PROBLEMS

Unless it is otherwise specified, all spaces mentioned here are assumed to be metric. Solutions to most of the exercises listed below can be found in the literature. In particular, solutions to 2, 6, 7 and 15 may be found in papers by David Bellamy.

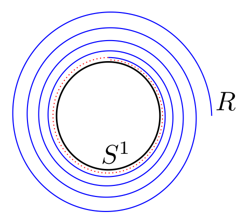
1. Suppose L is an arc and $\epsilon > 0$. Then there is a positive integer $N(L, \epsilon)$ such that, for each collection $\mathcal{C}$ of $N(L, \epsilon)$ subarcs of L whose interiors are mutually disjoint, at least one element of $\mathcal{C}$ has diameter less than ϵ.

2. Let X be a continuum. Show that there is a compactification of $[0, \infty)$ with the remainder homeomorphic to X.

3. Let Y be a compactification of $[0, \infty)$ with a continuum X as the remainder, and let Z be a compactification of $[0, \infty)$ with a continuum P as the remainder. Is it true that each continuous surjection $f : X \to P$ can be extended to a continuous surjection $f^* : Y \to Z$?

4. Let f be a continuous surjection of a continuum X onto a continuum P. Show that, for each a compactification Y of $[0, \infty)$ with X as the remainder, there is a compactification Z of $[0, \infty)$ with P as the remainder such that $f : X \to P$ can be extended to a continuous surjection $f^* : Y \to Z$.

5. Let S^1 denote the unit circle in the complex plane $\mathbb{C}$. Let $R = \left\{ (1 + 2^{-t})\, e^{\mathbf{i}t} \in \mathbb{C} \ \mid \ t \in [0, \infty) \right\}$ where $\mathbf{i} = \sqrt{-1}$. Set $Y = S^1 \cup R$.
Show that any orientation preserving homeomorphism h of S^1 onto itself can be extended to a homeomorphism h^* of Y onto itself.
Would the same be possible without the assumption that h is orientation preserving on S^1?



$Y = S^1 \cup R$

6. Let X be a continuum with an arc component dense. Show that there is a compactification Z of $[0, \infty)$ with X as the remainder such that there is a retraction of Z onto X.

7. Let X be a Peano continuum. Let Z be a compactification of $[0, \infty)$ with X as the remainder. Prove that there is a retraction of Z onto X.

All bonding maps in all inverse sequences mentioned here are continuous.

8. Let $X = \varprojlim \{X_i, f_i\}_{i=0}^{\infty}$ where $f_i : X_{i+1} \to X_i$ for each i. Let π_i denote the projection of X into X_i. Suppose $A \subset X$ and $x \in X \setminus \mathrm{cl}\,(A)$. Show that there is an integer $n \geq 0$ such that $\pi_n\,(x) \notin \mathrm{cl}\,(\pi_n\,(A))$.

9. Suppose f is continuous map of a continuum X into itself. Then, by $\varprojlim \{X, f\}$, denoted by X_f, we understand the inverse limit of the inverse sequence $\{X_i, f_i\}_{i=0}^{\infty}$ where $X_i = X$ and $f_i = f$ for each i. By the shift on X_f we understand the map $s : X_f \to X_f$ defined by $s((x_0, x_1, \dots)) = (f(x_0), f(x_1), \dots)$. Prove that s is a homeomorphism of X_f onto itself.

10. Suppose f is a mapping of a Peano continuum X into itself, $X_f = \varprojlim \{X, f\}$ and s is the shift map of X_f. Then, there is a compactification Z of the half-line

$[0, \infty)$ with X_f as the remainder such that the shift map s extends to a homeomorphism of Z onto itself.
Hint. Consider $Y = X \cup A$ where A is an arc such that $X \cap A$ is one of the endpoints of A. Define $g : Y \to Y$ so that $Z = Y_g = \varprojlim \{Y, g\}$ has the required properties.

11. Suppose $X = \varprojlim \{X_i, f_i\}_{i=0}^{\infty}$ where $X_i = [0,1]$ for each i. Let Y be a compactification of $[0, \infty)$ with X as the remainder. Show that Y may be expressed as $\varprojlim \{Y_i, g_i\}_{i=0}^{\infty}$ where, for each i, $Y_i = [0, 2]$ and g_i restricted to $[0, 1]$ is equal to f_i.

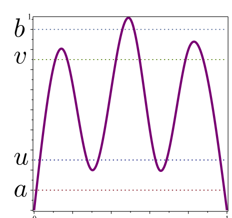
12. Let $\{U_0, U_1, \ldots, U_n\}$ be a finite open cover of a normal space X. Show that X can be covered by open sets $V_0, V_1, \ldots, V_n$ such that $\mathrm{cl}\,(V_i) \subset U_i$ for each $i = 0, \ldots, n$.
Remark. It follows from the above that if X is connected and $\{U_0, U_1, \ldots, U_n\}$ is a chain, then $V_0, V_1, \ldots, V_n$ is a taut chain.

13. Suppose X is a chainable continuum. Then, for each integer $i \geq 0$, there is a 2^{-i}-chain $\mathcal{V}^{(i)} = \left(V_0^{(i)}, V_1^{(i)}, \ldots, V_{n_i}^{(i)}\right)$ covering X in such way that $V_0^{(i+1)} \subset V_{n_i}^{(i)}$.

14. Prove that any nondegenerate arcwise connected chainable continuum is an arc.

15. Let X_0 be a compactification of $[0, \infty)$ with an arbitrary remainder. Using Problem 6 construct an inverse sequence $\{X_i, r_i\}_{i=0}^{\infty}$ such that, for each nonnegative integer i, X_{i+1} a compactification of $[0, \infty)$ with X_i as the remainder, and $r_i : X_{i+1} \to X_i$ is a retraction. Show that the inverse limit $\varprojlim \{X_i, r_i\}_{i=0}^{\infty}$ is indecomposable.

The following definition is illustrated by the figure on the right. The figure shows an example of a function crooked about the quadruple (a, b, u, v). The definition may be summarized in the following way. When traveling on the graph of the function from level a to level b, we first have to reach level v, then return to level u, and only after that we may reach level b.



Definition: Suppose $0 \leq a < u < v < b \leq 1$. We say that a continuous function $f : [0,1] \to [0,1]$ is crooked about the quadruple (a, b, u, v) if each component C_a of $[0,1] \setminus f^{-1}(v)$ such that $a \in f(C_a)$ and each component C_b of $[0,1] \setminus f^{-1}(u)$ such that $b \in f(C_b)$ are disjoint.

16. Consider an inverse sequence $\{X_i, f_i\}_{i=0}^{\infty}$ where $X_i = [0,1]$ and $f_i : X_{i+1} \to X_i$ for each i. For all integers $0 \leq i < j$ let $f_{ij} : X_j \to X_i$ be defined defined by $f_{ij} = f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$. Prove that $\varprojlim \{X_i, f_i\}_{i=0}^{\infty}$ is hereditarily indecomposable if and only if for all numbers a, b, u and v such that $0 \leq a < u < v < b \leq 1$, and for each nonnegative integer i there is an integer $j > i$ such that f_{ij} is crooked about the quadruple (a, b, u, v).

17. Suppose that g_0 and g_1 are two maps of a continuum Y into a solenoid Σ such that

(1) $\pi_0 \circ g_0(y)$ and $\pi_0 \circ g_1(y)$ are not antipodal for each $y \in Y$, and
(2) there exists $y_0 \in Y$ such that $g_0(y_0) = g_1(y_0)$.

Show that g_0 and g_1 are homotopic in Σ. Would the same be true without assuming (2)?

ASSORTED DEFINITIONS AND THEOREMS

All spaces mentioned here are metric. A *continuum* is a connected compact space. A space is *separable* if it contains a countable dense subset. Every compact metric space is separable. Every separable metric space is homeomorphic to a subset of the Hilbert cube $[0,1]^\infty$.

A space X is *locally connected at a point* p if for every neighborhood U of p in X, the component of p in U contains p in its interior. A space X is *locally connected* if it is locally connected at each point. A *Peano continuum* is a continuum that is locally connected.

An *arc* is a continuum homeomorphic to $[0,1]$. A space X is *arcwise connected* if any two of its points can be joined by an arc contained in X. X is *locally arcwise connected at its point* p if for every neighborhood U of p in X, there is a neighborhood V of p in X such that for every $v \in V \setminus \{p\}$ there is an arc in U containing both p and v. X is *locally arcwise connected* if it is locally arcwise connected at each point.

Theorem (Mazurkiewicz–Moore–Menger)**.** *Every locally connected complete space is locally arcwise connected.*

Theorem (Hahn–Mazurkiewicz-Sierpiński)**.** *Suppose X is a nonempty continuum. Then the following three conditions are equivalent.*

1. *X is a continuous image of* $[0,1]$.
2. *For every $\epsilon > 0$, X can be expressed as the union of finitely many subcontinua with diameter $< \epsilon$.*
3. *X is locally connected.*

We say that Z is a *compactification of the half-line* $[0,\infty)$ with X as the *remainder* provided that Z and X are continua, $X \subset Z$, $R = Z \setminus X$ is dense in Z and homeomorphic to $[0,\infty)$. For brevity, we also say that Z is a compactification of $[0,\infty)$ with the remainder X in the case where Z is a compactification of $[0,\infty)$ with the remainder homeomorphic to X.

All bonding maps in all inverse sequences mentioned here are continuous. By an *inverse sequence* we understand a sequence $\{X_i, f_i\}_{i=0}^\infty$ where $f_i : X_{i+1} \to X_i$ is a continuous for each $i = 0, 1, \ldots$. The mappings f_i are called *bonding maps*, and the spaces X_i are called factor spaces (or component spaces or coordinate spaces). The *inverse limit* of $\{X_i, f_i\}_{i=0}^\infty$, denoted by $\varprojlim \{X_i, f_i\}_{i=0}^\infty$, is defined by

$$\varprojlim \{X_i, f_i\}_{i=0}^\infty = \left\{ (x_0, x_1, \ldots) \in \prod_{i=0}^\infty X_i \;\mid\; f_i(x_{i+1}) = x_i \text{ for all } i = 0, 1, \ldots \right\}$$

where the topology on $\varprojlim \{X_i, f_i\}_{i=0}^\infty$ is inherited from the product $\prod_{i=0}^\infty X_i$. Let π_i denote the projection of $\varprojlim \{X_i, f_i\}_{i=0}^\infty$ into X_i.

Solenoids. Recall that S^1 denote the unit circle in the complex plane. For any positive integer n, let $\varphi_n : S^1 \to S^1$ be defined by the formula $\varphi_n(z) = z^n$. Clearly, φ_n is an n-fold covering map of S^1 onto itself. Let $\sigma = (n_0, n_1, \ldots)$ be an arbitrary sequence of positive integers. For each nonnegative integer i, let S_i be a copy of S^1, and let $f_i : S_{i+1} \to S_i$ be defined by $f_i = \varphi_{n_i}$. Let Σ denote the inverse limit $\varprojlim \{S_i, f_i\}_{i=0}^\infty$ and let π_i denote the projection of Σ to S_i. We say that Σ is the solenoid associated with the sequence σ. When σ is not specified we simply say that

Σ is a solenoid. The dyadic solenoid is the solenoid associated with the sequence $(2, 2, 2, \dots)$.

Theorem (Lebesgue's number lemma)**.** *For each open cover of a compact space X there exists a number $\lambda > 0$ such that every subset of X having diameter less than λ is contained in some member of the cover. Such a number λ is called a Lebesgue number of this cover.*

Let $\mathcal{C} = \{U_0, U_1, \dots, U_n\}$ be a finite nonempty sequence of open subsets of a space S. We say that $\mathcal{C}$ is a *chain* in S provided that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. U_i is called the i-th link of $\mathcal{C}$. By $\operatorname{mesh}(\mathcal{C})$ we understand the maximum diameter of its links. We say that a chain $\mathcal{C}$ is an ϵ-chain if $\operatorname{mesh}(\mathcal{C}) < \epsilon$. A continuum P is *chainable* if, for each $\epsilon > 0$, it can be covered by an ϵ-chain.

We say that a chain $\mathcal{C} = \{U_0, U_1, \dots, U_n\}$ is *taut* provided that $\operatorname{cl}(U_i) \cap \operatorname{cl}(U_j) \neq \emptyset$ if and only if $|i - j| \leq 1$.

Suppose $f : X \to Y$ is continuous and $\epsilon > 0$. Then f is called an ϵ-map if $\operatorname{diam}\left(f^{-1}(y)\right) < \epsilon$ for each $y \in Y$.

Theorem. *Suppose X is a nonempty continuum. Then the following three conditions are equivalent.*

(1) *X is chainable.*
(2) *For every $\epsilon > 0$, there is an ϵ-map $f_\epsilon : X \to [0, 1]$.*
(3) *X may be expressed as $\varprojlim \{X_i, f_i\}_{i=0}^{\infty}$ where, for each i, $X_i = [0, 1]$.*

A continuum is *decomposable* if it is not the union of two of its proper subcontinua, it is *indecomposable* otherwise. A continuum is *hereditarily indecomposable* if it does not contain a decomposable subcontinuum.

Theorem (Bing)**.** *All hereditarily indecomposable nondegenerate chainable continua are homeomorphic.*

The *pseudo-arc* is a hereditarily indecomposable nondegenerate chainable continuum.