INVERSE LIMITS OF FAMILIES OF SET-VALUED FUNCTIONS

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Abstract. In this paper we investigate inverse limits of two related parameterized families of upper semi-continuous set-valued functions. We include a theorem one consequence of which is that certain inverse limits with a single bonding function from one of these families are the closure of a topological ray (usually with indecomposable remainder) as well as a theorem giving a new sufficient condition that an inverse limit with set-valued functions is an indecomposable continuuum. It is shown that some, but not all, functions from these families produce chainable continua. This expands the list of examples of chainable continua produced by set-valued functions that are not mappings. The paper includes theorems on constructing subcontinua of inverse limits as well as theorems on expressing inverse limits with set-valued functions as inverse limits with mappings.

1. Introduction

In the theory of inverse limits with mappings consideration of inverse limits of families of mappings was inspired, in part, by dynamical systems. Following in the tradition of investigating the topology of inverse limits of families of mappings, in this paper we consider inverse limits of a parameterized family of set-valued functions. As in the theory of inverse limits with mappings, this investigation has led to some interesting phenomena in continuum theory such as abrupt changes in the topology of the inverse limit over small neighborhoods of parameter values. Some studies of inverse limits of families of set-valued functions has already occurred. For instance, the examples presented in [2, Sections 2.4 and 2.8.1] are concerned with families of inverse limits although not much change takes place in the topology of the inverse limit over large sets of parameter values (big changes do occur at the extreme values of the parameters). In this article we present a family of examples in which extreme changes occur quite often as the parameter varies. By-products of this study are some "new" chainable continua (i.e., new to the study of inverse limits with set-valued functions that are not mappings) as well as some examples of set-valued functions that are not mappings having inverse limits that are closures of topological rays with non-degenerate indecomposable remainders (suggesting reasons to consider [2, Problem 6.47, p. 80].

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2. Preliminaries

By a *compactum* we mean a compact subset of a metric space; by a *continuum* we mean a connected compactum. If X is a continuum we denote the collection of closed subsets of X by 2^X; $C(X)$ denotes the connected elements of 2^X. If each of X and Y is a continuum, a function $f : X \to 2^Y$ is said to be *upper semi-continuous at the point x of X* provided that if V is an open subset of Y that contains $f(x)$ then there is an open subset U of X containing x such that if t is a point of U then $f(t) \subseteq V$. A function $f : X \to 2^Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of X. If $X_0 \subseteq X$ and $f : X_0 \to 2^Y$ is a set-valued function, by the *graph* of f, denoted $G(f)$, we mean $\{(x, y) \in X \times Y \mid y \in f(x)\}$; $G(f)^{-1} = G(f^{-1})$. It is known that if M is a subset of $X \times Y$ such that X is the projection of M to its set of first coordinates then M is closed if and only if M is the graph of an upper semi-continuous function [5, Theorem 2.1]. If $f : X \to 2^Y$ is a set-valued function and $A \subseteq X$, we let $f(A) = \{y \in Y \mid$ there is a point $x \in X$ such that $y \in f(x)\}$; we say that f is *surjective* provided $f(X) = Y$. In the case that f is upper semi-continuous and single-valued (i.e., $f(t)$ is degenerate for each $t \in X$), f is a continuous function. We call a continuous function a *mapping*; if $f : X \to Y$ is a mapping, we denote that f is surjective by $f : X \twoheadrightarrow Y$.

We denote by $\mathbb{N}$ the set of positive integers. If $\boldsymbol{s} = s_1, s_2, s_3, \ldots$ is a sequence, we normally denote the sequence in boldface type and its terms in italics. Because every metric space has an equivalent metric that is bounded by 1, we assume throughout that all of our spaces have metrics bounded by 1. Suppose $\boldsymbol{X}$ is a sequence of compact metric spaces and $f_n : X_{n+1} \to 2^{X_n}$ is an upper semi-continuous function for each $n \in \mathbb{N}$. We call the pair $\{\boldsymbol{X}, \boldsymbol{f}\}$ an *inverse limit sequence*. By the *inverse limit* of the inverse limit sequence $\{\boldsymbol{X}, \boldsymbol{f}\}$, denoted $\varprojlim\{\boldsymbol{X}, \boldsymbol{f}\}$, or for short, $\varprojlim \boldsymbol{f}$, we mean $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for each positive integer $i\}$. Inverse limits are nonempty and compact [5, Theorem 3.2]; they are metric spaces being subsets of the metric space $\prod_{i>0} X_i$. It is known that if each bonding function in an inverse limit sequence on continua is continuum-valued then the inverse limit is a continuum [6, Theorem 125]. We use the metric d on $\prod_{i>0} X_i$ given by $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i>0} d_i(x_i, y_i)/2^i$. In the case that each f_n is a mapping our definition of the inverse limit reduces to the usual definition of an inverse limit on compacta with mappings. If $A \subseteq \mathbb{N}$, we denote by p_A the projection of $\prod_{n>0} X_n$ onto $\prod_{n\in A} X_n$ given $p_A(\boldsymbol{x}) = \boldsymbol{y}$ provided $y_i = x_i$ for each $i \in A$. If $A = \{n\}$, $p_{\{n\}}$ is normally denoted p_n. In the case that $A \subseteq B \subseteq \mathbb{N}$, we normally also denote the restriction of p_A to $\prod_{n\in B} X_n$ by p_A inferring by context that we are using this restriction. We denote the projection from the inverse limit into the ith factor space by π_i and, more generally, for $A \subseteq \mathbb{N}$, we denote by π_A the restriction of p_A to the inverse limit.

For a inverse limit sequence $\{\boldsymbol{X}, \boldsymbol{f}\}$, a sequence $\boldsymbol{G}$ of sets traditionally used in the proof that $\varprojlim \boldsymbol{f}$ is nonempty and compact is $G_n = \{\boldsymbol{x} \in \prod_{k>0} X_k \mid x_i \in f_i(x_{i+1})$ for $1 \le i \le n\}$ for each $n \in \mathbb{N}$. This sequence also plays a key role in connectedness proofs. We adopt and use throughout this article the notation $G'_n = \{\boldsymbol{x} \in \prod_{k=1}^{n+1} X_k \mid x_i \in f_i(x_{i+1})$ for

$1 \leq i \leq n\}$. Note that for $A = \{1, 2, \ldots, n+1\}$, $G'_n = p_A(G_n)$. For a finite sequence of functions $f_1, f_2, \ldots, f_n$ it is often convenient to denote G'_n by $G'(f_1, f_2, \ldots, f_n)$. Of course, $G_n = G'(f_1, f_2, \ldots, f_n) \times \prod_{i>n+1} X_i$ and $G'(f) = G(f^{-1}) = G'_1$. We denote the closure of a set A by $\mathrm{Cl}(A)$; $\mathcal{Q}$ denotes the Hilbert cube, $[0,1]^\infty$. If $\boldsymbol{X}$ is a sequence of compacta and $\boldsymbol{f}$ is a sequence of functions such that $f_i : X_{i+1} \to 2^{X_i}$ for each $i \in \mathbb{N}$ and i and j are positive integers with $i < j$, $f_{i\,j} : X_j \to 2^{X_i}$ denotes the composition $f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$; we adopt the usual convention that $f_{i\,i}$ denotes the identity on X_i.

Theorem 2.1. *Suppose X and Y are compact metric spaces, $f : X \to C(Y)$ is an upper semi-continuous function, and H is a connected subset of X. Then, $\{(x, y) \in G(f) \mid x \in H\}$ is connected.*

Proof. Let $K = \{(x, y) \in G(f) \mid x \in H\}$ and suppose K is the union of two mutually separated sets A and B. There exist two mutually exclusive open sets U and V such that $A \subseteq U$ and $B \subseteq V$. For each $x \in H$, $\{x\} \times f(x)$ is a subcontinuum of $A \cup B$ so it is a subset of one of them. Let $H_A = \{x \in H \mid \{x\} \times f(x) \subseteq A\}$ and $H_B = \{x \in H \mid \{x\} \times f(x) \subseteq B\}$. Note that H_A and H_B are mutually exclusive and their union is H. If $x \in H_A$ then, because $f(x)$ is compact, there are open sets O_x and R_x such that $\{x\} \times f(x) \subseteq O_x \times R_x$ and $O_x \times R_x \subseteq U$. Because $f(x)$ is a subset of the open set R_x and f is upper semi-continuous, there is an open set W containing x and lying in O_x such that if $s \in W$ then $f(s) \subseteq R_x$. Therefore, if $s \in W \cap H$, then $s \in H_A$ so $x \notin \mathrm{Cl}(H_B)$. Similarly, if $x \in H_B$ then $x \notin \mathrm{Cl}(H_A)$, so H_A and H_B are mutually separated, a contradiction. Thus, K is connected. □

The following theorem is a corollary of Theorem 2.1. It is similar to [2, Theorem 2.4, p. 16].

Theorem 2.2. *Suppose each of X and Y is a compact metric space and H is a connected subset of X. If $f : X \to C(Y)$ is upper semi-continuous, then $f(H)$ is connected.*

Proof. The set $f(H)$ is the image under the projection p_2 of the connected set $\{(x, y) \in G(f) \mid x \in H\}$. □

Theorem 2.3. *Suppose X, Y, and Z are metric spaces. If $f : X \to C(Y)$ and $g : Y \to C(Z)$ are upper semi-continuous, then $g \circ f(x)$ is connected for each $x \in X$, i.e., $g \circ f$ is an upper semi-continuous function from X into $C(Z)$.*

Proof. It is well known that $g \circ f$ is upper semi-continuous. If $x \in X$ then $f(x)$ is a continuum. By Theorem 2.2, $g(f(x))$ is connected and thus is a continuum. But, $g(f(x)) = g \circ f(x)$. □

We close this section of preliminary results with a lemma that amounts to little more than an observation. We often make use of this lemma without specific reference to it.

Lemma 2.4. *Suppose $\boldsymbol{X}$ is a sequence of continua and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function for each $i \in \mathbb{N}$. If p is a point of X_{n+1} such that $f_{i\,n+1}(p)$ is a single point for each integer $i, 1 \le i \le n$, then there is only one point $\boldsymbol{x}$ of G'_n such that $x_{n+1} = p$.*

3. Subcontinua of an inverse limit

The ability to determine subcontinua of an inverse limit is often key to determining the structure of the entire inverse limit. In the theory of inverse limits with mappings, identifying subcontinua of the inverse limit can be as simple as finding a sequence of subcontinua of the factor spaces each term of which is mapped into the previous term by the bonding map and taking the inverse limit; in fact, each subcontinuum of an inverse limit with mappings is the inverse limit of its projections. For inverse limits with set-valued functions, the matter is more complicated; for one thing, a proper subcontinuum can project onto the entire factor space in each factor space. In our first example, we revisit such an example from [2] where in Example 2.14 on page 30 a topological conjugate of the bonding function was discussed. Following this example we discuss determining some of its subcontinua.

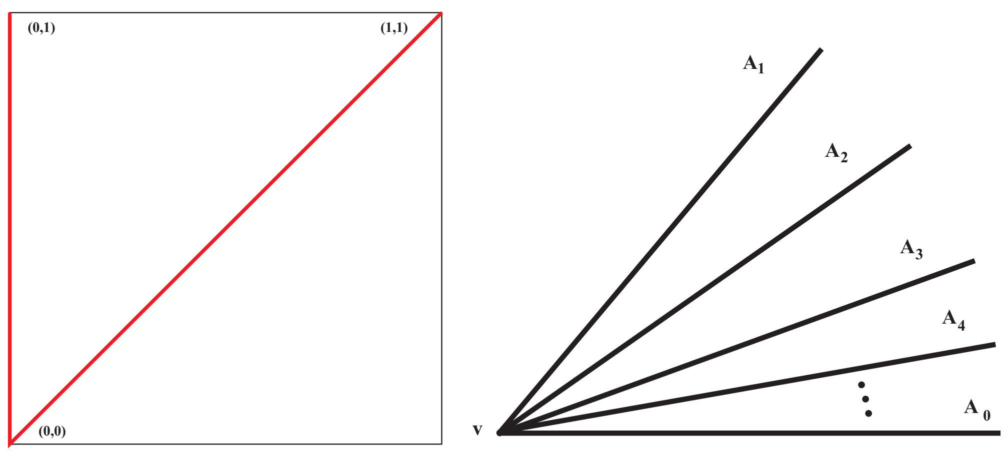

Figure 1. The graph of the bonding function f and its inverse limit in Example 3.1

Example 3.1. *Let $f : [0,1] \to C([0,1])$ be the upper semi-continuous function whose graph consists of two straight line intervals, one from $(1,0)$ to $(0,0)$ and the other from $(0,0)$ to $(1,1)$. The inverse limit, $\varprojlim \boldsymbol{f}$, is a fan with vertex $\boldsymbol{v} = (0,0,0,\dots)$ containing a proper subcontinuum H such that $\pi_n(H) = [0,1]$ for each $n \in \mathbb{N}$.*

Proof. Let $M = \varprojlim \boldsymbol{f}$ and $A_0 = \{\boldsymbol{x} \in M \mid x_i = x_1 \text{ for each } i \in \mathbb{N}\}$. For $j \in \mathbb{N}$, let $A_j = \{\boldsymbol{x} \in M \mid x_i = x_1 \text{ for } 1 \le i \le j \text{ and } x_i = 0 \text{ for } i > j\}$. Each A_n is an arc for $n \ge 0$ and each point of A_0 is a limit point of a sequence $\boldsymbol{z_1}, \boldsymbol{z_2}, \boldsymbol{z_3}, \dots$ of points of M such that $\boldsymbol{z_n} \in A_n$ for each $n \in \mathbb{N}$. It follows that $M = A_0 \cup A_1 \cup A_2 \cup \cdots$ is a fan with vertex

$(0,0,0,\dots)$ and $H = A_0$ is a proper subcontinuum of M such that $\pi_n(H) = [0,1]$ for each $n \in \mathbb{N}$. $\square$

Subcontinua of the fan M from Example 3.1. By using sequences of bonding functions having graphs that are subsets of $G(f)$ we obtain a few subcontinua of M.

- Let $g_1 = \mathrm{Id}$ (the identity on $[0,1]$) and $g_i = f$ for each $i > 1$. Then, $\varprojlim \boldsymbol{g} = A_0 \cup A_2 \cup A_3 \cup A_4 \cup \cdots$ (deleting from M all of the points of A_1 except $\boldsymbol{v}$).
- Let $g_i = \mathrm{Id}$ for each odd integer i and $g_i = f$ for each even integer i. Then, $\varprojlim \boldsymbol{g} = A_0 \cup A_2 \cup A_4 \cup A_6 \cup \cdots$.
- Let g be the upper semi-continuous function such that $G(g) = G(f) \cap \big([0,1/2]^2\big)$. Then, $\varprojlim \boldsymbol{g}$ is a proper subfan of M each arm of which lies in an arm of M.
- Let φ be the upper semi-continuous function whose graph consists of the vertical line from $(1,0)$ to $(0,0)$. Let $g_1 = \varphi$ and $g_i = f$ for $i > 1$. Then, $\varprojlim \boldsymbol{g} = A_1$. Letting $g_1 = \mathrm{Id}$, $g_2 = \varphi$, and $g_i = f$ for $i > 2$ results in A_2 as the inverse limit. Using $g_1 = f, g_2 = \varphi$, and $g_i = f$ for $i > 2$ yields $A_1 \cup A_2$.

Other subcontinua of inverse limits with set-valued functions appear not to arise in these ways. Part of the problem seems to lie in the fact that the the Subsequence Theorem fails for inverse limits with set-valued functions. We now turn to additional means of obtaining subcontinua of an inverse limit. The technique developed in Theorem 3.7 is used in Example 9.1.

Our next theorem is a convenient restatement of the classic theorem characterizing upper semi-continuous functions as having graphs that are closed subsets of a product space [6, Theorem 105]. Here we have reversed the roles of X and Y.

Lemma 3.2. *Suppose each of X and Y is a compact metric space and H is a subset of $X \times Y$ such that if $y \in Y$ there is a point $x \in X$ such that $(x,y) \in X \times Y$. Then H is closed if and only if there is an upper semi-continuous function $F : Y \to 2^X$ such that $H = G(F^{-1})$.*

An immediate consequence of this lemma is the following lemma stated in a form for use in this paper. If $X_1, X_2, X_3, \dots$ is a sequence of compact metric spaces and $m \in \mathbb{N}$, we denote by Π_m the product $X_1 \times X_2 \times \cdots \times X_m$.

Lemma 3.3. *Suppose $\boldsymbol{X}$ is a sequence of compact metric spaces, n is a positive integer, and H is a closed subset of Π_{n+1}. If $p_{n+1}(H) = X_{n+1}$, then there is an upper semi-continuous function $F : X_{n+1} \to 2^{\Pi_n}$ such that H is homeomorphic to $G(F^{-1})$.*

Proof. Let $\hat{H} = \{(\boldsymbol{y}, x) \in \Pi_n \times X_{n+1} \mid (y_1, y_2, \dots, y_n, x) \in H\}$. Then, $\hat{H}$ is a closed subset of $\Pi_n \times X_{n+1}$ that projects onto X_{n+1}. By Lemma 3.3, there is an upper semi-continuous function $F : X_{n+1} \to 2^{\Pi_n}$ such that $\hat{H} = G(F^{-1})$. Thus, H is homeomorphic to $G(F^{-1})$. $\square$

The points of $G(F^{-1})$ in Theorem 3.3 are ordered pairs the first term of which is an n-tuple. The homeomorphism between $G(F^{-1})$ and H merely ignores the parentheses of the n-tuple. In the remainder of the paper, it will be convenient not make explicit mention of this homeomorphism.

Remark 3.4. *In the case that $\boldsymbol{X}$ is a sequence of metric spaces and $f_i : X_{i+1} \to 2^{X_i}$ is a surjective upper semi-continuous for each $i \in \mathbb{N}$, the function F from Lemma 3.3 is the upper semi-continuous function Van Nall derives from $G'(f_1, f_2, \ldots, f_n)$ and denotes by F_n in* [9]. *Furthermore, $G'(f_1, f_2, \ldots, f_n) = G(F_n^{-1})$ where $F_n : X_{n+1} \to 2^{Y_n}$ for $Y_1 = X_1$ and $Y_n = G'(f_1, f_2, \ldots, f_{n-1})$ if $n > 1$.*

Suppose $\{\boldsymbol{X}, \boldsymbol{f}\}$ is an inverse limit sequence with surjective bonding functions. Throughout the remainder of this paper, we adopt Nall's notation that F_n denotes the upper semi-continuous function such that $G(F_n^{-1}) = G'(f_1, f_2, \ldots, f_n)$. As a convenience of notation, when $n = 1$, we let the otherwise meaningless symbol $G'(f_1, f_2, \ldots, f_{n-1})$ denote X_1.

Theorem 3.5. *Suppose $\boldsymbol{X}$ is a sequence of compact metric spaces and $f_i : X_{i+1} \to C(X_i)$ upper semi-continuous and surjective for each positive integer i. Then, for $n \in \mathbb{N}$, F_n is an upper semi-continuous function from X_{n+1} into $C(G'(f_1, f_2, \ldots, f_{n-1}))$.*

Proof. Observe that $F_1 = f_1 : X_2 \to C(X_1)$. Inductively, assume that k is a positive integer such that $F_k : X_{k+1} \to C(P_k)$. Choose $t \in X_{k+2}$. It is sufficient to show that $F_{k+1}(t)$ is connected. Because $f_{k+1}(t)$ is connected, by Theorem 2.1, $K = \{(s, \boldsymbol{y}) \in G(F_k) \mid s \in f_{k+1}(t)\}$ is connected. But K is homeomorphic to $\{(F_k(s), s) \in P_{k+2} \mid s \in f_{k+1}(t)\}$, a set homeomorphic to $F_{k+1}(t)$. $\square$

Theorem 3.6. *Suppose $\boldsymbol{X}$ is a sequence of continua and $\boldsymbol{f}$ is a sequence of surjective upper semi-continuous functions such that $f_i : X_{i+1} \to C(X_i)$ for each positive integer i. If $n \in \mathbb{N}$ and K is a connected subset of X_{n+1}, then $\{\boldsymbol{x} \in G'(f_1, f_2, \ldots, f_n) \mid x_{n+1} \in K\}$ is connected.*

Proof. By Theorem 3.5, F_n is continuum-valued, so, by Theorem 2.1, $H = \{(x, \boldsymbol{y}) \in X_{n+1} \times P_n \mid x \in K \text{ and } \boldsymbol{y} \in F_n(K)\}$ is connected. However, $\{\boldsymbol{x} \in G'(f_1, f_2, \ldots, f_n) \mid x_{n+1} \in K\}$ is homeomorphic to H. $\square$

In our next theorem, we obtain another means of constructing a proper subcontinuum of an inverse limit with set-valued functions. We make use of the techniques of this theorem in Example 9.1.

Theorem 3.7. *Suppose $\boldsymbol{X}$ is a sequence of continua and $f_i : X_{i+1} \to C(X_i)$ is a surjective upper semi-continuous function for each positive integer i. Suppose further that n is a positive integer and H is a closed proper subcontinuum of $G'(f_1, f_2, \ldots, f_n)$ such that $p_{n+1}(H) = X_{n+1}$. Let $Y_1 = G'(f_1, f_2, \ldots, f_{n-1})$ and $g_1 = F$ where $H = G(F^{-1})$. For $i > 1$, let $Y_i = X_{n+i}$ and $g_i = f_{n+i}$. If F is continuum-valued, then $\varprojlim \boldsymbol{g}$ is a proper subcontinuum of $\varprojlim \boldsymbol{f}$.*

Proof. Because $M = \varprojlim \boldsymbol{f}$ is a continuum, it follows that $Y_1 = \pi_A(M)$ where $A = \{1, 2, \ldots, n\}$ is a continuum. Because each g_i is continuum-valued and all the factor spaces are continua, $\varprojlim \boldsymbol{g}$ is a continuum. Because H is a proper subset of $G'(f_1, f_2, \ldots, f_n)$ such that $p_{n+1}(H) = X_{n+1}$, it follows that there is a point of $\boldsymbol{z} \in G'(f_1, \ldots, f_{n-1})$ that is not in $g_1(X_{n+1})$. For each $i \in \mathbb{N}$, f_i is surjective so there is a point $\boldsymbol{x} \in M$ such that $p_A(\boldsymbol{x}) = \boldsymbol{z}$. The point $\boldsymbol{x}$ does not belong to $\varprojlim \boldsymbol{g}$, thus, $\varprojlim \boldsymbol{g}$ is a proper subcontinuum of M. $\square$

We close this section with a theorem giving one other means of constructing subcontinua of inverse limits with set-valued functions, in this case by embedding copies of the sets G'_n into an inverse limit with a single bonding function through imposing conditions on the bonding function. This theorem is due to Marsh [8, Corollary 3.1] although his statement is somewhat different and more general; we state it in a form for use in this paper.

Theorem 3.8. (Marsh) *Suppose X is a continuum, $f : X \to 2^X$ is an upper semi-continuous function, Y is a compact subset of X, and $g : Y \to 2^X$ is an upper semi-continuous function such that $G(g) \subseteq G(f)$ and g^{-1} is a mapping of X into Y. If $n \in \mathbb{N}$, then $\varphi : G'_n \to \varprojlim \boldsymbol{f}$ given by $\varphi(\boldsymbol{x}) = (x_1, x_2, \ldots, x_{n+1}, g^{-1}(x_{n+1}), g^{-2}(x_{n+1}), g^{-3}(x_{n+1}), \ldots)$ is a homeomorphism.*

In Section 3 of [8] Marsh has several additional theorems that speak to recognizing subcontinua of inverse limits with a single set-valued bonding function. In the next section we include one additional theorem on obtaining subcontinua of an inverse limit with set-valued functions.

4. Inverse limits with set-valued functions as inverse limits with mappings

There are ways to express inverse limits with set-valued functions as inverse limits with mappings. Generally, in order to do so, one has to give up something. The price normally involves (even in the single bonding function case) changing to a sequence of (perhaps) more complicated factor spaces and using a sequence of bonding mappings. However, in case these new factor spaces are relatively simple spaces, the price can be well worth it.

Our next theorem allows us to express closed subsets of inverse limits with set-valued functions as inverse limits with mappings. Of course, this result encompasses proper subcontinua as well as the entire inverse limit. Recall that we have adopted the convention that $G'(f_1, f_2, \ldots, f_{i-1}) = X_1$ for $i = 1$.

Theorem 4.1. *Suppose $\boldsymbol{X}$ is a sequence of compacta, $f_i : X_{i+1} \to 2^{X_i}$ is a surjective upper semi-continuous function for each positive integer i, and K is a closed subset of $\varprojlim \boldsymbol{f}$. For $i \in \mathbb{N}$ let $A_i = \{1, 2, \ldots, i\}$. Then,*

(1) *if $K_i = \pi_{A_i}(K)$ and $g_i = p_{A_i}|K_{i+1}$ for $i \in \mathbb{N}$, K is homeomorphic to $\varprojlim\{\boldsymbol{K}, \boldsymbol{g}\}$,*
(2) *if K_i is a closed subset (resp., subcontinuum) of $G'(f_1, f_2, \ldots, f_{i-1})$, $g_i = p_{A_i}|K_{i+1}$, and $g_i(K_{i+1}) \subseteq K_i$ for each $i \in \mathbb{N}$, then $\varprojlim\{\boldsymbol{K}, \boldsymbol{g}\}$ is a homeomorphic to a closed subset (resp., subcontinuum) of $\varprojlim \boldsymbol{f}$.*

Proof. For (1), note that g_i is a mapping of K_{i+1} onto K_i. Let $N = \varprojlim \boldsymbol{g}$. Then, $h : K \twoheadrightarrow N$ given by $h(\boldsymbol{x}) = (x_1, (x_1, x_2), (x_1, x_2, x_3), \dots)$ is a homeomorphism.

Statement (2) holds because inverse limits on compacta (resp., continua) with mappings are compacta (resp., continua) and φ embeds $\varprojlim \boldsymbol{g}$ in $\varprojlim \boldsymbol{f}$ where φ is given by $\varphi\big((x_1,(x_1,x_2,),(x_1,x_2,x_3),\dots\big) = (x_1, x_2, x_3, \dots)$. □

Corollary 4.2. *Suppose $\boldsymbol{X}$ is a sequence of compacta and $f_i : X_{i+1} \to 2^{X_i}$ is a surjective upper semi-continuous function for each positive integer i. Then, $\varprojlim \boldsymbol{f}$ is homeomorphic to an inverse limit on the sequence of spaces $X_1, G'(f_1), G'(f_1, f_2), G'(f_1, f_2, f_3), \dots$ with bonding functions that are mappings.*

Next we state another theorem due to Marsh [8, Corollary 2.3], although his statement of the theorem is slightly different and contains more information in its conclusion. In Theorem 4.3 one needs a little more hypothesis than is needed in Theorem 4.1, but one gains substantial information about the bonding mappings that can be of particular value in proving fixed point theorems.

Theorem 4.3. (Marsh) *Suppose $\boldsymbol{X}$ is a sequence of compacta and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function for each i. Suppose further there is a positive integer k such that, for $i \geq k$, $G(f_i)$ contains the graph of an upper semi-continuous function g_i such that g_i^{-1} is a mapping of X_i into X_{i+1}. Then, $\varprojlim \boldsymbol{f}$ is homeomorphic to an inverse limit on copies of the spaces G'_n that lie in $\varprojlim \boldsymbol{f}$ with bonding mappings that are retractions.*

The topic of representing inverse limits with set-valued functions as inverse limits with mappings has been considered before. For instance, because inverse limits can be viewed as intersections of closed sets, they admit a natural representation as an inverse limit, [6, Theorem 171, p. 122]. Corollary 4.2 is essentially a restatement of this theorem that usefully reduces the dimension of the factor spaces (in the case that each X_n is finite dimensional).

5. Closures of topological rays

In the theory of inverse limits with mappings, finding dense topological rays has played a rather significant role in determining the nature of the inverse limit. This led the author to ask when inverse limits with upper semi-continuous bonding functions that are not mappings have dense rays, see [2, Question 6.43]: *What are sufficient conditions on a single bonding function on $[0,1]$ so that the inverse limit is the closure of a toplogical ray?* In this section we provide a partial answer to this question.

Theorem 5.1. *Suppose $f : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function such that $f(0) = 0$, a and c are numbers such that $0 < a < c < 1$, $h : [0,c] \twoheadrightarrow [0,1]$ is a homeomorphism of $[0,c]$ onto $[0,1]$ with no fixed point in $[a,c]$ such that $h = f|[0,c]$, and $f([a,1]) \subseteq [a,1]$. Let $g = f|[a,1]$, $K = \varprojlim \boldsymbol{g}$, and $M = \varprojlim \boldsymbol{f}$. If G'_n is an arc for each* *positive integer n, then M is a chainable continuum that is the closure of a topological ray R with remainder K.*

Proof. Because each G'_n is an arc, it follows from Corollary 4.2 that M is chainable. For each $n \in \mathbb{N}$, using Theorem 3.8, let φ_n be the Marsh homeomorphism from G'_n into M given by $\varphi_n(\boldsymbol{x}) = (x_1, x_2, \dots, x_{n+1}, h^{-1}(x_{n+1}), h^{-2}(x_{n+1}), \dots)$ and let $\alpha_n = \varphi_n(G'_n)$. Because each G'_n is an arc, α_n is an arc. Furthermore, $\alpha_n \subseteq \alpha_{n+1}$. Let $R = \bigcup_{i>0} \alpha_i$. Then, R is a topological ray. Suppose $\boldsymbol{x} \in K$. Let $\boldsymbol{y_i} = (x_1, x_2, \dots, x_{i+1},\ h^{-1}(x_{i+1}), h^{-2}(x_{i+1}), \dots)$. Note that $\boldsymbol{y_i} \notin K$ because $h^{-1}(x_{i+1}), h^{-2}(x_{i+1}), h^{-3}(x_{i+1}), \dots$ converges to a fixed point of h, a point not in $[a, 1]$. Then, $\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3}, \dots$ is a sequence of points of R that converges to $\boldsymbol{x}$. □

The condition in Theorem 5.1 that G'_n be an arc can be difficult to determine by looking at the graph of the bonding function. This leads to the following question.

Question 5.2. *What are sufficient conditions on a single set-valued bonding function on $[0, 1]$ so that G'_n is an arc for each $n \in \mathbb{N}$?*

The reader interested in this question should note that in Example 7.1 below, by letting the parameter $a = 1/2^n$, one obtains a function for which G'_{n+2} is not an arc but G'_i is an arc for each integer $i, 1 \le i \le n + 1$.

Nall has shown that the arc is the only finite graph that is an inverse limit on $[0, 1]$ with a single upper semi-continuous function. Results in this section indicate that the following question could be of interest.

Question 5.3. *Is there an upper semi-continuous function $f : [0, 1] \to 2^{[0,1]}$ such that $\varprojlim \boldsymbol{f}$ is the closure of a topological ray with remainder a simple triod?*

6. Indecomposability and the full projection property

In this section we prove a slightly more general version of Theorem 4.3 of [1] (see Theorem 6.2 below). Also, Lemma 6.1 below is a stronger version of Lemma 4.2 from that paper. In the following version of the two-pass condition, we define it for a sequence of upper semi-continuous functions intead of a single function and we omit the requirement that the mutually exlusive connected sets U and V be open.

Suppose $\boldsymbol{X}$ is a sequence of continua and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function for each positive integer i. The sequence $\boldsymbol{f}$ is said to satisfy the *two-pass condition* provided if n is a positive integer then there are an integer $m > n$ and two mutually exclusive connected subsets U and V of X_m such that (1) $f_{i\,m}|U$ and $f_{i\,m}|V$ are mappings for each integer $i, n \le i \le m$, and (2) $\mathrm{Cl}(f_{n\,m}(U)) = \mathrm{Cl}(f_{n\,m}(V)) = X_n$. For constant sequences of bonding functions (i.e., $X_i = X$ and $f_i = f : X \to 2^X$ for each $i \in \mathbb{N}$), the two-pass condition reduces to the following: there are a positive integer m and two mutually exclusive connected subsets U and V of X such that $f^i|U$ and $f^i|V$ are mappings for $1 \le i \le m$ and $\mathrm{Cl}(f^m(U)) = \mathrm{Cl}(f^m(V)) = X$.

The following lemma slightly strengthens the statement of Lemma 4.2 of [1] by relaxing the requirement that the connected sets U and V be open. The hypothesis that U and V be open was used in the proof given in [1] but it was not necessary, so almost no changes to that proof are required. The proof is short and we include it for the convenience of the reader.

Lemma 6.1. *Suppose T is an arc or a simple n-od for some positive integer n and H and K are two proper subcontinua of T whose union is T. If U and V are two mutually exclusive connected subsets of T then one of U and V is a subset of one of H and K.*

Proof. If T is an arc, let J denote a separating point of T; if T is an n-od, let J denote its junction point. The point J cannot belong to both U and V; suppose $J \notin U$. Denote by A the end point of T such that $U \subseteq [J, A]$. Assume $A \in H$. If $J \in H$ then $U \subseteq H$. If $J \notin H$ and U is not a subset of either H or K, then $H \cap K \subseteq U$. If $A \in U$, then $T - U \subseteq K$ so $V \subseteq K$. If $A \notin U$, then $T - U$ is the union of two mutually separated sets C and D with $A \in C$. Then, $V \subseteq C$ or $V \subseteq D$. Because $C \subseteq H$ and $D \subseteq K$, we have $V \subseteq H$ or $V \subseteq K$. □

If $\{\boldsymbol{X}, \boldsymbol{f}\}$ is an inverse limit sequence and $M = \varprojlim \boldsymbol{f}$, we say that $\{\boldsymbol{X}, \boldsymbol{f}\}$ has the *full projection property* provided that if H is a subcontinuum of M such that $\pi_i(H) = X_i$ for infinitely many integers i then $H = M$ (Example 3.1 fails to have the full projection property). Only minor changes need to be made in the proof of Theorem 4.3 of [1] to show that the following theorem holds. Again its proof is relatively short but we include it for the convenience of the reader.

Theorem 6.2. *Suppose $T_1, T_2, T_3, \ldots$ is a sequence such that if i is a positive integer, then T_i is an arc or a simple n_i-od for some positive integer n_i and, for each positive integer j, $f_j : T_{j+1} \to 2^{T_j}$ is an upper semi-continuous function. If the sequence $\boldsymbol{f}$ satisfies the two-pass condition and $\{\boldsymbol{T}, \boldsymbol{f}\}$ has the full projection property then $\varprojlim \boldsymbol{f}$ is indecomposable.*

Proof. Let $M = \varprojlim \boldsymbol{f}$ and suppose M is the union of two proper subcontinua H and K. Because $\{\boldsymbol{T}, \boldsymbol{f}\}$ has the full projection property, there is a positive integer n such that if $j \geq n$ then $\pi_j(H) \neq T_j$ and $\pi_j(K) \neq T_j$. Because the sequence $\boldsymbol{f}$ satisfies the two-pass condition, there are an integer $m > n$ and two mutually exclusive connected subsets U and V of T_m such that $f_{i\,m}|U$ and $f_{i\,m}|V$ are mappings for each $i, n \leq i \leq m$, and $\mathrm{Cl}(f_{n\,m}(U)) = T_n$ and $\mathrm{Cl}(f_{n\,m}(V)) = T_n$. Because $T_m = \pi_m(H) \cup \pi_m(K)$, by Lemma 6.1, one of U and V is a subset of one of $\pi_m(H)$ and $\pi_m(K)$. Suppose $U \subseteq \pi_m(H)$. We show that $f_{n\,m}(U) \subseteq \pi_n(H)$. To that end, if t is a point of $f_{n\,m}(U)$ there is a point $s \in U$ such that $f_{n\,m}(s) = t$. Because $s \in \pi_m(H)$, there is a point $\boldsymbol{x} \in H$ such that $x_m = s$. Then, because $f_{i\,m}|U$ is a mapping for $n \leq i \leq m$, $x_n = t$ so $t \in \pi_n(H)$. However, $\mathrm{Cl}(f_{n\,m}(U)) = T_n$ contradicting that $\pi_n(H) \neq T_n$. □

Other studies of indecomposability of inverse limits with set-valued functions include [7], [10], [11], and [12]. As was the case in [1], none of these address detecting indecomposability of the inverse limit by looking at compositions of the bonding functions as in Theorem 6.2.

7. EXAMPLES

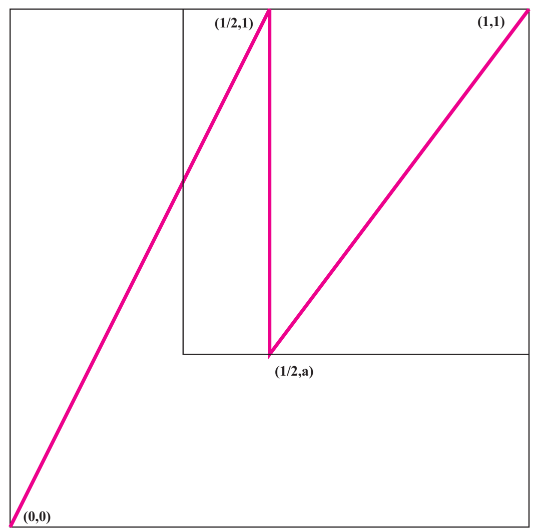

FIGURE 2. The graph of a typical bonding function in Example 7.1

In this section we consider examples of inverse limits with a single bonding function chosen from a one-parameter family of set-valued functions. Specifically, for $a \in [0,1]$, let $f_a : [0,1] \to C([0,1])$ be the upper semi-continuous set-valued function whose graph consists of three straight line intervals, one from $(0,0)$ to $(1/2,1)$, one from $(1/2,1)$ to $(1/2,a)$, and one from $(1/2,a)$ to $(1,1)$. For each $a \in [0,1]$ the function f_a has a graph without flat spots, thus its inverse limit is a tree-like continuum [4, Corollary 4.1]. Here we determine the parameter values for which the inverse limit is chainable, i.e., homeomorphic to an inverse limit on arcs with bonding functions that are mappings. It is known that for $a = 1/2$ the inverse limit is not chainable because it contains a triod [2, Example 3.11] and for $a = 0$ the inverse limit is indecomposable [2, Example 3.9] and chainable [3, Example 5.1]. For $a = 1$, f_a is a mapping and the inverse limit is an arc [6, Example 11].

Example 7.1. *Suppose a is a number, $0 \leq a < 1$. Let f_a be the upper semi-continuous set-valued function given by $f_a(t) = 2t$ for $0 \leq t < 1/2$, $f_a(1/2) = [a,1]$, and $f_a(t) = 2(1-a)(x-1)+1$ for $1/2 < t \leq 1$. Then, $\varprojlim \boldsymbol{f_a}$ is chainable if and only if $f_a^n(a) \neq 1/2$ for each $n \in \mathbb{N}$. Moreover, if $f_a^n(a) \neq 1/2$ for each $n \in \mathbb{N}$, $\varprojlim \boldsymbol{f_a}$ is the closure of a topological ray with remainder $\varprojlim \boldsymbol{g_a}$ where $g_a = f|[a,1]$. If $1/2 < a \leq 1$, $\varprojlim \boldsymbol{f}$ is an arc.*

Proof. Note that $G(f_a^{-1})$ is the union of three mappings φ_1, φ_2, and φ_3 where
$\varphi_1 : [0,1] \to [0,1]$ is given by $\varphi_1(x) = x/2$,
$\varphi_2 : [a,1] \to [0,1]$ is given by $\varphi_2(x) = 1/2$,
$\varphi_3 : [a,1] \to [0,1]$ is given by $\varphi_3(x) = (x-1)/(2(1-a)) + 1$.

Suppose $f_a^n(a) \neq 1/2$ for each $n \in \mathbb{N}$. We show that G'_n is an arc for each n. From this it follows by Theorem 4.3 that $\varprojlim \boldsymbol{f_a}$ is chainable and by Theorem 5.1 that $\varprojlim \boldsymbol{f_a}$ is the closure of a topological ray with remainder $\varprojlim \boldsymbol{g_a}$.

To see that G'_n is an arc for each $n \in \mathbb{N}$, we proceed inductively. Note that $G'_1 = G(f_a^{-1})$ is an arc containing $(1,1)$. Assume k is an integer such that G'_k is an arc containing $(1,1,\ldots,1)$. By Lemma 2.4, only one point of G'_k has last coordinate a because $f_a^i(a)$ is a single point for $1 \leq i \leq k$. It follows that $A = G'_k \cap \big([0,1]^k \times [a,1]\big)$ is an arc. Define homeomorphisms $\Phi_1 : G'_k \to [0,1]^{k+2}, \Phi_2 : A \to [0,1]^{k+2}$, and $\Phi_3 : A \to [0,1]^{k+2}$ by $\Phi_i(\boldsymbol{x}) = (x_1, x_2, \ldots, x_{k+1}, \phi_i(x_{k+1}))$ for $i = 1,2,3$. Let A_1 be the arc $\Phi_1(G'_k)$ and A_i be the arc $\Phi_i(A)$ for $i = 2,3$; $G'_{k+1} = A_1 \cup A_2 \cup A_3$ and $(1,1,\ldots,1) \in A_3$. Moreover, $A_1 \cap A_2 = \{(1,1,\ldots,1,1/2)\}$, $A_2 \cap A_3 = \{(f_a^k(a),\ldots,a,1/2)\}$, and $A_1 \cap A_3 = \emptyset$. It follows that G'_{k+1} is an arc containing $(1,1,\ldots,1)$ and the induction is complete.

To see that $\varprojlim \boldsymbol{f_a}$ is an arc for $1/2 < a \leq 1$, observe that in this parameter range $f_a^n(a) \neq 1/2$ for $n \in \mathbb{N}$. Thus, by Theorem 5.1 the inverse limit is the closure of a ray with remainder $\varprojlim \boldsymbol{g_a}$. But, $\varprojlim \boldsymbol{g_a}$ is $(1,1,\ldots,1)$ so $\varprojlim \boldsymbol{f_a}$ is an arc.

On the other hand, suppose n is a positive integer such that $f_a^n(a) = 1/2$ but, in case $n > 1$, $f_a^i(a) \neq 1/2$ for $i < n$. Note that $a \leq 1/2$ because, if $a > 1/2$ and n is a positive integer, $f_a^n(t) > t$ for $1/2 \leq t < 1$. We show that $\varprojlim \boldsymbol{f_a}$ contains a triod. We may assume that $a < 1/2$ because it is known that $\varprojlim \boldsymbol{f_a}$ contains a triod for $a = 1/2$ [2, Example 3.11]. Because $G(f_a)$ contains the inverse of the mapping φ_1, by Theorem 3.8, $\varprojlim \boldsymbol{f_a}$ contains a copy of G'_{n+2}. Hence, in order to show that the inverse limit contains a triod, it is sufficient to show that G'_{n+2} contains a triod. For $t \in [0,1]$, $f_a^{-1}(t)$ contains only one, two, or three points. Therefore, $f_a^{-1}(1/2) \cup f_a^{-2}(1/2) \cup \cdots \cup f_a^{-k}(1/2)$ is finite for each positive integer k. Because $a \in f_a^{-n}(1/2)$, there is a point b, $a < b < 1/2$, such that $(a,b] \cap \big(f_a^{-1}(1/2) \cup f_a^{-2}(1/2) \cup \cdots \cup f_a^{-(n+1)}(1/2)\big) = \emptyset$. Thus, $f_a^i|(a,b]$ is a mapping for $1 \leq i \leq n+1$. For $t \in (a,b]$, $1/2 < f_a^n(t)$ because $f_a^i|(a,b]$ is order preserving for each $i, 1 \leq i \leq n$. Let

$\alpha = \{\boldsymbol{x} \in G'_{n+2} \mid \boldsymbol{x} = (t, 1/2, f_a^{n-1}(a), \ldots, a, 1/2) \text{ for } a \leq t \leq 1\}$,

$\beta = \mathrm{Cl}\big(\{\boldsymbol{x} \in G'_{n+2} \mid \boldsymbol{x} = (f_a^{n+1}(t), \ldots, f_a(t), t, 1/2) \text{ for } a < t < b\}\big)$,

$\gamma = \mathrm{Cl}\big(\{\boldsymbol{x} \in G'_{n+2} \mid \boldsymbol{x} = (f_a^{n+1}(t), \ldots, f_a(t), t, \varphi_3(t)) \text{ for } a < t < b\}\big)$.

Each of α, β, and γ is an arc. Because $\lim_{t\to a^+} f_a^n(t) = 1/2$ and $f_a^n(t) > 1/2$ for $t \in (a,b]$, we have $\lim_{t\to a^+} f_a^{n+1}(t) = a$. It follows that $\boldsymbol{p} = (a, 1/2, f_a^{n-1}(a), \ldots, a, 1/2) \in \alpha \cap \beta \cap \gamma$. In fact, $\alpha \cap \beta \cap \gamma = \{\boldsymbol{p}\}$ because $\boldsymbol{p}$ is the only point of either β or γ with next-to-last coordinate a and it is the only point of γ with last coordinate $1/2$. The point $(1, 1/2, f_a^{n-1}(a), \ldots, a,\ 1/2) \in \alpha - (\beta \cup \gamma)$; $(f_a^n(b), \ldots,\ f_a(b), b, 1/2) \in \beta - (\alpha \cup \gamma)$; $(f_a^n(b), \ldots, f_a(b),\ b, \varphi_3(b)) \in \gamma - (\alpha \cup \beta)$. It follows that $\alpha \cup \beta \cup \gamma$ is a simple triod lying in G'_{n+2}. $\square$

Remark 7.2. *In Example 7.1, for $a \neq 1$, each bonding function has a graph consisting of one vertical line and two straight line intervals. Because of the linearity, in order for a to satisfy the condition $f_a^n(a) = 1/2$, the parameter satisfies a polynomial equation with rational coefficients. Thus, the chainability of the inverse limit occurs for uncountably many values for a, $0 < a < 1/2$.*

For $1/2 < a < 1$, $\varprojlim \boldsymbol{f_a}$ is an arc, but we do not address the question whether, for two different parameter values in $[0, 1/2]$, the inverse limits are topologically different. However, the following seems to be an interesting problem.

Question 7.3. *Suppose a is a number, $0 \leq a < 1$. Let f_a be the upper semi-continuous set-valued function given by $f_a(t) = 2t$ for $0 \leq t < 1/2$, $f_a(1/2) = [a, 1]$, and $f_a(t) = 2(1-a)(x-1)+1$ for $1/2 < t \leq 1$. If $0 \leq b < c \leq 1/2$, are $\varprojlim \boldsymbol{f_b}$ and $\varprojlim \boldsymbol{f_c}$ topologically different?*

8. Cores

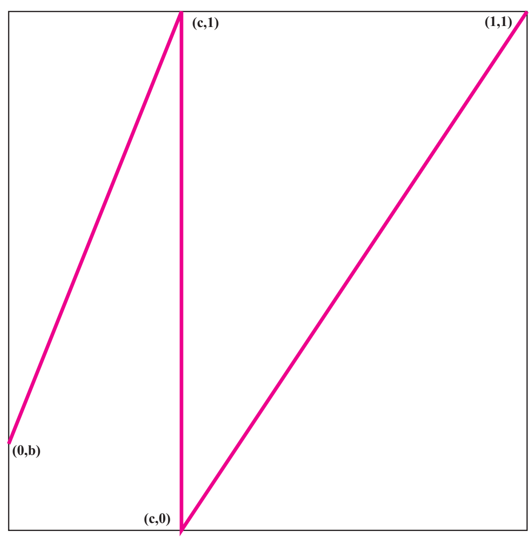

Figure 3. The graph of a typical bonding function in Example 8.2

In this section and the next we address a natural question: in Example 7.1, what is the nature of $\varprojlim \boldsymbol{g_a}$ where $g_a = f_a|[a, 1]$? Because of the similarity to the case with inverse limits of mappings, we refer to g_a as the *core* of f_a and $\varprojlim \boldsymbol{g_a}$ as the *core* of $\varprojlim \boldsymbol{f_a}$. In our next example we consider a two-parameter family of interval-valued upper semi-continuous functions on $[0, 1]$. Each core of a bonding function from Example 7.1 produces a continuum homeomorphic to an element of a member of this family. Thus, by what is shown in Example 8.2, if a is a parameter value, $0 < a < 1/2$, such that $f_a^n(a) \neq 1/2$ for each $n \in \mathbb{N}$ then the core of $\varprojlim \boldsymbol{f_a}$ is an indecomposable chainable continuum.

In [7], Kelly and Meddaugh prove the following theorem; we make use of it in our next example.

Theorem 8.1. (Kelly and Meddaugh) *Suppose $\boldsymbol{X}$ is a sequence of continua, $f_i : X_{i+1} \to 2^{X_i}$ is upper semi-continuous for each $i \in \mathbb{N}$, and $\varprojlim \boldsymbol{f}$ is a continuum. If, for each $n \in \mathbb{N}$, there exist points a and b in X_{n+1} such that $G'(f_1, f_2, \ldots, f_n)$ is irreducible between the sets $\{\boldsymbol{x} \in G'(f_1, f_2, \ldots, f_n) \mid x_{n+1} = a\}$ and $\{\boldsymbol{x} \in G'(f_1, f_2, \ldots, f_n) \mid x_{n+1} = b\}$, then $\{\boldsymbol{X}, \boldsymbol{f}\}$ has the full projection property.*

Example 8.2. *Let b and c be numbers, $0 \leq b < 1$ and $0 < c < 1$. Let $g_{b,c} : [0,1] \to C([0,1])$ be the upper semi-continuous function whose graph consists of three straight line intervals, one from $(0,b)$ to $(c,1)$, one from $(c,1)$ to $(c,0)$, and one from $(c,0)$ to $(1,1)$. Then, if $g_{b,c}^n(0) \neq c$ for each $n \in \mathbb{N}$, $\varprojlim \boldsymbol{g_{b,c}}$ is an indecomposable chainable continuum.*

Proof. We show that, for each positive integer n, G'_n is an arc that is irreducible from $\{\boldsymbol{x} \in G'_n \mid x_{n+1} = 0\}$ to $\{\boldsymbol{x} \in G'_n \mid x_{n+1} = 1\}$. We proceed inductively. Note that G'_1 is an arc irreducible from $\{(x,y) \in [0,1]^2 \mid y = 0\}$ to $\{(x,y) \in [0,1]^2 \mid y = 1\}$. In fact, the graph of f^{-1} is the union of three mappings: two homeomorphisms $\varphi_1 : [b,1] \to [0,c]$ and $\varphi_3 : [0,1] \twoheadrightarrow [c,1]$ and a constant mapping $\varphi_2 : [0,1] \twoheadrightarrow \{c\}$. Suppose k is a positive integer such that G'_k is an arc that is irreducible from $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = 0\}$ to $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = 1\}$. Because $g_{b,c}(1) = 1$ and $g_{b,c}^n(0) \neq c$ for each $n \in \mathbb{N}$, it follows that $(g_{b,c}^{k-1}(b), \ldots, g_{b,c}(b), b, 0)$ and $(1, 1, \ldots, 1)$ are the only points of G'_k having last coordinate 0 and 1, respectively, so they are the endpoints of G'_k. Let $\alpha = \{\boldsymbol{x} \in G'_k \mid x_{k+1} \geq b\}$; α is connected by Theorem 3.6 so α is an arc. Then, G'_{k+1} is the union of three arcs, $A_1 = \{\boldsymbol{x} \in G'_{k+1} \mid (x_1, \ldots, x_{k+1}) \in \alpha$ and $x_{k+2} = \varphi_1(x_{k+1})\}$ and $A_j = \{\boldsymbol{x} \in G_{k+1} \mid (x_1, \ldots, x_{k+1}) \in G'_k$ and $x_{k+2} = \varphi_j(x_{k+1})\}$ for $j \in \{2,3\}$. Because $A_1 \cap A_2 = \{(1, 1, \ldots, 1, 1/2)\}$ and $A_2 \cap A_3 = \{(g_{b,c}^{k-1}(b), \ldots, g_{b,c}(b), b, 0, 1/2)\}$ while $A_1 \cap A_3 = \emptyset$, it follows that G'_{k+1} is an arc with endpoints $\boldsymbol{p_0} = (g_{b,c}^k(b), \ldots, g_{b,c}(b), b, 0)$ and $\boldsymbol{p_1} = (1, 1, \ldots, 1)$. Because $\boldsymbol{p_0}$ and $\boldsymbol{p_1}$ are the only points of G'_{k+1} having last coordinate 0 and 1, respectively, G'_{k+1} is irreducible from $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 0\}$ to $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 1\}$.

Let $Y_n = G'_n$ for each $n \in \mathbb{N}$. Because Y_n is an arc for each $n \in \mathbb{N}$ that is irreducible from $\{\boldsymbol{x} \in G'_n \mid x_{n+1} = 0\}$ to $\{\boldsymbol{x} \in G'_n \mid x_{n+1} = 1\}$, it follows by Theorem 8.1 that $\{\boldsymbol{Y}, \boldsymbol{g_{b,c}}\}$ has the full projection property. Because $G(f)$ contains the graph of the inverse of the mapping φ_3, M is chainable by Theorem 4.3.

To see that M is indecomposable, we use Theorem 6.2 by showing that the sequence $f_1, f_2, f_3, \ldots$ satisfies the two-pass condition where $f_i = g_{b,c}$ for each $i \in \mathbb{N}$. There are a point $p \geq \max\{b, c\}$ and a positive integer k such that $g_{b,c}^k(p) = c$ and $g_{b,c}^i(p) > c$ for $1 \leq i < k$. Let $U = (g_{b,c}|[0,c))^{-1}((p,1))$ and $V = (g_{b,c}|(c,1))^{-1}((p,1))$. Then, U and V are mutually exclusive connected subsets of $[0,1]$ such that $\mathrm{Cl}(f^{k+2}(U)) = [0,1]$ and $\mathrm{Cl}(f^{k+2}(V)) = [0,1]$. So, M is indecomposable. $\square$

Question 8.3. *Suppose c is a fixed parameter value, $0 < c < 1$. If $0 \leq a < b < 1$, are $\varprojlim \boldsymbol{g_{a,c}}$ and $\varprojlim \boldsymbol{g_{b,c}}$ topologically different?*

9. Nonchainable examples

In this section we complete a study of the class of examples from Example 8.2 by showing that the continua obtained using a single bonding function from the family in Example 8.2 having the property that $g^n_{b,c}(b) = c$ for some positive integer n are decomposable tree-like continua that are not chainable. For reasons similar to those showing that only countably many of the examples from Example 7.1 are not chainable, for a fixed value of c, there are only countably many values of b such that $\varprojlim \boldsymbol{g_{b,c}}$ is not chainable.

Example 9.1. *Suppose $0 < c < 1$ and $0 < b < 1$. Let $g_{b,c} : [0,1] \to C([0,1])$ be the upper semi-continuous function whose graph consists of three straight line intervals, one from $(0,b)$ to $(c,1)$ along with one from $(c,1)$ to $(c,0)$ and one from $(c,0)$ to $(1,1)$. If $g^n_{b,c}(0) = c$ for some positive integer n, then $\varprojlim \boldsymbol{g_{b,c}}$ is a decomposable tree-like continuum that is not chainable.*

Proof. For simplicity of notation, let $f = g_{b,c}$ and denote by M the inverse limit, $\varprojlim \boldsymbol{f}$. That M is tree-like is a consequence of the fact that f is an upper semi-continuous function on $[0,1]$ without flat spots [4, Corollary 4.1].

We show that M is not chainable by showing that M contains a triod. This part of the proof is similar to the construction of a triod given in the proof for Example 7.1. There is a positive number d such that $f^j((0,d])$ does not contain c for $1 \leq j \leq n$ because $f^{-1}(c) \cup f^{-2}(c) \cup \cdots \cup f^{-n}(c)$ is a finite set containing 0. Let $A = \{\boldsymbol{x} \in M \mid x_{n+2} = 0$ and $x_{n+k} = c$ for $k \geq 3\}$; $B = \mathrm{Cl}(\{\boldsymbol{x} \in M \mid x_{n+2} \in (0,d]$ and $x_{n+k} = c$ for $k \geq 3\})$; $C = \mathrm{Cl}(\{\boldsymbol{x} \in M \mid x_{n+2} \in (0,d], x_{n+3} = (1-c)(x_{n+2}-1)+1$ and $x_{n+k} = c$ for $k \geq 4\})$. By the choice of d, $\boldsymbol{z} = (0, c, f^{n-1}(0), \dots, b, 0, c, c, c, \dots)$ is the only point of either B or C having 0 as its (n+2)nd coordinate. Each of A, B, and C is an arc and $A \cap B \cap C = A \cap B = A \cap C = B \cap C = \{\boldsymbol{z}\}$. Let $T = A \cup B \cup C$. That T is a triod follows from the fact that $(1, c, \dots, b, 0, c, c, c) \in T - (B \cup C)$, $(f^{n+1}(d), \dots, d, c, c, c) \in T - (A \cup C)$, and $(f^{n+1}(d), \dots, d, (1-c)(d-1)+1, c, c, c) \in T - (A \cup B)$.

To see that M is decomposable, we show that M contains a proper subcontinuum with interior. Let $K = \mathrm{Cl}(\{\boldsymbol{x} \in G'_{n+1} \mid x_{n+2} > 0\})$. Note that K is a continuum because $\{\boldsymbol{x} \in G'_{n+1} \mid x_{n+2} > 0\}$ is connected by Theorem 3.6. Further, $(0, c, f^{n-1}(0), \dots, b, 0)$ is the only point of K having last coordinate 0, and, because $(1, 1, \dots, 1) \in K$, $p_{n+2}(K) = [0,1]$. Therefore, because K is a closed subset of G'_{n+1}, by Lemma 3.3 there is an upper semi-continuous function $F : X_{n+2} \to 2^{G'_n}$ such that $G'_{n+1} = G(F^{-1})$. Because $f : [0,1] \to C([0,1])$, by Theorem 3.5, the Nall function $F_{n+1} : X_{n+2} \to C(G'_n)$. However, because $F(t) = F_{n+1}(t)$ for $t > 0$ and $F(0)$ is a single point, $F : X_{n+2} \to C(G'_n)$. Let $X_1 = G'_n$ and $X_i = [0,1]$ for $i > 1$; let $\varphi_1 = F$ and $\varphi_i = f$ for $i > 1$. Then, $\{\boldsymbol{X}, \boldsymbol{\varphi}\}$ is an inverse sequence on continua with continuum-valued functions so $\varprojlim \boldsymbol{\varphi}$ is a continuum that is homeomorphic to $H = \mathrm{Cl}(\{\boldsymbol{x} \in M \mid x_{n+2} > 0\})$, a subcontinuum of M containing the open set $M \cap \big([0,1]^{n+1} \times (0,1] \times \mathcal{Q}\big)$. The point $(1, c, f^{n-1}(0), \dots, b, 0, c, c, c, \dots)$ is a point of $M - H$ so H is a proper subcontinuum of M with interior. $\square$

It is interesting to observe that the proper subcontinuum H constructed in Example 9.1 has the property that $\pi_n(H) = [0,1]$ for each $n \in \mathbb{N}$ so the inverse limit sequence fails to have the full projection property.

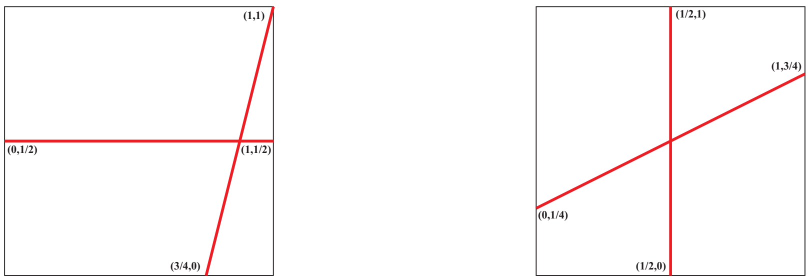

FIGURE 4. The graphs of the bonding functions f_1 (left) and f_2 (right) in Example 10.1

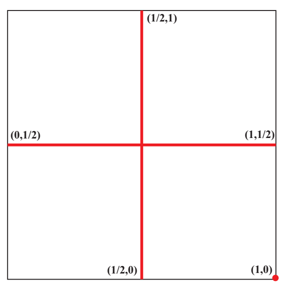

FIGURE 5. The graph of $f_1 \circ f_2$ in Example 10.1

10. CORRIGENDUM

There is an error in the books [2] and [6]. The statements of Theorem 126 on page 90 of [6] and Theorem 2.8 on page 18 of [2] are incorrect. In private correspondence with the author Mark Marsh provided the following example that illustrates the error.

Example 10.1. (Marsh) *Let $f_1 : [0,1] \to 2^{[0,1]}$ be given by $f_1(t) = 1/2$ for $0 \le t < 3/4$ and $f_1(t) = \{1/2, 4t-3\}$ for $3/4 \le t \le 1$. Note that $f_1^{-1} : [0,1] \to C([0,1])$. Let $f_2 : [0,1] \to C([0,1]$ be given by $f_2(t) = 1/2t + 1/4$ for $t \ne 1/2$ and $f_2(1/2) = [0,1]$. For $n > 2$, let f_n be the identity on $[0,1]$. Then $(1,0)$ is an isolated point for $f_1 \circ f_2$ and, thus, $\varprojlim \boldsymbol{f}$ is not connected.*

The following is a corrected statement of Theorem 126 in [6].

Theorem. *Suppose* $\{X_i, f_i\}$ *is an inverse limit sequence on Hausdorff continua with upper semi-continuous bonding functions such that* f_i *is Hausdorff continuum-valued for each* $i \in \mathbb{N}$ (*or* $f_i(X_{i+1})$ *is connected with* $f_i^{-1} : f_i(X_i) \to X_{i+1}$ *Hausdorff continuum-valued*) for each $i \in \mathbb{N}$ *then* $\varprojlim \boldsymbol{f}$ *is a Hausdorff continuum.*

The following is a corrected statement of Theorem 2.8 in [2].

Theorem. *Suppose* $\boldsymbol{X}$ *is a sequence of subintervals of* $[0, 1]$ *and* $\boldsymbol{f}$ *is a sequence of upper semi-continuous functions such that* $f_i : X_{i+1} \to 2^{X_i}$ *for each positive integer* i*. Suppose further that* f_i *has connected values for each* $i \in \mathbb{N}$ (*or for each* $i \in \mathbb{N}$, $f_i(X_{i+1})$ *is connected and* $f_i^{-1}(x)$ *is an interval for each* $x \in f_i(X_{i+1})$)*. Then,* $\varprojlim \boldsymbol{f}$ *is a continuum.*

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