EXERCISES

Assume all spaces are compact metric spaces (*compacta*). Denote the closed subsets of a space X by 2^X and the connected elements of 2^X by $C(X)$. A function $f : X \to 2^Y$ is said to be *upper semi-continuous* at the point x of X provided if V is an open set in Y containing $f(x)$ then there is an open set U in X containing x such that if $t \in U$ then $f(t) \subseteq V$; f is *upper semi-continuous* provided it is upper semi-continuous at each point of X. If $f : X \to 2^Y$ is an upper semi-continuous function, let $G(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$; $G(f)$ is called the *graph* of f. If $f : X \to 2^Y$ is upper semi-continuous and $H \subseteq X$, then $f(H)$ denotes $\{y \in Y \mid$ there is a point $x \in X$ such that $y \in f(x)\}$; we say that f is *surjective* if $f(X) = Y$. If $f : X \to 2^Y$ and $g : Y \to 2^Z$ are upper semi-continuous, by $g \circ f$ we mean the function from X into 2^Z given by $g \circ f(x) = z$ provided there is a point $y \in Y$ such that $y \in f(x)$ and $z \in g(y)$.

If $\boldsymbol{X} = X_1, X_2, X_3, \ldots$ is a sequence of compacta and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function for each positive integer i, by the *inverse limit* of the pair $\{\boldsymbol{X}, \boldsymbol{f}\}$ of sequences is meant $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for each positive integer $i\}$; we denote this inverse limit by $\varprojlim \boldsymbol{f}$. If A is a set of positive integers, we denote by p_A the projection of $\prod_{i>0} X_i$ into $\prod_{i\in A} X_i$ where $p_A(\boldsymbol{x}) = \boldsymbol{y}$ if and only if $y_i = x_i$ for each $i \in A$; if $A \subseteq \{1, 2, \ldots, n\}$, we also use p_A to denote the projection of $\prod_{i=1}^{n} X_i$ onto $\prod_A X_i$. If $A = \{n\}$, we denote p_A by p_n. We use π_A to denote $p_A | \varprojlim \boldsymbol{f}$.

The first few exercises should be familiar to anyone who has looked at inverse limits with set-valued functions in the past. If you are studying inverse limits with set-valued functions for the first time, you should verify some of this background material most of which can be found in the book on inverse limits with Mahavier or my Springer Brief. Later exercises contain material not found in those sources.

Exercise 1. *Suppose X and Y are compacta and $M \subseteq X \times Y$ such that $p_1(M) = X$. Then, M is closed if and only if there is an upper semi-continuous function $f : X \to 2^Y$ such that $M = G(f)$.*

Exercise 2. *Suppose X and Y are continua and $f : X \to C(Y)$ is upper semi-continuous. Then, $G(f)$ is a continuum.*

Suppose $\boldsymbol{X}$ is a sequence of compacta and $f_i : X_{i+1} \to 2^{X_i}$ is upper semi-continuous for each positive integer i. If n is a positive integer, let $G_n = \{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for $1 \leq i \leq n\}$. We denote the similarly defined subset of the finite product $\prod_{i=1}^{n+1}$ by G'_n or often by $G'(f_1, f_2, \ldots, f_n)$.

Exercise 3. *Suppose $\boldsymbol{X}$ is a sequence of compacta, $f_i : X_{i+1} \to 2^{X_i}$ is upper semi-continuous for each positive integer i, and n is a positive integer. Then, G_n is closed and $G_{n+1} \subseteq G_n$. Moreover, $\varprojlim \boldsymbol{f} = \bigcap_{i>0} G_i$ so $\varprojlim \boldsymbol{f}$ is a nonempty compactum.*

Exercise 4. *Suppose $\boldsymbol{X}$ is a sequence of continua, $f_i : X_{i+1} \to C(X_i)$ is upper semi-continuous for each positive integer i, and n is a positive integer. Then, G_n is connected. Thus, $\varprojlim \boldsymbol{f}$ is a continuum.*

One of the best ways to begin to understand inverse limits is to study examples.

Exercise 5. *Following this list of exercises is a list of set-valued functions on the interval $[0,1]$. Choose some of those functions and prove that their inverse limits have the stated properties.*

Exercise 6. *Suppose X and Y are continua, $f : X \to C(Y)$ is upper semi-continuous, and H is a connected subset of X. Then, $\{(x,y) \in G(f) \mid x \in H\}$ is connected; thus, $f(H)$ is connected.*

Exercise 7. *If $f : X \to 2^Y$ and $g : Y \to 2^Z$ are upper semi-continuous, then $g \circ f$ is upper semi-continuous.*

Exercise 8. *Suppose $f : X \to C(Y)$ and $g : Y \to C(Z)$ are upper semi-continuous. Then, $g \circ f$ is an upper semi-continuous function from X into $C(Z)$.*

Exercise 9. *Suppose $\boldsymbol{X}$ is a sequence of compacta, n is a positive integer, and H is a closed subset of Π_{n+1} where $\Pi_j = \prod_{i=1}^{j} X_i$. If $p_{n+1}(H) = X_{n+1}$ then there is an upper semi-continuous function $F : X_{n+1} \to 2^{\Pi_n}$ such that H is homeomorphic to $G(F^{-1})$.*

Exercise 10. *Suppose $\boldsymbol{X}$ is a sequence of continua and $f_i : X_{i+1} \to C(X_i)$ is a surjective upper semi-continuous function for each positive integer i. If n is a positive integer and $F : X_{n+1} \to 2^{\Pi_n}$ is the upper semi-continuous function such that $G(F^{-1})$ is homeomorphic to $G'(f_1, f_2, \dots, f_n)$, then F is continuum-valued, i.e., $F : X_{n+1} \to C(G'(f_1, f_2, \dots, f_n))$.*

Exercise 11. *Suppose $\boldsymbol{X}$ is a sequence of continua, $f_i : X_{i+1} \to C(X_i)$ is upper semi-continuous for each positive integer i, H is a closed proper subset of $G'(f_1, f_2, \dots, f_n)$ such that $p_{n+1}(H) = X_{n+1}$, and $F : X_{n+1} \to 2^{\Pi_n}$ is the upper semi-continuous function such that H is homeomorphic to $G(F^{-1})$. Let $Y_1 = G'(f_1, f_2, \dots, f_{n-1})$ and $Y_j = X_{n+j-1}$ for $j > 1$; let $g_1 = F$ and $g_j = f_{n+j-1}$ for $j > 1$. If F is continuum-valued, $\varprojlim \boldsymbol{g}$ is a proper subcontinuum of $\varprojlim \boldsymbol{f}$.*

Exercise 12. *Suppose $\boldsymbol{X}$ is a sequence of compacta and, for each positive integer i, $f_i : X_{i+1} \to 2^{X_i}$ is upper an semi-continuous function whose graph contains the graph of an upper semi-continuous function g_i such that $g_i^{-1} : X_i \to X_{i+1}$ is a mapping. Then, $\varprojlim \boldsymbol{f}$ contains a copy of $G'(f_1, f_2, \dots, f_n)$ for each positive integer n.*

SOME SAMPLE FUNCTIONS

(1) Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = \{0,1\}$ for each $t \in [0,1]$. Then $\varprojlim \boldsymbol{f}$ is a Cantor set.

(2) Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = \{t, 1-t\}$ for each $t \in [0,1]$. Then, $\varprojlim \boldsymbol{f}$ is a cone over the Cantor set.

(3) Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = t$ for $0 \le t < 1$ and $f(1) = \{0,1\}$. Then, $\varprojlim \boldsymbol{f}$ is the union of an arc and a simple covergent sequence.

(4) Let $f : [0,1] \to C([0,1])$ be given by $f(t) = t$ for $0 \le t < 1$ and $f(1) = [0,1]$. Then, $\varprojlim \boldsymbol{f}$ is a fan.

(5) Let $f : [0,1] \to C([0,1])$ be given by $f(t) = 0$ for $0 \le t < 1$ and $f(1) = [0,1]$. Then, $\varprojlim \boldsymbol{f}$ is an arc.

(6) Let $f : [0,1] \to C([0,1])$ be given by $f(0) = [0,1]$ and $f(t) = 0$ for $0 < t \le 1$. Then, $\varprojlim \boldsymbol{f}$ is infinite dimensional.

(7) Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = \{1/2 + t, 1/2 - t\}$ for $0 \le t \le 1/2$ and $f(t) = \{3/2 - t, t - 1/2\}$ for $1/2 < t \le 1$. Then, $\varprojlim \boldsymbol{f}$ contains a simple closed curve.

(8) Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = t/2$ for $0 \le t < 1/2$ and $f(t) = \{t/2, 2t-1\}$ for $1/2 \le t \le 1$. Then, $\varprojlim \boldsymbol{f}$ is not connected even though $G(f)$ is connected.

(9) Let $f : [0,1] \to C([0,1])$ be given by $f(t) = 0$ for $0 \le t < 1/2$, $f(1/2) = [0,1/2]$, $f(t) = 1/2$ for $1/2 < t < 1$ and $f(1) = [1/2,1]$. Then, $\varprojlim \boldsymbol{f}$ contains a 2-cell.