The Mountain Climbing Theorem and Inverse Limits With Set Valued Functions

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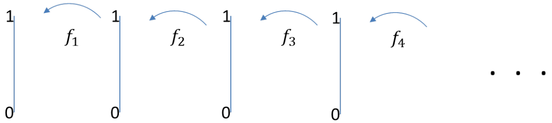
The Hilbert Cube

- $[0,1]^\infty = \{(x_0, x_1, x_2, \ldots) \; : \; x_i \in [0,1]\}$
- $d(\hat{x}, \hat{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}}$

Inverse Limits

- $f_i : [0,1] \to [0,1]$ for $i = 1, 2, 3 \cdots$
- $\varprojlim f_i = \{(x_0, x_1, x_2, \ldots) \in [0,1]^\infty \ : \ x_{i-1} = f_i(x_i)\}$



Set Valued Functions

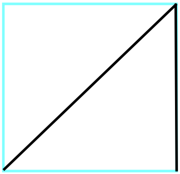
- $2^{[0,1]}$ is the set of all closed subsets of $[0, 1]$
- A function $F : [0, 1] \to 2^{[0,1]}$ is upper semi-continuous if and only if $\Gamma(F) = \{(y, x) : y \in F(x)\}$ is a closed subset of $[0, 1] \times [0, 1]$.
- Let $\pi_v(\Gamma(F)) = \{x : (y, x) \in \Gamma(F)\}$ and $\pi_h(\Gamma(F)) = \{y : (y, x) \in \Gamma(F)\}$
- A function $F : [0, 1] \to 2^{[0,1]}$ is surjective if $\pi_h(\Gamma(F)) = [0, 1]$

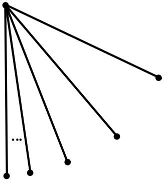
Inverse Limits with Set Valued Functions

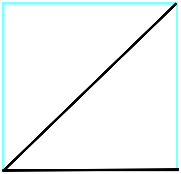
- Assume $F : [0,1] \to 2^{[0,1]}$ is surjective and upper semi-continuous.
- $\varprojlim(\mathbf{F}) = \{(x_0, x_1, x_2, \ldots) \in [0,1]^\infty \ : \ x_{i-1} \in F_i(x_i)\}$ is a nonempty compact subset of $[0,1]^\infty$

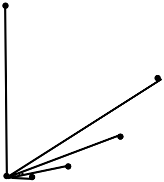
Some Examples

graph of F | $\varprojlim(F)$



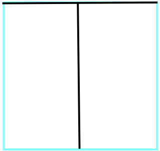


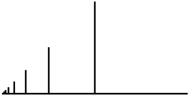


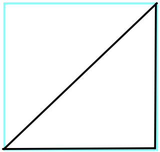


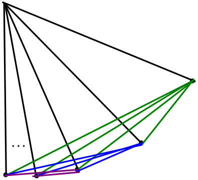
More Examples

graph of F | $\varprojlim f(F)$









The Shift Map σ

- $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, x_3, \ldots)$.
- If $F : [0,1] \to 2^{[0,1]}$ is surjective, then $\sigma(\varprojlim(\mathbf{F})) = \varprojlim(\mathbf{F})$

Finite Mahavier Products

- Suppose for each $i \in \{1, 2, 3, \ldots, n\}$ that $F_i : [0, 1] \to 2^{[0,1]}$ is upper semi-continuous.
- Define $\Gamma(F_1, F_2, \ldots, F_n) = \{(x_0, x_1, x_2, \ldots, x_n) \mid x_{i-1} \in F_i(x_i)\}$
- Let $G_i = \{(y, x) \mid y \in f(x)\}$
- The Mahavier Product of the closed sets $G_1, G_2, \ldots, G_n$ denoted $G_1 \bigstar G_2 \bigstar \ldots \bigstar G_n$ is another name for $\Gamma(F_1, F_2, F_3, \ldots, F_n)$.
- Then $\bigstar_{i=1}^{\infty} G_i = \varprojlim (\mathbf{F})$

A Lemma

Lemma

Suppose G is a closed subset of $[0,1]^m$*, and H is a closed subset of* $[0,1]^n$*, that contains a continuum K such that* $\pi_0(K) = [0,1]$*. Then for each continuum L in G, there is a continuum* L^* *in G★H such that* $\pi_{\{0,m-1\}}(L^*) = L$ *and* $\pi_{\{m-1,m+n-2\}}(L^*) \subset K$*.*

A two pass condition

Definition

A closed subset G of $[0,1]^m$ satisfies the two pass condition if G contains two disjoint continua J_0 and J_1 such that $\pi_0(J_0) = \pi_0(J_1) = [0,1]$.

A Theorem

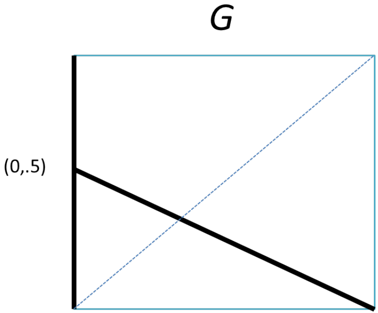
Theorem

Suppose G is a closed subset of $[0,1]^m$ *that satisfies the two pass condition. If L is any continuum in* $\bigstar_{i=1}^{n} G$*, then there is an uncountable pairwise disjoint collection* $\mathcal{C}$ *of continua in* $\bigstar_{i=1}^{\infty} G$ *such that if* $L^* \in \mathcal{C}$*, then* $\pi_{\{0,n-1\}}(L^*) = L$

Consequences

- Suppose $F : [0,1] \to 2^{[0,1]}$ is surjective and upper semi-continuous with $G = \Gamma(F)$
- Suppose for some n that $\star_{i=1}^{n} G$ satisfies the two pass condition.
- Then for any continuum K in $\varprojlim(\mathbf{F}) = \star_{i=1}^{\infty} G$ and any $\epsilon > 0$, there is an uncountable pairwise disjoint collection of continua having Hausdorff distance from K less than ϵ
- $\sigma : \varprojlim(\mathbf{F}) \to \varprojlim(\mathbf{F})$ has positive entropy.
- $\varprojlim(\mathbf{F})$ is either indecomposable or $\varprojlim(\mathbf{F})$ contains an infinite-odd.
- $\varprojlim(\mathbf{F})$ is either 1/2-indecomposable or $\varprojlim(\mathbf{F})$ contains a subcontinuum that is not unicoherent.

Example



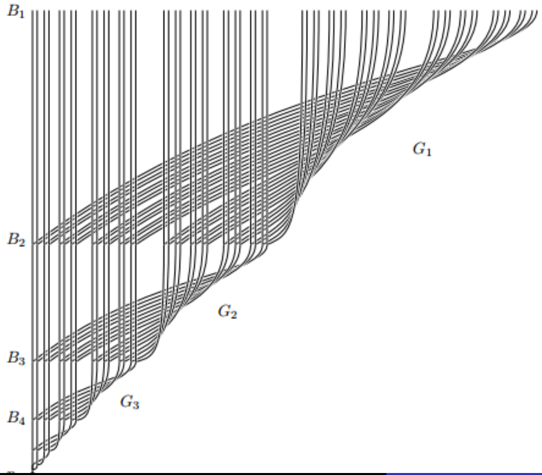
$J_0 = \{(x, 0, 0) \mid x \in [0, 1]\}$ and $J_1 = \{(x, 0, 1) \mid x \in [0, 1]\}$
$\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots)$

Properties of $\bigstar_{i=1}^{\infty} G$

- $\bigstar_{i=1}^{\infty} G$ is one-dimensional and has trivial shape and therefore is tree-like.
- $\bigstar_{i=1}^{\infty} G$ is hereditarily unicoherent.
- $\bigstar_{i=1}^{\infty} G$ is $\frac{1}{2}$-indecomposable.
- $\bigstar_{i=1}^{\infty} G$ is hereditarily decomposable.
- $\bigstar_{i=1}^{\infty} G$ is hereditarily arc-connected (a dendroid)
- $\bigstar_{i=1}^{\infty} G \setminus \{\mathbf{x}\}$ has uncountable many arc components and they are all dense.
- $\sigma : \bigstar_{i=1}^{\infty} G \to \bigstar_{i=1}^{\infty} G$ has positive entropy.

Like this example from Piotr Minc (2000)

BOTTLENECKS IN DENDROIDS



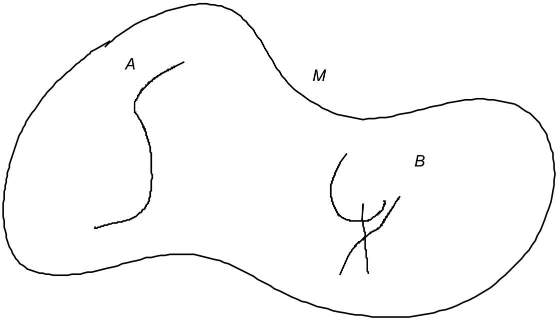
The Mountain Climbing Theorem

Theorem

Suppose $f : [0,1] \to [0,1]$ *and* $g : [0,1] \to [0,1]$ *are piece wise linear continuous functions such that* $f(0) = g(0) = 0$ *and* $f(1) = g(1) = 1$*. Then there are surjective continuous functions* $\alpha : [0,1] \to [0,1]$ *and* $\beta : [0,1] \to [0,1]$ *such that* $\alpha(0) = \beta(0) = 0$ *,* $\alpha(1) = \beta(1) = 1$*, and* $f(\alpha(t)) = g(\beta(t))$ *for each* $t \in [0,1]$

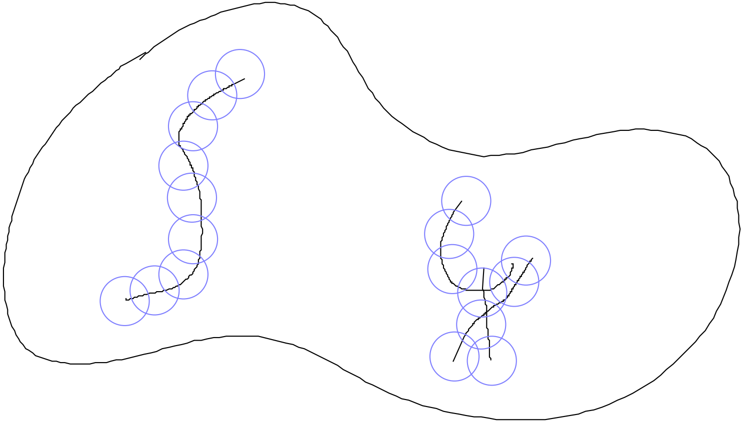
$\pi_1(A) = [0,1]$

A is a subcontinuum of $M = \varprojlim(\mathbf{F})$ and $\pi_1(A) = [0,1]$



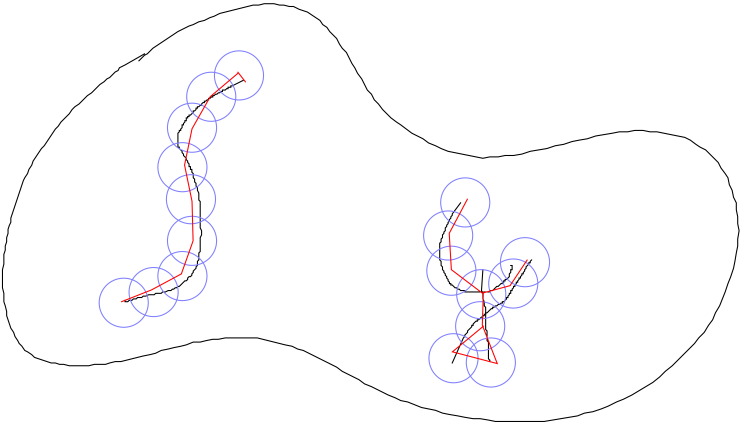
B is any subcontinuum of M.

Cover A and B with balls of radius $\frac{1}{i}$

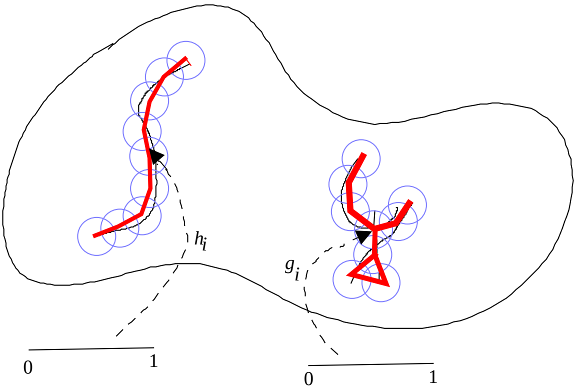


Nerves of the covers

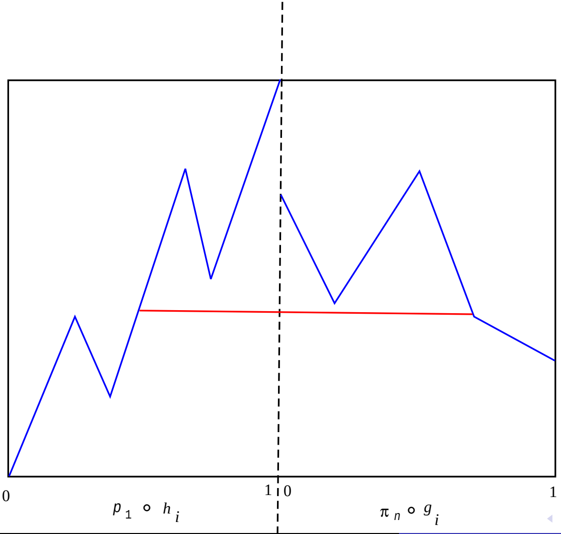
Consider the nerves of these covers.



Map [0, 1] onto these nerves with p.w. linear maps

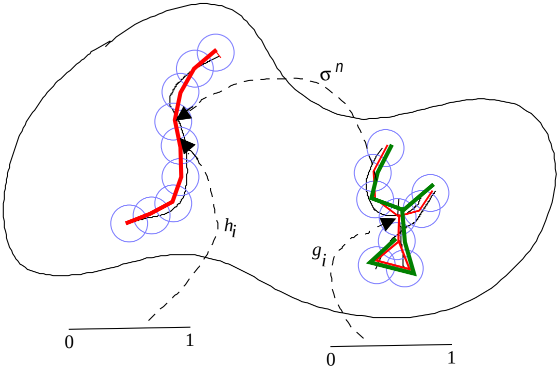


The Mountain Climbing Theorem



Technicalitites

- There are functions $\alpha_i : [0,1] \to [0,1]$ and $\beta_i : [0,1] \twoheadrightarrow [0,1]$ such that $\pi_1 \circ h_i \circ \alpha_i = \pi_n \circ g_i \circ \beta_i$.
- For each $t \in [0,1]$ let $\Psi_i(t)$ be the point in $\prod_{i=1}^{\infty} [0,1]$ whose first n coordinates are the same as the first n coordinates of $g_i \circ \beta_i(t)$ and whose coordinates from n to ∞ are the same as the coordinates starting from the beginning of $h_i \circ \alpha_i(t)$.
- Some subsequence of $\{\Psi_i([0,1])\}$ converges to a continuum W in M such that $d_H(W, B) < \frac{1}{2^n}$ and $\sigma^n(W) \subset A$.



σ n
h_i
g_i
0
1
0
1

The dynamics of σ

Lemma

Suppose $M = \varprojlim(\mathbf{F})$ *where* $F : [0,1] \to 2^{[0,1]}$. *If* A *and* B *are subcontinua of* M *such that* $\pi_1(A) = [0,1]$, *then for each* $\delta > 0$ *there is a continuum* W *in* M *such that* $d_H(W, B) < \delta$ *and* $\sigma^n(W) \subset A$ *for each* $n > \frac{1}{\delta}$.

Theorem

If G is a continuum in $[0,1]\times[0,1]$ *such that* $\pi_1(G)=[0,1]$ *and* $\pi_{\{0,n\}}$ *is the projection of* $\bigstar_{i=1}^{\infty} G$ *onto* $\bigstar_{i=1}^{n} G$*, then every subcontinuum of* $\bigstar_{i=1}^{n} G$ *is the image under* $\pi_{\{0,n\}}$ *of a continuum in* $\bigstar_{i=1}^{n} G$*. In other words* $\pi_{\{0,n\}}$ *is weakly confluent.*

Excercises

E1 Let $G = \{(x,x) \in I^2 \mid x \in [0,1]\} \cup \{(x,0) \in I^2 \mid x \in [0,1]\} \cup \{(1,y) \in I^2 \mid y \in [0,1]\}$. Show that $\bigstar_{i=1}^{n} G$ is homeomorphic to K_{n+2}, the complete graph with $n+2$ vertices.

E2 A compact subset M of the Hilbert cube, I^∞, has the crossover property if whenever $(x_1, x_2, x_3 \ldots)$ and $(y_1, y_2, y_3, \ldots)$ are both elements of M and there is an integer n such that $x_n = y_n$, then $(x_1, x_2, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots)$ is also an element of M. Show that if a compact subset M of the Hilbert cube has the crossover property and if and only if there is a collection of closed closed sets $G_1, G_2, G_3, \ldots$ in $[0,1] \times [0,1]$ such that $M = \bigstar_{i=1}^{\infty} G_i$.

More Excercises

- E3 Show that if a compact subset M of the Hilbert cube has the crossover property and $\sigma(M) = M$ if and only if there is a single closed subset G of $[0,1] \times [0,1]$ such that $M = \bigstar_{i=1}^{\infty} G$
- E4 Find a proof of the Mountain Climbing Theorem online, and read about other applications.
- E5 Show that $\bigstar_{i=1}^{\infty}$ is homeomorphic to an inverse limit with continuous single valued bonding maps between the spaces $G, G \bigstar G, G \bigstar G \bigstar G, \ldots$.
- E6 Show that if G is a continuum in $[0,1] \times [0,1]$ such that $\pi_1(G) = \pi_2(G) = [0,1]$ and G contains the graph of a continuous function defined on all of $[0,1]$, then for each n, $\bigstar_{i=1}^{n} G$ is homeomorphic to a subcontinuum of $\bigstar_{i=1}^{\infty} G$.

Questions

Q1 Suppose G is a closed set in $[0,1] \times [0,1]$ such that $\pi_1(G) = \pi_2(G) = [0,1]$ and $\bigstar_{i=1}^{\infty} G$ is a treelike (chainable) continuum. Is it true that $\bigstar_{i=1}^{n} G$ is a treelike (chainable) continuum for each n? (I can prove this (both treelike and chainable) for the case when G is a finite graph with straight edges).

Q2 Suppose G is a closed set in $[0,1] \times [0,1]$ such that $\pi_1(G) = \pi_2(G) = [0,1]$ and $\bigstar_{i=1}^{\infty} G$ is a treelike (chainable) continuum. Is it rue that $\bigstar_{i=1}^{\infty} G^{-1}$ is a treelike (chainable) continuum? (Positive answers to the questions above imply positive answers to this question.)

More Questions

Q3 Suppose G is a decomposable (hereditarily decomposable) continuum in $[0,1] \times [0,1]$ such that $\pi_1(G) = \pi_2(G) = [0,1]$. Is it true that $G \bigstar G$ is decomposable (hereditarily decomposable)?

Q4 Suppose G is an indecomposable continuum in $[0,1] \times [0,1]$. Is it true that $G \bigstar G$ is disconnected?