Infinite-dimensional topology
Prerequisites and introduction
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1. Background

Lemma 1.1 (Lemma 1.1.1 in the book). *Let X be a compact subspace of a space Y and let $\mathscr{U}$ be a collection of open subsets of Y which covers X. Then there exists $\delta > 0$ with the property that every $A \subseteq Y$ with $\operatorname{diam}(A) < \delta$ and which moreover intersects X, is contained in an element $U \in \mathscr{U}$.*

Proof. Suppose, to the contrary, that such δ does not exist. Then for every $n \in \mathbb{N}$ we can find a subset A_n of Y such that

(1) $\operatorname{diam}(A_n) < {}^1\!/_n$,
(2) A_n intersects X, say $x_n \in A_n \cap X$,
(3) A_n is not contained in any element of $\mathscr{U}$.

Since X is compact, every sequence in X has a convergent subsequence, so without loss of generality we may assume that $x = \lim_{n\to\infty} x_n$ exists and belongs to X of course. There exists $U \in \mathscr{U}$ such that $x \in U$. Since U is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. In addition, there exists $N \in \mathbb{N}$ such that $x_m \in B(x, {}^\varepsilon\!/_2)$ for every $m \geq N$. Now choose $m \geq N$ so large that ${}^1\!/_m \leq {}^\varepsilon\!/_2$. Since the diameter of A_m is less than ${}^1\!/_m \leq {}^\varepsilon\!/_2$, it now follows easily that $A_m \subseteq B(x, \varepsilon) \subseteq U$, which is a contradiction. $\square$

The number δ in the above lemma is called a *Lebesgue number* for $\mathscr{U}$.

If X and Y are spaces then $C(X, Y)$ denotes the set of all continuous functions from X to Y. It will be convenient to topologize $C(X, Y)$ and interesting subsets of it with a natural topology.

Let X be compact. For all $f, g \in C(X, Y)$ put

$$d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

Observe that $d(f, g) \in [0, \infty)$.

Lemma 1.2 (Lemma 1.3.1 in the book). *Let X and Y be spaces with X compact. Then $d\colon C(X, Y) \times C(X, Y) \to [0, \infty)$ is a metric.*

We shall endow $C(X, Y)$ with the topology induced by d. It can be shown easily that $C(X, Y)$ is seperable ([1, Proposition 1.3.3]). There is a problem because the topology on $C(X, Y)$ was defined with the help of the metric d. This is not really a problem, as the next result shows.

Lemma 1.3 (Lemma 1.3.2 in the book). *Let X and Y be spaces with X compact. In addition, let d_1 and d_2 be admissible metrics for Y. Then the topologies on $C(X, Y)$ induced by d_1 and d_2 are the same.*

Proof. For each $\varepsilon > 0, y \in Y$ and $i \in \{1,2\}$ we put

$$B_i(y,\varepsilon) = \{z \in Y : d_i(y,z) < \varepsilon\}.$$

Take $f \in C(X,Y)$ and $\varepsilon > 0$, arbitrarily. Since $f[X]$ is compact, the open cover

$$\mathscr{U} = \{B_1(f(x), {}^1\!/\!_4\varepsilon) : x \in X\}$$

has a d_2-Lebesgue number, say δ (Lemma 1.1). Without loss of generality, $\delta < {}^{\varepsilon}\!/\!_4$. Now take $g \in C(X,Y)$ such that $d_2(f,g) < \delta$. For each $x \in X$ we have $d_2(f(x), g(x)) < \delta$, so there exists $p_x \in X$ such that $\{f(x), g(x)\} \subseteq B_1(f(p_x), {}^{\varepsilon}\!/\!_4)$. Consequently, for each $x \in X$ we have

$$d_1(f(x), g(x)) < {}^{\varepsilon}\!/\!_2,$$

from which it follows that $d_1(f,g) \le {}^{\varepsilon}\!/\!_2 < \varepsilon$. We conclude that

$$\{g \in C(X,Y) : d_2(f,g) < \delta\} \subseteq \{g \in C(X,Y) : d_1(f,g) < \varepsilon\}$$

and hence that the topology on $C(X,Y)$ induced by d_2 is finer than the topology on $C(X,Y)$ induced by d_1. By interchanging the roles of d_1 and d_2 in the above argument we obtain the desired result. $\square$

We now turn to completeness properties of $C(X,Y)$.

Proposition 1.4 (Proposition 1.3.4 in the book)**.** *Let X and (Y,d) be compact spaces. Let $(f_n)_n$ be a d-Cauchy sequence in $C(X,Y)$. Then the function $f\colon X \to Y$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous. Moreover, $f = \lim_{n\to\infty} f_n$ (in $C(X,Y;d)$).*

Proof. To begin with, let us establish the following

Claim 1. $\forall\varepsilon > 0\, \exists N \in \mathbb{N}$ such that for every $x \in X$ and $m \ge N$, $d(f(x), f_m(x)) < \varepsilon$.

Take $N \in \mathbb{N}$ such that $d(f_n, f_m) < {}^{\varepsilon}\!/\!_2$ for all $n,m \ge N$. We claim that N is as required. Let $x \in X$. Since $d(f_n(x), f_m(x)) < {}^{\varepsilon}\!/\!_2$ for all $n,m \ge N$ and since $f(x)$ is equal to $\lim_{n\to\infty} f_n(x)$, we obtain that for every $m \ge N$, $d(f(x), f_m(x)) \le {}^{\varepsilon}\!/\!_2 < \varepsilon$.

We conclude that the sequence $(f_n)_n$ converges uniformly to f on X.

We shall now prove that f is continuous. Take $x \in X$ and $\varepsilon > 0$ arbitrarily. By the above, there exists $N \in \mathbb{N}$ such that $d(f(x), f_m(x)) < {}^{\varepsilon}\!/\!_3$ for all $m \ge N$. Since f_N is continuous at x, there exists $\delta > 0$ such that if $d(x,z) < \delta$ then $d(f_N(x), f_N(z)) < {}^{\varepsilon}\!/\!_3$. Now take $z \in X$ with $d(x,z) < \delta$. Then

$$\begin{aligned} d(f(x), f(z)) &\le d(f(x), f_N(x)) + d(f_N(x), f_N(z)) + d(f_N(z), f(z)) \\ &< {}^{\varepsilon}\!/\!_3 + {}^{\varepsilon}\!/\!_3 + {}^{\varepsilon}\!/\!_3 \\ &= \varepsilon. \end{aligned}$$

We conclude that f is continuous at x.

It remains to prove that $f = \lim_{n\to\infty} f_n$ (in $C(X,Y)$). However, this follows easily from the claim. $\square$

Corollary 1.5 (Corollary 1.3.5 in the book)**.** *Let X and (Y,d) be compact spaces. Then the metric d on $C(X,Y)$ is complete.*

Now let X and Y be spaces and define

$$\mathscr{S}(X,Y) = \{f \in C(X,Y) : f \text{ is surjective}\}.$$

There are spaces X and Y for which $\mathscr{S}(X,Y)$ is empty, see Exercise 1.

Proposition 1.6 (Proposition 1.3.7 in the book)**.** *Let X and Y be compact spaces. Then $\mathscr{S}(X,Y)$ is closed in $C(X,Y)$.*

Proof. Assume that $f \notin \mathscr{S}(X,Y)$, i.e., there exists a point $y \in Y \setminus f[X]$. By compactness, $\varepsilon = d(y, f[X]) > 0$. It is a triviality to verify that $B(f,\varepsilon) \cap \mathscr{S}(X,Y) = \emptyset$. We conclude that $C(X,Y) \setminus \mathscr{S}(X,Y)$ is open in $C(X,Y)$. $\square$

Let X and Y be spaces with X compact, and let $\varepsilon > 0$. A function $f \in C(X,Y)$ is called an ε-map if for every $y \in Y$,

$$\operatorname{diam}(f^{-1}(y)) < \varepsilon.$$

Put $C_\varepsilon(X,Y) = \{f \in C(X,Y) : f \text{ is an } \varepsilon\text{-map}\}$ and $\mathscr{S}_\varepsilon(X,Y) = C_\varepsilon(X,Y) \cap \mathscr{S}(X,Y)$, respectively. In addition, let $\mathscr{G}_\varepsilon(X,Y) = \mathscr{S}(X,Y) \setminus \mathscr{S}_\varepsilon(X,Y)$.

Lemma 1.7 (Lemma 1.3.8 in the book)**.** *Let X and Y be spaces with X compact and let $\varepsilon > 0$. Then $C_\varepsilon(X,Y)$ is an open subspace of $C(X,Y)$. Consequently, $\mathscr{G}_\varepsilon(X,Y)$ is closed in $C(X,Y)$.*

Proof. Take $f \in C_\varepsilon(X,Y)$. Since X is compact, $f\colon X \to f[X]$ is a closed map (Exercise 2). Consequently there exists for every $y \in f[X]$ an open neighborhood U_y (in $f[X]$) such that

$$\operatorname{diam}(f^{-1}(U_y)) < \varepsilon$$

(Exercise 3). Let $\delta > 0$ be a Lebesgue number for the open covering $\{U_y : y \in f[X]\}$ of $f[X]$ (Lemma 1.1). Let $g \in C(X,Y)$ be such that $d(g,f) < \delta/2$. We claim that $g \in C_\varepsilon(X,Y)$. To this end, take an arbitrary $y \in Y$. Since $d(f,g) < \delta/2$ it follows easily that $\operatorname{diam}(fg^{-1}(y)) < \delta$. Consequently there exists a point $z \in f[X]$ such that $fg^{-1}(y) \subseteq U_z$ which implies that $\operatorname{diam}(f^{-1}fg^{-1}(y)) < \varepsilon$. Since $g^{-1}(y) \subseteq f^{-1}fg^{-1}(y)$, we conclude that $\operatorname{diam}(g^{-1}(y)) < \varepsilon$, i.e., g is an ε-map. $\square$

Let X and Y be spaces and let $\mathscr{H}(X,Y)$ denote the set of all homeomorphisms from X onto Y considered as a subspace of $C(X,Y)$. If $X = Y$ then for $\mathscr{H}(X,X)$ we shall simply write $\mathscr{H}(X)$. As usual, $\mathscr{H}(X)$ is called the autohomeomorphism group of X.

Lemma 1.8 (Lemma 1.3.9 in the book)**.** *Let X and Y be compact spaces. Then $\mathscr{H}(X,Y) = \bigcap_{n=1}^\infty \mathscr{S}_{1/n}(X,Y)$. As a consequence, $\mathscr{H}(X,Y)$ is a G_δ-subset of $\mathscr{S}(X,Y)$ and hence of $C(X,Y)$.*

Proof. That $\mathscr{H}(X,Y) \subseteq \bigcap_{n=1}^\infty \mathscr{S}_{1/n}(X,Y)$ is a triviality. Pick $f \in \bigcap \mathscr{S}_{1/n}(X,Y)$. Then f is a $1/n$-map for every n, hence f is one-to-one. Since f is onto, the compactness of X implies that f is a homeomorphism (Exercise 4). $\square$

Let X be a compact space. The above lemma implies that $\mathscr{H}(X)$ is a G_δ-subset of $C(X,X)$.

Exercises

(1) Give an example of two compact spaces X and Y such that $\mathscr{S}(X,Y)=\emptyset$.
(2) Let X and Y be compact spaces and let $f\colon X\to Y$ be continuous. Prove that f is a *closed* map, i.e., that $f[A]$ is closed in Y for every closed subset A of X.
(3) Let $f\colon X\to Y$ be a closed map, let $A\subseteq Y$ and let U be an open neighborhood of $f^{-1}[A]$ in X. Prove that there is an open neighborhood V of A in Y such that $f^{-1}[V]\subseteq U$.
(4) Let X be compact and let $f\colon X\to Y$ be one-to-one. Prove that if $f[X]$ is dense in Y then f is a homeomorphism.
(5) Let X be a compact space. Prove that the function $\xi\colon \mathscr{H}(X)\times\mathscr{H}(X)\to\mathscr{H}(X)$ defined by $\xi(f,g)=f\circ g^{-1}$ is continuous (i.e., $\mathscr{H}(X)$ is a topological group).
(6) Prove that the function $f\colon I\to I$ defined by

$$f(x)=\begin{cases} 2x & (0\le x\le 1/4),\\ 1/2 & (1/4\le x\le 3/4),\\ 2x-1 & (3/4\le x\le 1).\end{cases}$$

belongs to the closure of $\mathscr{H}(I)$ in $C(I,I)$.
(7) Prove that $\mathscr{H}(I)$ has exactly two components.
(8) Let X, Y and Z be compact spaces, let $f\in C(Z,X)$ and let $g,h\in C(X,Y)$. Prove that $d(g\circ f,h\circ f)\le d(g,h)$. In addition, prove that if f is surjective then $d(g\circ f,h\circ f)=d(g,h)$.

References

[1] J. van Mill, *Infinite-dimensional topology: prerequisites and introduction*, North-Holland Publishing Co., Amsterdam, 1989.