$, we conclude that (X, τ) is almost ω-irresolvable too. This proves that (2) holds. □

Hence, the consistency strength of the existence of a Tychonoff crowded almost irresolvable (respectively, almost ω-irresolvable) space is greater than the existence of a measurable cardinal.

Theorem 3.19. *If an infinite cardinal κ carries a uniform aωi-maximal independent family of size 2^κ, then the following holds:*

(1) *κ has uncountable cofinality,*
(2) *$\kappa \neq \omega_1$, and*
(3) *CH fails, i.e., $\mathfrak{c} > \omega_1$.*

Proof. (1) is a corollary of Proposition 3.1.

To prove (2) and (3) assume that κ is as described in the hypothesis. By Theorem 3.16, κ carries a uniform ai-maximal independent family of size 2^κ so we proceed as in the proof of (1) $\to$ (2) in Theorem 3.16 to get a uniform nice independent family $\mathcal{C}$ on κ which is globally ai-maximal independent. Theorem 3.15 implies that I, the family of all subsets of $X_\mathcal{C}$ with empty interior, is an ω_1-complete ideal on κ which contains all singletons. Moreover, if $\mathcal{A} \subseteq \mathcal{P}(\kappa) \setminus I$ is pairwise disjoint, then $\{\operatorname{int} A : A \in \mathcal{A}\}$ is a cellular family of size $|\mathcal{A}|$. Since $X_\mathcal{C}$ is ccc (Remark 2.4), we conclude that I is σ-saturated.

According to [9, Lemma 10.13], there is no ω_1-complete σ-saturated ideal on ω_1. This proves (2).

Now we shall prove (3). Apply [9, Lemma 10.9] to obtain that either there is a set $Y \subseteq \kappa$ such that $I \restriction Y := \{A \in I : A \subseteq Y\}$ is a prime ideal or there exists an ω_1-complete σ-saturated ideal on some cardinal $\lambda \leq \mathfrak{c}$.

Let $Y \subseteq \kappa$ be an arbitrary subset. We will argue that $I \restriction Y$ is not a prime ideal. If $Y \in I$, then $I \restriction Y$ is not an ideal because $Y \in I \restriction Y$. When $Y \notin I$, there is $p \in \mathrm{Fn}(2^\kappa, 2)$ such that $\mathcal{C}(p) \subseteq Y$. Let $\alpha \in 2^\kappa \setminus \operatorname{dom} p$ and define $A := \mathcal{C}(p \cup \{(\alpha, 0)\})$. Then $A \notin I$ and $\mathcal{C}(p \cup \{(\alpha, 1)\}) \subseteq Y \setminus A$, i.e., $Y \setminus A \notin I$. In other words, I is not prime.

The two previous paragraphs imply that there is a cardinal $\lambda \leq \mathfrak{c}$ which carries an ω_1-complete σ-saturated ideal. Clearly $\lambda \neq \omega$ and [9, Lemma 10.13] guarantees that $\lambda \neq \omega_1$. Thus $\omega_1 < \lambda \leq \mathfrak{c}$. $\square$

Note that a corollary of the previous result is that if CH holds, then no cardinal κ carries a uniform aωi-maximal independent family of size 2^κ. The same conclusion is consistent with $\neg$CH. Indeed, [5, Theorem 4.1] states that if there are no Souslin trees, then every ccc crowded Hausdorff space is almost ω-resolvable (see the discussion following Definition 3.2).

In [12, p. 79] and [14, Theorem 3.3] it is shown that if κ is measurable and the ground model satisfies CH, then the generic extension yield by $\mathrm{Fn}(\kappa, 2, \omega_1)$ contains a Baire OHI space X with $\Delta(X) = |X|$ (compare with part (3) of Theorem 3.16). But in the generic extension no cardinal κ carries a uniform aωi-maximal independent family of size 2^κ because CH holds in it.

At this stage we do not know if the existence of uniform aωi-maximal independent families is consistent with ZFC, but if this were the case, we would be able to answer the following two questions in the negative.

(1) [17, Questions 5.8] *Is the topology generated by the union of a chain of almost ω-resolvable topologies for a set X always almost ω-resolvable?*

Assume that κ carries a uniform ai-maximal independent family of size 2^κ. Corollary 3.8 provides us with a nice ai-maximal independent family $\mathcal{C} = \{(C^0_\alpha, C^1_\alpha) : \alpha < 2^\kappa\}$. For each integer n let $\mathcal{C}_n := \mathcal{C} \setminus \{(C^0_\alpha, C^1_\alpha) : n \leq \alpha < \omega\}$. Denote by τ_n the topology which has $\{\mathcal{C}_n(p) : p \in \mathrm{Fn}(2^\kappa \setminus n, 2)\}$ as a base. Then $\{\tau_n : n < \omega\}$ is an increasing sequence of topologies. Moreover, $\{C^0_n, C^1_n\}$ is a partition of (κ, τ_n) into two disjoint dense sets for all $n < \omega$ and therefore τ_n is resolvable. On the other hand, the topology generated by $\bigcup_n \tau_n$ coincides with the topology of $X_\mathcal{C}$ and so it is almost ω-irresolvable (see the discussion at the beginning of Section 3).

(2) [3, Problem 7.4] *Is every regular ccc submaximal space strongly σ-discrete?*

Let $\mathcal{C}$ be a nice ai-maximal independent family. Thus $X_{\mathcal{C}}$ is crowded and submaximal. Also, Proposition 3.9 implies that this space is not σ-discrete. Since $X_{\mathcal{C}}$ is homeomorphic to $D_{\mathcal{C}}$, a dense subspace of the product 2^{2^κ}, we have that $X_{\mathcal{C}}$ is Tychonoff and ccc.

We do have a ZFC answer for [16, Problem 3.8]: *Does any submaximal space contain a dense maximal space?* According to [11, Theorem 4.1], 2^{2^κ} has a dense subspace Y which is submaximal. On the other hand, an immediate consequence of [1, Corollary 2.2] is that no dense subspace of 2^{2^κ} is maximal. Therefore Y is a submaximal space which contains no dense maximal space.

4. Some combinatorics

The following result suggests that if one adds enough random reals, the generic extension may contain an ai-maximal independent family.

Theorem 4.1. *Let $\mathcal{B}$ be a uniform independent family of size 2^κ on a cardinal κ and let $\mathcal{C}$ be the family which was constructed in the proof of Theorem 3.7. If $m : \mathcal{P}(\kappa) \to [0, 1]$ is a σ-additive measure such that $m(\mathcal{C}(p)) = 2^{-|p|}$, for each $p \in \mathrm{Fn}(2^\kappa, 2)$, then $\mathcal{C}$ is globally ai-maximal independent.*

Proof. Denote by I the ideal of null sets, i.e., $x \in I$ iff $m(x) = 0$. Since m is σ-additive, I is ω_1-complete so, according to Theorem 3.15, we only need to show that I coincides with the collection of all subsets of $X_{\mathcal{C}}$ with void interior. Set $\mathbb{P} := \mathrm{Fn}(2^\kappa, 2)$.

Observe that if $A \subseteq X_{\mathcal{C}}$ and $p \in \mathbb{P}$ satisfy $\mathcal{C}(p) \subseteq A$, then $m(A) > 0$. Hence all null sets have empty interior.

Now let A be a subset of $X_{\mathcal{C}}$ with empty interior. If $A = \emptyset$, $m(A) = 0$ so let us assume that $A \neq \emptyset$. Since all finite subsets of $X_{\mathcal{C}}$ are closed, we have that $\kappa \setminus A$ is infinite. Our plan is to show that $m(A) \leq 2^{-i}$ for all $i < \omega$.

For the rest of the argument we will follow the notation introduced in the proof of Theorem 3.7. Let $n < \omega$ be arbitrary.

Suppose first that $|A| < \kappa$. Fix a set $H \subseteq \kappa \setminus A$ with $|H| = n$ and for each $\alpha \in H$ let $\bar{\alpha} \in J_{A,\alpha}$ be arbitrary. Thus, if we let $p := \{(\bar{\alpha}, 0) : \alpha \in H\}$, then $A \subseteq \mathcal{C}(p)$ and $|p| = n$. Clearly, $m(A) \leq 2^{-n}$.

When $|A| = \kappa$, there exists $\beta < 2^\kappa$ with $A = F_\beta$. Let $H \subseteq K_\beta$ be such that $|H| = n$ and $f_\beta[H] \subseteq \kappa \setminus A$. Thus $q := H \times \{0\} \in \mathbb{P}$ and $A \subseteq \mathcal{C}(q)$. $\square$

We finish this section with a combinatorial characterization of the existence of uniform ai-maximal independent families.

Given a poset $\mathbb{P}$, we will denote by $B(\mathbb{P})$ its Boolean completion, i.e., $B(\mathbb{P})$ is a complete Boolean algebra which contains $\mathbb{P}$ as a dense subset. As usual, given a set $S \subseteq B(\mathbb{P})$, $\bigvee S$ and $\bigwedge S$ represent the supremum and the infimum of S in $B(\mathbb{P})$, respectively.

Remark 4.2. If $b \in B(\mathbb{P})$, then $b = \bigvee\{p \in \mathbb{P} : p \leq b\}$ and therefore $B(\mathbb{P}) = \{\bigvee S : S \subseteq \mathbb{P}\}$.

The following fact (see, for example, [13, II Exercise 19]) will be used later.

Remark 4.3. If $S, T \subseteq \mathbb{P}$, then $\bigvee S \leq \bigvee T$ iff for all $p \in S$ and for each $q \leq p$ there exists $r \in T$ with $r \mid q$.

Theorem 4.4. *The following are equivalent for any infinite cardinal κ.*

(1) *There exists a uniform ai-maximal independent family on κ of size 2^κ.*
(2) *κ carries an ω_1-complete ideal I for which the quotient Boolean algebra $\mathcal{P}(\kappa)/I$ is isomorphic to $B(\mathrm{Fn}(2^\kappa, 2))$ and $[\kappa]^{<\kappa} \subseteq I$.*

Proof. We will show first that (2) implies (1). Set $\mathbb{P} := \mathrm{Fn}(2^\kappa, 2)$ and suppose that $f : B(\mathbb{P}) \to \mathcal{P}(\kappa)/I$ is an isomorphism.

Let $\{A_\xi : \xi < 2^\kappa\}$ be an enumeration of I where each A_ξ is listed infinitely many times.

Let $\xi < 2^\kappa$ be arbitrary. Fix $B^0_\xi \subseteq \kappa$ such that $f(\{(\xi, 0)\}) = [B^0_\xi]$, where $[B^0_\xi]$ denotes the equivalence class of B^0_ξ modulo I, and let $B^1_\xi := \kappa \setminus B^0_\xi$. Since $\{(\xi, 1)\}$ is the Boolean complement of $\{(\xi, 0)\}$ in $B(\mathbb{P})$, we have that $f(\{(\xi, 1)\}) = [B^1_\xi]$. Define $C^0_\xi := B^0_\xi \setminus A_\xi$ and $C^1_\xi := B^1_\xi \cup A_\xi$.

We will argue that $\mathcal{C} := \{(C^0_\xi, C^1_\xi) : \xi < 2^\kappa\}$ is uniform ai-maximal independent.

In order to prove that $\mathcal{C}$ is uniform independent let $p \in \mathbb{P}$ be arbitrary. Observe that if $\xi < 2^\kappa$ and $i < 2$, then $[C^i_\xi] = [B^i_\xi]$. Therefore the equality $f(p) = [\mathcal{C}(p)]$ follows from the fact $p = \bigwedge\{\{(\alpha, p(\alpha))\} : \alpha \in \mathrm{dom}\, p\}$. In particular, $\mathcal{C}(p) \notin I$ and so $|\mathcal{C}(p)| = \kappa$.

Let $\{Y_n : n \in \omega\}$ be a partition of κ. Since I is an ω_1-complete ideal on κ, there is $m < \omega$ with $Y_m \notin I$. Let $b \in B(\mathbb{P})$ and $p \in \mathbb{P}$ be so that $f(b) = [Y_m]$ and $p \leq b$. Thus $[\mathcal{C}(p)] = f(p) \leq [Y_m]$, i.e., $\mathcal{C}(p) \setminus Y_m \in I$. For some $\xi \in 2^\kappa \setminus \mathrm{dom}\, p$ we get $A_\xi = \mathcal{C}(p) \setminus Y_m$ and so $q := p \cup \{(\xi, 0)\}$ satisfies $\mathcal{C}(q) \subseteq Y_m$.

Assume (1). Proceeding as in the proof of (1) → (2) in Theorem 3.16, there is a nice independent family $\mathcal{C}$ on κ for which $X_\mathcal{C}$ is Baire submaximal and $\Delta(X_\mathcal{C}) = \kappa$. Thus I, the ideal of nowhere dense subsets of $X_\mathcal{C}$, is an ω_1-complete ideal on κ and coincides with the collection of all subsets of $X_\mathcal{C}$ with empty interior. Moreover, each element of I is closed in $X_\mathcal{C}$ and $[\kappa]^{<\kappa} = [X_\mathcal{C}]^{<\kappa} \subseteq I$.

For each $x \subseteq \kappa$ let $x^* := \{p \in \mathbb{P} : \mathcal{C}(p) \subseteq x\}$. Define $h : \mathcal{P}(\kappa) \to B(\mathbb{P})$ by $h(x) := \bigvee x^*$. We will show that the following holds:

(a) for all $x, y \in \mathcal{P}(\kappa)$, $x \setminus y \in I$ iff $h(x) \leq h(y)$; and
(b) h is onto.

Notice that if (a) and (b) are true, then h induces an isomorphism from $\mathcal{P}(\kappa)/I$ onto $B(\mathbb{P})$.

Observe that if $p \in x^*$ and $q \leq p$, then $q \in x^*$. Therefore we apply Remark 4.3 to obtain that $h(x) \leq h(y)$ iff for each $p \in x^*$ there is $q \in y^*$ with $p \mid q$.

Let us prove (a). Suppose that $x \setminus y \in I$ and let $p \in x^*$ be arbitrary. Then $x \setminus y$ is closed, $\mathcal{C}(p) \nsubseteq x \setminus y$, and $\mathcal{C}(p) \subseteq x$. Hence $\mathcal{C}(p) \setminus x = \emptyset$ and $\mathcal{C}(p) \setminus (x \setminus y) = (\mathcal{C}(p) \setminus x) \cup (\mathcal{C}(p) \cap y) = \mathcal{C}(p) \cap y$ is a non-empty open set. There is $q \in \mathbb{P}$ so that $\mathcal{C}(q) \subseteq \mathcal{C}(p) \cap y$. Clearly $q \in y^*$ and $q \mid p$ (Lemma 2.3). By the observation made in the previous paragraph: $h(x) \leq h(y)$.

Now suppose that $x \setminus y \notin I$. Then $\mathcal{C}(p) \subseteq x \setminus y$ for some $p \in \mathbb{P}$. In particular, $p \in x^*$. Notice that for all $q \in y^*$ we have $\mathcal{C}(q) \subseteq y$ and thus $\mathcal{C}(p) \cap \mathcal{C}(q) = \emptyset$, i.e., $p \perp q$ (Lemma 2.3). This shows that $h(x) \nleq h(y)$ and so (a) is proved.

According to Remark 4.2, h is onto if for each $S \subseteq \mathbb{P}$ there is $x \subseteq \kappa$ such that $h(x) = \bigvee S$. So let $S \subseteq \mathbb{P}$ be arbitrary and define $x := \bigcup\{\mathcal{C}(p) : p \in S\}$. Clearly, $S \subseteq x^*$ and hence $\bigvee S \leq h(x)$. We will use Remark 4.3 to show that $h(x) \leq \bigvee S$. If $p \in x^*$, then $\mathcal{C}(p) \subseteq x$ and hence $\mathcal{C}(p) \cap \mathcal{C}(q) \neq \emptyset$ for some $q \in S$. Thus $p \mid q$ according to Lemma 2.3. $\square$

It is worth noticing that the argument given for (2) → (1) in the previous theorem shows that the existence of an ω_1-complete ideal, I, on κ for which the quotient $\mathcal{P}(\kappa)/I$ is isomorphic to $B(\mathrm{Fn}(2^\kappa, 2))$ implies the existence of an ai-maximal independent family on κ of size 2^κ.

5. Questions

This section is dedicated to some problems we consider interesting.

Problem 5.1. Are the following statements consistent with ZFC?

(1) There is a cardinal κ which carries a uniform aωi-independent family of size 2^κ.
(2) There is a cardinal κ which carries a uniform aωi-independent family of size λ with $\lambda < 2^\kappa$.
(3) For some cardinal λ, 2^λ contains a dense almost ω-irresolvable subspace but no dense almost irresolvable subspace?

Problem 5.2. Is it always the case that the existence of an ai-maximal (respectively, aωi-maximal) independent family implies the existence of a uniform one?

If $\mathcal{B}$ is an arbitrary uniform independent family on a cardinal κ of size 2^κ, the construction described in the proof of Theorem 3.7 shows how to modify $\mathcal{B}$ to obtain a uniform nice independent family $\mathcal{C}$ on κ. One may wonder if this process preserves algebraic structures.

Problem 5.3. Is it true that if $D_\mathcal{B}$ is a topological subgroup of the product 2^{2^κ}, then so is $D_\mathcal{C}$?

We have two remarks regarding this question.

First, it is proved in [3, Corollary 8.16] that the cardinality of a ccc nodec topological group is not greater than $\mathfrak{c}$. Therefore $D_\mathcal{C}$ is not a topological subgroup of 2^{2^κ} when $\kappa > \mathfrak{c}$, independently of the properties that $\mathcal{B}$ has.

Second, [1, Corollary 3.8] states that if Y is a homogeneous submaximal space with $|Y| = \Delta(Y)$ and $|Y|$ is a non-measurable cardinal, then Y is strongly σ-discrete. Thus $X_\mathcal{C}$ is not homogeneous when $\mathcal{B}$ is ai-maximal independent and κ is non-measurable (see the proof of Corollary 3.8 and Proposition 3.9).

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