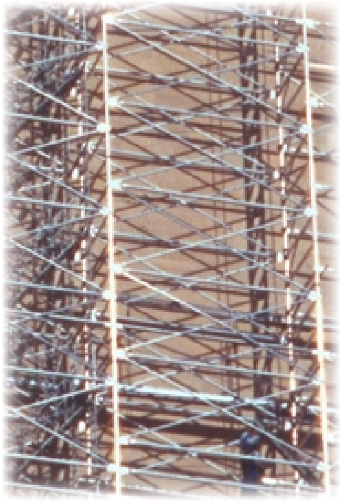

CHAPTER 4

Vector Spaces

4.1 SPACES AND SUBSPACES

After matrix theory became established toward the end of the nineteenth century, it was realized that many mathematical entities that were considered to be quite different from matrices were in fact quite similar. For example, objects such as points in the plane $\Re^2$, points in 3-space $\Re^3$, polynomials, continuous functions, and differentiable functions (to name only a few) were recognized to satisfy the same additive properties and scalar multiplication properties given in §3.2 for matrices. Rather than studying each topic separately, it was reasoned that it is more efficient and productive to study many topics at one time by studying the common properties that they satisfy. This eventually led to the axiomatic definition of a vector space.

A vector space involves four things—two sets $\mathcal{V}$ and $\mathcal{F}$, and two algebraic operations called vector addition and scalar multiplication.

- $\mathcal{V}$ is a nonempty set of objects called ***vectors***. Although $\mathcal{V}$ can be quite general, we will usually consider $\mathcal{V}$ to be a set of n-tuples or a set of matrices.
- $\mathcal{F}$ is a scalar field—for us $\mathcal{F}$ is either the field $\Re$ of real numbers or the field $\mathcal{C}$ of complex numbers.
- Vector addition (denoted by $\mathbf{x}+\mathbf{y}$) is an operation between elements of $\mathcal{V}$.
- Scalar multiplication (denoted by $\alpha\mathbf{x}$) is an operation between elements of $\mathcal{F}$ and $\mathcal{V}$.

The formal definition of a vector space stipulates how these four things relate to each other. In essence, the requirements are that vector addition and scalar multiplication must obey exactly the same properties given in §3.2 for matrices.

Vector Space Definition

The set $\mathcal{V}$ is called a ***vector space over*** $\mathcal{F}$ when the vector addition and scalar multiplication operations satisfy the following properties.

(A1) $\mathbf{x}+\mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. This is called the ***closure property for vector addition.***

(A2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$. This is the ***closure property for scalar multiplication.***

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$ for every $\alpha \in \mathcal{F}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

A theoretical algebraic treatment of the subject would concentrate on the logical consequences of these defining properties, but the objectives in this text are different, so we will not dwell on the axiomatic development.[23] Neverthe-

[23] The idea of defining a vector space by using a set of abstract axioms was contained in a general theory published in 1844 by Hermann Grassmann (1808–1887), a theologian and philosopher from Stettin, Poland, who was a self-taught mathematician. But Grassmann's work was originally ignored because he tried to construct a highly abstract self-contained theory, independent of the rest of mathematics, containing nonstandard terminology and notation, and he had a tendency to mix mathematics with obscure philosophy. Grassmann published a complete revision of his work in 1862 but with no more success. Only later was it realized that he had formulated the concepts we now refer to as linear dependence, bases, and dimension. The Italian mathematician Giuseppe Peano (1858–1932) was one of the few people who noticed Grassmann's work, and in 1888 Peano published a condensed interpretation of it. In a small chapter at the end, Peano gave an axiomatic definition of a vector space similar to the one above, but this drew little attention outside of a small group in Italy. The current definition is derived from the 1918 work of the German mathematician Hermann Weyl (1885–1955). Even though Weyl's definition is closer to Peano's than to Grassmann's, Weyl did not mention his Italian predecessor, but he did acknowledge Grassmann's "epoch making work." Weyl's success with the idea was due in part to the fact that he thought of vector spaces in terms of geometry, whereas Grassmann and Peano treated them as abstract algebraic structures. As we will see, it's the geometry that's important.

less, it is important to recognize some of the more significant examples and to understand why they are indeed vector spaces.

Example 4.1.1

Because **(A1)–(A5)** are generalized versions of the five additive properties of matrix addition, and **(M1)–(M5)** are generalizations of the five scalar multiplication properties given in §3.2, we can say that the following hold.

- The set $\Re^{m\times n}$ of $m \times n$ real matrices is a vector space over $\Re$.
- The set $\mathcal{C}^{m\times n}$ of $m \times n$ complex matrices is a vector space over $\mathcal{C}$.

Example 4.1.2

The ***real coordinate spaces***

$$\Re^{1\times n} = \{(\, x_1 \quad x_2 \quad \cdots \quad x_n \,),\ x_i \in \Re\} \quad \text{and} \quad \Re^{n\times 1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ x_i \in \Re \right\}$$

are special cases of the preceding example, and these will be the object of most of our attention. In the context of vector spaces, it usually makes no difference whether a coordinate vector is depicted as a row or as a column. When the row or column distinction is irrelevant, or when it is clear from the context, we will use the common symbol $\Re^n$ to designate a coordinate space. In those cases where it is important to distinguish between rows and columns, we will explicitly write $\Re^{1\times n}$ or $\Re^{n\times 1}$. Similar remarks hold for complex coordinate spaces.

Although the coordinate spaces will be our primary concern, be aware that there are many other types of mathematical structures that are vector spaces—this was the reason for making an abstract definition at the outset. Listed below are a few examples.

Example 4.1.3

With function addition and scalar multiplication defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x),$$

the following sets are vector spaces over $\Re$:

- The set of functions mapping the interval $[0, 1]$ into $\Re$.
- The set of all real-valued continuous functions defined on $[0, 1]$.
- The set of real-valued functions that are differentiable on $[0, 1]$.
- The set of all polynomials with real coefficients.

Example 4.1.4

Consider the vector space $\Re^2$, and let

$$\mathcal{L} = \{(x, y) \mid y = \alpha x\}$$

be a line through the origin. $\mathcal{L}$ is a subset of $\Re^2$, but $\mathcal{L}$ is a special kind of subset because $\mathcal{L}$ also satisfies the properties **(A1)–(A5)** and **(M1)–(M5)** that define a vector space. This shows that it is possible for one vector space to properly contain other vector spaces.

Subspaces

Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{V}$ over $\mathcal{F}$ (symbolically, $\mathcal{S} \subseteq \mathcal{V}$). If $\mathcal{S}$ is also a vector space over $\mathcal{F}$ using the same addition and scalar multiplication operations, then $\mathcal{S}$ is said to be a ***subspace*** of $\mathcal{V}$. It's not necessary to check all 10 of the defining conditions in order to determine if a subset is also a subspace—only the closure conditions **(A1)** and **(M1)** need to be considered. That is, a nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if

(A1) $\quad \mathbf{x}, \mathbf{y} \in \mathcal{S} \implies \mathbf{x} + \mathbf{y} \in \mathcal{S}$

and

(M1) $\quad \mathbf{x} \in \mathcal{S} \implies \alpha\mathbf{x} \in \mathcal{S}$ for all $\alpha \in \mathcal{F}$.

Proof. If $\mathcal{S}$ is a subset of $\mathcal{V}$, then $\mathcal{S}$ automatically inherits all of the vector space properties of $\mathcal{V}$ except **(A1)**, **(A4)**, **(A5)**, and **(M1)**. However, **(A1)** together with **(M1)** implies **(A4)** and **(A5)**. To prove this, observe that **(M1)** implies $(-\mathbf{x}) = (-1)\mathbf{x} \in \mathcal{S}$ for all $\mathbf{x} \in \mathcal{S}$ so that **(A5)** holds. Since $\mathbf{x}$ and $(-\mathbf{x})$ are now both in $\mathcal{S}$, **(A1)** insures that $\mathbf{x} + (-\mathbf{x}) \in \mathcal{S}$, and thus $\mathbf{0} \in \mathcal{S}$. ■

Example 4.1.5

Given a vector space $\mathcal{V}$, the set $\mathcal{Z} = \{\mathbf{0}\}$ containing only the zero vector is a subspace of $\mathcal{V}$ because **(A1)** and **(M1)** are trivially satisfied. Naturally, this subspace is called the ***trivial subspace.***

Vector addition in $\Re^2$ and $\Re^3$ is easily visualized by using the ***parallelogram law,*** which states that for two vectors $\mathbf{u}$ and $\mathbf{v}$, the sum $\mathbf{u} + \mathbf{v}$ is the vector defined by the diagonal of the parallelogram as shown in Figure 4.1.1.

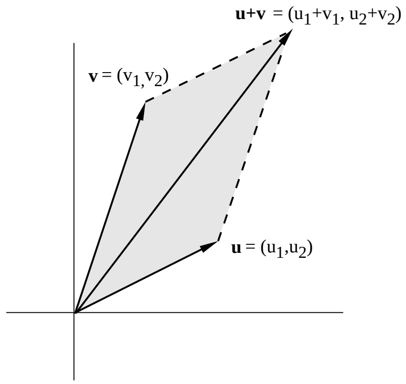

FIGURE 4.1.1

We have already observed that straight lines through the origin in $\Re^2$ are subspaces, but what about straight lines not through the origin? No—they cannot be subspaces because subspaces must contain the zero vector (i.e., they must pass through the origin). What about *curved* lines through the origin—can some of them be subspaces of $\Re^2$? Again the answer is "No!" As depicted in Figure 4.1.2, the parallelogram law indicates why the closure property **(A1)** cannot be satisfied for lines with a curvature because there are points $\mathbf{u}$ and $\mathbf{v}$ on the curve for which $\mathbf{u}+\mathbf{v}$ (the diagonal of the corresponding parallelogram) is not on the curve. Consequently, the only proper subspaces of $\Re^2$ are the trivial subspace and lines through the origin.

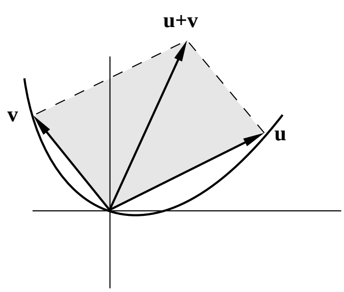

FIGURE 4.1.2

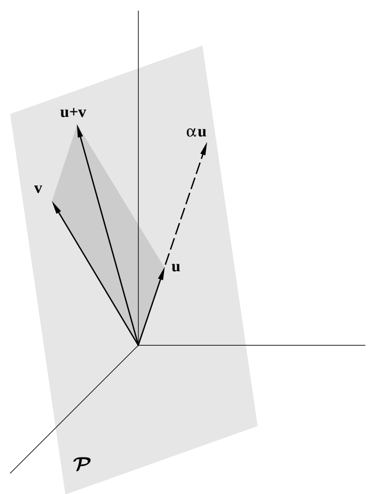

FIGURE 4.1.3

In $\Re^3$, the trivial subspace and lines through the origin are again subspaces, but there is also another one—planes through the origin. If $\mathcal{P}$ is a plane through the origin in $\Re^3$, then, as shown in Figure 4.1.3, the parallelogram law guarantees that the closure property for addition **(A1)** holds—the parallelogram defined by any two vectors in $\mathcal{P}$ is also in $\mathcal{P}$ so that if $\mathbf{u}, \mathbf{v} \in \mathcal{P}$, then $\mathbf{u}+\mathbf{v} \in \mathcal{P}$. The closure property for scalar multiplication **(M1)** holds because multiplying any vector by a scalar merely stretches it, but its angular orientation does not change so that if $\mathbf{u} \in \mathcal{P}$, then $\alpha\mathbf{u} \in \mathcal{P}$ for all scalars α. Lines and surfaces in $\Re^3$ that have curvature cannot be subspaces for essentially the same reason depicted in Figure 4.1.2. So the only proper subspaces of $\Re^3$ are the trivial subspace, lines through the origin, and planes through the origin.

The concept of a subspace now has an obvious interpretation in the visual spaces $\Re^2$ and $\Re^3$ —*subspaces are the flat surfaces passing through the origin.*

Flatness

Although we can't use our eyes to see "flatness" in higher dimensions, our minds can conceive it through the notion of a subspace. From now on, think of flat surfaces passing through the origin whenever you encounter the term "subspace."

For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ from a vector space $\mathcal{V}$, the set of all possible linear combinations of the $\mathbf{v}_i$'s is denoted by

$$span(\mathcal{S}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r \mid \alpha_i \in \mathcal{F}\}.$$

Notice that $span(\mathcal{S})$ is a subspace of $\mathcal{V}$ because the two closure properties **(A1)** and **(M1)** are satisfied. That is, if $\mathbf{x} = \sum_i \xi_i\mathbf{v}_i$ and $\mathbf{y} = \sum_i \eta_i\mathbf{v}_i$ are two linear combinations from $span(\mathcal{S})$, then the sum $\mathbf{x}+\mathbf{y} = \sum_i(\xi_i+\eta_i)\mathbf{v}_i$ is also a linear combination in $span(\mathcal{S})$, and for any scalar β, $\beta\mathbf{x} = \sum_i(\beta\xi_i)\mathbf{v}_i$ is also a linear combination in $span(\mathcal{S})$.

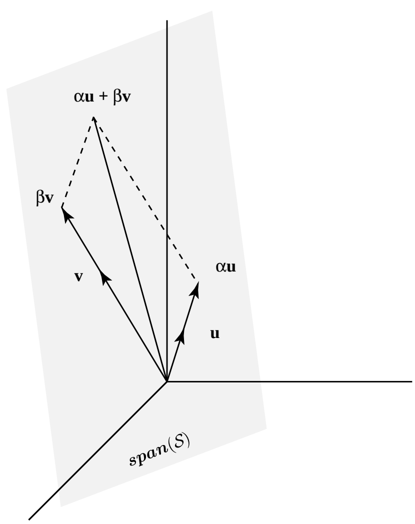

FIGURE 4.1.4

For example, if $\mathbf{u} \neq \mathbf{0}$ is a vector in $\Re^3$, then $span\{\mathbf{u}\}$ is the straight line passing through the origin and $\mathbf{u}$. If $\mathcal{S} = \{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors in $\Re^3$ not lying on the same line, then, as shown in Figure 4.1.4, $span(\mathcal{S})$ is the plane passing through the origin and the points $\mathbf{u}$ and $\mathbf{v}$. As we will soon see, *all* subspaces of $\Re^n$ are of the type $span(\mathcal{S})$, so it is worthwhile to introduce the following terminology.

Spanning Sets

- For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, the subspace

$$span(\mathcal{S}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r\}$$

  generated by forming all linear combinations of vectors from $\mathcal{S}$ is called the ***space spanned by*** $\mathcal{S}$.
- If $\mathcal{V}$ is a vector space such that $\mathcal{V} = span(\mathcal{S})$, we say $\mathcal{S}$ is a ***spanning set*** for $\mathcal{V}$. In other words, $\mathcal{S}$ ***spans*** $\mathcal{V}$ whenever each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{S}$.

Example 4.1.6

(i) In Figure 4.1.4, $\mathcal{S} = \{\mathbf{u}, \mathbf{v}\}$ is a spanning set for the indicated plane.

(ii) $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$ spans the line $y = x$ in $\Re^2$.

(iii) The unit vectors $\left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ span $\Re^3$.

(iv) The unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\Re^n$ form a spanning set for $\Re^n$.

(v) The finite set $\{1, x, x^2, \ldots, x^n\}$ spans the space of all polynomials such that $\deg p(x) \leq n$, and the infinite set $\{1, x, x^2, \ldots\}$ spans the space of all polynomials.

Example 4.1.7

Problem: For a set of vectors $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ from a subspace $\mathcal{V} \subseteq \Re^{m\times 1}$, let $\mathbf{A}$ be the matrix containing the $\mathbf{a}_i$'s as its columns. Explain why $\mathcal{S}$ spans $\mathcal{V}$ if and only if for each $\mathbf{b} \in \mathcal{V}$ there corresponds a column $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$ (i.e., if and only if $\mathbf{Ax} = \mathbf{b}$ is a consistent system for every $\mathbf{b} \in \mathcal{V}$).

Solution: By definition, $\mathcal{S}$ spans $\mathcal{V}$ if and only if for each $\mathbf{b} \in \mathcal{V}$ there exist scalars α_i such that

$$\mathbf{b} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_n\mathbf{a}_n = \big(\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\big)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix} = \mathbf{Ax}.$$

Note: This simple observation often is quite helpful. For example, to test whether or not $\mathcal{S} = \{(1 \;\; 1 \;\; 1),\, (1 \;\; -1 \;\; -1),\, (3 \;\; 1 \;\; 1)\}$ spans $\Re^3$, place these rows as columns in a matrix $\mathbf{A}$, and ask, "Is the system

$$\begin{pmatrix}1 & 1 & 3\\ 1 & -1 & 1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$$

consistent for *every* $\mathbf{b} \in \Re^3$?" Recall from (2.3.4) that $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. In this case, $rank\,(\mathbf{A}) = 2$, but $rank[\mathbf{A}|\mathbf{b}] = 3$ for some $\mathbf{b}$'s (e.g., $b_1 = 0, b_2 = 1, b_3 = 0$), so $\mathcal{S}$ doesn't span $\Re^3$. On the other hand, $\mathcal{S}' = \{(1 \;\; 1 \;\; 1),\, (1 \;\; -1 \;\; -1),\, (3 \;\; 1 \;\; 2)\}$ is a spanning set for $\Re^3$ because

$$\mathbf{A} = \begin{pmatrix}1 & 1 & 3\\ 1 & -1 & 1\\ 1 & -1 & 2\end{pmatrix}$$

is nonsingular, so $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$ (the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$).

As shown below, it's possible to "add" two subspaces to generate another.

Sum of Subspaces

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then the ***sum*** of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the set of all possible sums of vectors from $\mathcal{X}$ with vectors from $\mathcal{Y}$. That is,

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{y} \in \mathcal{Y}\}.$$

- The sum $\mathcal{X} + \mathcal{Y}$ is again a subspace of $\mathcal{V}$. (4.1.1)
- If $\mathcal{S}_X, \mathcal{S}_Y$ span $\mathcal{X}, \mathcal{Y}$, then $\mathcal{S}_X \cup \mathcal{S}_Y$ spans $\mathcal{X} + \mathcal{Y}$. (4.1.2)

Proof. To prove (4.1.1), demonstrate that the two closure properties **(A1)** and **(M1)** hold for $\mathcal{S} = \mathcal{X} + \mathcal{Y}$. To show **(A1)** is valid, observe that if $\mathbf{u}, \mathbf{v} \in \mathcal{S}$, then $\mathbf{u} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{v} = \mathbf{x}_2 + \mathbf{y}_2$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{Y}$, and therefore $\mathbf{u} + \mathbf{v} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) \in \mathcal{S}$. To verify **(M1)**, observe that $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication so that $\alpha\mathbf{x}_1 \in \mathcal{X}$ and $\alpha\mathbf{y}_1 \in \mathcal{Y}$ for all α, and consequently $\alpha\mathbf{u} = \alpha\mathbf{x}_1 + \alpha\mathbf{y}_1 \in \mathcal{S}$ for all α. To prove (4.1.2), suppose $\mathcal{S}_X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\mathcal{S}_Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$, and write

$$\mathbf{z} \in span\,(\mathcal{S}_X \cup \mathcal{S}_Y) \Longleftrightarrow \mathbf{z} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i + \sum_{i=1}^{t} \beta_i \mathbf{y}_i = \mathbf{x} + \mathbf{y} \text{ with } \mathbf{x} \in \mathcal{X},\ \mathbf{y} \in \mathcal{Y}$$
$$\Longleftrightarrow \mathbf{z} \in \mathcal{X} + \mathcal{Y}. \quad \blacksquare$$

Example 4.1.8

If $\mathcal{X} \subseteq \Re^2$ and $\mathcal{Y} \subseteq \Re^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \Re^2$. This follows from the parallelogram law—sketch a picture for yourself.

Exercises for section 4.1

4.1.1. Determine which of the following subsets of $\Re^n$ are in fact subspaces of $\Re^n$ $(n > 2)$.

(a) $\{\mathbf{x} \mid x_i \geq 0\}$, (b) $\{\mathbf{x} \mid x_1 = 0\}$, (c) $\{\mathbf{x} \mid x_1 x_2 = 0\}$,

(d) $\left\{\mathbf{x} \,\middle|\, \sum_{j=1}^{n} x_j = 0\right\}$, (e) $\left\{\mathbf{x} \,\middle|\, \sum_{j=1}^{n} x_j = 1\right\}$,

(f) $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \text{ where } \mathbf{A}_{m\times n} \neq \mathbf{0} \text{ and } \mathbf{b}_{m\times 1} \neq \mathbf{0}\}$.

4.1.2. Determine which of the following subsets of $\Re^{n\times n}$ are in fact subspaces of $\Re^{n\times n}$.

(a) The symmetric matrices. (b) The diagonal matrices.
(c) The nonsingular matrices. (d) The singular matrices.
(e) The triangular matrices. (f) The upper-triangular matrices.
(g) All matrices that commute with a given matrix $\mathbf{A}$.
(h) All matrices such that $\mathbf{A}^2 = \mathbf{A}$.
(i) All matrices such that $trace\,(\mathbf{A}) = 0$.

4.1.3. If $\mathcal{X}$ is a plane passing through the origin in $\Re^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, what is $\mathcal{X} + \mathcal{Y}$?

4.1.4. Why must a real or complex nonzero vector space contain an infinite number of vectors?

4.1.5. Sketch a picture in $\Re^3$ of the subspace spanned by each of the following.

(a) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ -9 \\ -6 \end{pmatrix} \right\}$, (b) $\left\{ \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$,

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

4.1.6. Which of the following are spanning sets for $\Re^3$?

(a) $\{(1\ \ 1\ \ 1)\}$ (b) $\{(1\ \ 0\ \ 0),\ (0\ \ 0\ \ 1)\}$,
(c) $\{(1\ \ 0\ \ 0),\ (0\ \ 1\ \ 0),\ (0\ \ 0\ \ 1),\ (1\ \ 1\ \ 1)\}$,
(d) $\{(1\ \ 2\ \ 1),\ (2\ \ 0\ \ -1),\ (4\ \ 4\ \ 1)\}$,
(e) $\{(1\ \ 2\ \ 1),\ (2\ \ 0\ \ -1),\ (4\ \ 4\ \ 0)\}$.

4.1.7. For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{V}$, explain why $span\,(\mathcal{M} \cup \mathcal{N}) = span\,(\mathcal{M}) + span\,(\mathcal{N})$.

4.1.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$.
(a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$.
(b) Show that the union $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace of $\mathcal{V}$.

4.1.9. For $\mathbf{A} \in \Re^{m \times n}$ and $\mathcal{S} \subseteq \Re^{n \times 1}$, the set $\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \mathcal{S}\}$ contains all possible products of $\mathbf{A}$ with vectors from $\mathcal{S}$. We refer to $\mathbf{A}(\mathcal{S})$ as the set of ***images*** of $\mathcal{S}$ under $\mathbf{A}$.
(a) If $\mathcal{S}$ is a subspace of $\Re^n$, prove $\mathbf{A}(\mathcal{S})$ is a subspace of $\Re^m$.
(b) If $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ spans $\mathcal{S}$, show $\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k$ spans $\mathbf{A}(\mathcal{S})$.

4.1.10. With the usual addition and multiplication, determine whether or not the following sets are vector spaces over the real numbers.
(a) $\Re$, (b) $\mathcal{C}$, (c) The rational numbers.

4.1.11. Let $\mathcal{M} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\}$ and $\mathcal{N} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r, \mathbf{v}\}$ be two sets of vectors from the same vector space. Prove that $span\,(\mathcal{M}) = span\,(\mathcal{N})$ if and only if $\mathbf{v} \in span\,(\mathcal{M})$.

4.1.12. For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, prove that $span\,(\mathcal{S})$ is the intersection of all subspaces that contain $\mathcal{S}$. **Hint:** For $\mathcal{M} = \bigcap_{\mathcal{S} \subseteq \mathcal{V}} \mathcal{V}$, prove that $span\,(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq span\,(\mathcal{S})$.

4.2 FOUR FUNDAMENTAL SUBSPACES

The closure properties **(A1)** and **(M1)** on p. 162 that characterize the notion of a subspace have much the same "feel" as the definition of a linear function as stated on p. 89, but there's more to it than just a "similar feel." Subspaces are intimately related to linear functions as explained below.

Subspaces and Linear Functions

For a linear function f mapping $\Re^n$ into $\Re^m$, let $\mathcal{R}(f)$ denote the ***range*** of f. That is, $\mathcal{R}(f) = \{f(\mathbf{x}) \,|\, \mathbf{x} \in \Re^n\} \subseteq \Re^m$ is the set of all "images" as $\mathbf{x}$ varies freely over $\Re^n$.

- The range of every linear function $f : \Re^n \to \Re^m$ is a subspace of $\Re^m$, and every subspace of $\Re^m$ is the range of some linear function.

For this reason, subspaces of $\Re^m$ are sometimes called ***linear spaces***.

Proof. If $f : \Re^n \to \Re^m$ is a linear function, then the range of f is a subspace of $\Re^m$ because the closure properties **(A1)** and **(M1)** are satisfied. Establish **(A1)** by showing that $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(f) \Rightarrow \mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(f)$. If $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(f)$, then there must be vectors $\mathbf{x}_1, \mathbf{x}_2 \in \Re^n$ such that $\mathbf{y}_1 = f(\mathbf{x}_1)$ and $\mathbf{y}_2 = f(\mathbf{x}_2)$, so it follows from the linearity of f that

$$\mathbf{y}_1 + \mathbf{y}_2 = f(\mathbf{x}_1) + f(\mathbf{x}_2) = f(\mathbf{x}_1 + \mathbf{x}_2) \in \mathcal{R}(f).$$

Similarly, establish **(M1)** by showing that if $\mathbf{y} \in \mathcal{R}(f)$, then $\alpha\mathbf{y} \in \mathcal{R}(f)$ for all scalars α by using the definition of range along with the linearity of f to write

$$\mathbf{y} \in \mathcal{R}(f) \Longrightarrow \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \in \Re^n \Longrightarrow \alpha\mathbf{y} = \alpha f(\mathbf{x}) = f(\alpha\mathbf{x}) \in \mathcal{R}(f).$$

Now prove that every subspace $\mathcal{V}$ of $\Re^m$ is the range of some linear function $f : \Re^n \to \Re^m$. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathcal{V}$ so that

$$\mathcal{V} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n \,|\, \alpha_i \in \mathcal{R}\}. \tag{4.2.1}$$

Stack the $\mathbf{v}_i$'s as columns in a matrix $\mathbf{A}_{m\times n} = \big(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \cdots \,|\, \mathbf{v}_n\big)$, and put the α_i's in an $n \times 1$ column $\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ to write

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \big(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \cdots \,|\, \mathbf{v}_n\big) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \mathbf{A}\mathbf{x}. \tag{4.2.2}$$

The function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is linear (recall Example 3.6.1, p. 106), and we have that $\mathcal{R}(f) = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \in \Re^{n\times 1}\} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n \,|\, \alpha_i \in \mathcal{R}\} = \mathcal{V}$. ∎

In particular, this result means that every matrix $\mathbf{A} \in \Re^{m\times n}$ generates a subspace of $\Re^m$ by means of the range of the linear function $f(\mathbf{x}) = \mathbf{Ax}$. Likewise, the transpose[24] of $\mathbf{A} \in \Re^{m\times n}$ defines a subspace of $\Re^n$ by means of the range of $f(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$. These two "range spaces" are two of the four fundamental subspaces defined by a matrix.

Range Spaces

The ***range of a matrix*** $\mathbf{A} \in \Re^{m\times n}$ is defined to be the subspace $R(\mathbf{A})$ of $\Re^m$ that is generated by the range of $f(\mathbf{x}) = \mathbf{Ax}$. That is,

$$R(\mathbf{A}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \Re^n\} \subseteq \Re^m.$$

Similarly, the range of $\mathbf{A}^T$ is the subspace of $\Re^n$ defined by

$$R\left(\mathbf{A}^T\right) = \{\mathbf{A}^T\mathbf{y} \,|\, \mathbf{y} \in \Re^m\} \subseteq \Re^n.$$

Because $R(\mathbf{A})$ is the set of all "images" of vectors $\mathbf{x} \in \Re^m$ under transformation by $\mathbf{A}$, some people call $R(\mathbf{A})$ the ***image space*** of $\mathbf{A}$.

The observation (4.2.2) that every matrix–vector product $\mathbf{Ax}$ (i.e., every image) is a linear combination of the columns of $\mathbf{A}$ provides a useful characterization of the range spaces. Allowing the components of $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n)^T$ to vary freely and writing

$$\mathbf{Ax} = \left(\mathbf{A}_{*1} \mid \mathbf{A}_{*2} \mid \cdots \mid \mathbf{A}_{*n}\right)\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix} = \sum_{j=1}^{n} \xi_j \mathbf{A}_{*j}$$

shows that the set of all images $\mathbf{Ax}$ is the same as the set of all linear combinations of the columns of $\mathbf{A}$. Therefore, $R(\mathbf{A})$ is nothing more than the space spanned by the columns of $\mathbf{A}$. That's why $R(\mathbf{A})$ is often called the ***column space*** of $\mathbf{A}$.

Likewise, $R\left(\mathbf{A}^T\right)$ is the space spanned by the columns of $\mathbf{A}^T$. But the columns of $\mathbf{A}^T$ are just the rows of $\mathbf{A}$ (stacked upright), so $R\left(\mathbf{A}^T\right)$ is simply the space spanned by the rows[25] of $\mathbf{A}$. Consequently, $R\left(\mathbf{A}^T\right)$ is also known as the ***row space*** of $\mathbf{A}$. Below is a summary.

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[24] For ease of exposition, the discussion in this section is in terms of real matrices and real spaces, but all results have complex analogs obtained by replacing $\mathbf{A}^T$ by $\mathbf{A}^*$.

[25] Strictly speaking, the range of $\mathbf{A}^T$ is a set of columns, while the row space of $\mathbf{A}$ is a set of rows. However, no logical difficulties are encountered by considering them to be the same.

Column and Row Spaces

For $\mathbf{A} \in \Re^{m\times n}$, the following statements are true.

- $R(\mathbf{A}) =$ the space spanned by the columns of $\mathbf{A}$ (column space).
- $R(\mathbf{A}^T) =$ the space spanned by the rows of $\mathbf{A}$ (row space).
- $\mathbf{b} \in R(\mathbf{A}) \iff \mathbf{b} = \mathbf{Ax}$ for some $\mathbf{x}$. (4.2.3)
- $\mathbf{a} \in R(\mathbf{A}^T) \iff \mathbf{a}^T = \mathbf{y}^T\mathbf{A}$ for some $\mathbf{y}^T$. (4.2.4)

Example 4.2.1

Problem: Describe $R(\mathbf{A})$ and $R(\mathbf{A}^T)$ for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

Solution: $R(\mathbf{A}) = span\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*3}\} = \{\alpha_1\mathbf{A}_{*1}+\alpha_2\mathbf{A}_{*2}+\alpha_3\mathbf{A}_{*3} \mid \alpha_i \in \Re\}$, but since $\mathbf{A}_{*2} = 2\mathbf{A}_{*1}$ and $\mathbf{A}_{*3} = 3\mathbf{A}_{*1}$, it's clear that every linear combination of $\mathbf{A}_{*1}$, $\mathbf{A}_{*2}$, and $\mathbf{A}_{*3}$ reduces to a multiple of $\mathbf{A}_{*1}$, so $R(\mathbf{A}) = span\{\mathbf{A}_{*1}\}$. Geometrically, $R(\mathbf{A})$ is the line in $\Re^2$ through the origin and the point $(1,2)$. Similarly, $R(\mathbf{A}^T) = span\{\mathbf{A}_{1*}, \mathbf{A}_{2*}\} = \{\alpha_1\mathbf{A}_{1*} + \alpha_2\mathbf{A}_{2*} \mid \alpha_1, \alpha_2 \in \Re\}$. But $\mathbf{A}_{2*} = 2\mathbf{A}_{1*}$ implies that every combination of $\mathbf{A}_{1*}$ and $\mathbf{A}_{2*}$ reduces to a multiple of $\mathbf{A}_{1*}$, so $R(\mathbf{A}^T) = span\{\mathbf{A}_{1*}\}$, and this is a line in $\Re^3$ through the origin and the point $(1,2,3)$.

There are times when it is desirable to know whether or not two matrices have the same row space or the same range. The following theorem provides the solution to this problem.

Equal Ranges

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same shape:

- $R(\mathbf{A}^T) = R(\mathbf{B}^T)$ if and only if $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. (4.2.5)
- $R(\mathbf{A}) = R(\mathbf{B})$ if and only if $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$. (4.2.6)

Proof. To prove (4.2.5), first assume $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ so that there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{B}$. To see that $R(\mathbf{A}^T) = R(\mathbf{B}^T)$, use (4.2.4) to write

$$\begin{aligned} \mathbf{a} \in R(\mathbf{A}^T) &\iff \mathbf{a}^T = \mathbf{y}^T\mathbf{A} = \mathbf{y}^T\mathbf{P}^{-1}\mathbf{PA} \quad \text{for some } \mathbf{y}^T \\ &\iff \mathbf{a}^T = \mathbf{z}^T\mathbf{B} \quad \text{for } \mathbf{z}^T = \mathbf{y}^T\mathbf{P}^{-1} \\ &\iff \mathbf{a} \in R(\mathbf{B}^T). \end{aligned}$$

Conversely, if $R(\mathbf{A}^T) = R(\mathbf{B}^T)$, then

$$span\{\mathbf{A}_{1*}, \mathbf{A}_{2*}, \ldots, \mathbf{A}_{m*}\} = span\{\mathbf{B}_{1*}, \mathbf{B}_{2*}, \ldots, \mathbf{B}_{m*}\},$$

so each row of $\mathbf{B}$ is a combination of the rows of $\mathbf{A}$, and vice versa. On the basis of this fact, it can be argued that it is possible to reduce $\mathbf{A}$ to $\mathbf{B}$ by using only row operations (the tedious details are omitted), and thus $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. The proof of (4.2.6) follows by replacing $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A}^T$ and $\mathbf{B}^T$. ∎

Example 4.2.2

Testing Spanning Sets. Two sets $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s\}$ in $\Re^n$ span the same subspace if and only if the nonzero rows of $\mathbf{E_A}$ agree with the nonzero rows of $\mathbf{E_B}$, where $\mathbf{A}$ and $\mathbf{B}$ are the matrices containing the $\mathbf{a}_i$'s and $\mathbf{b}_i$'s as *rows*. This is a corollary of (4.2.5) because zero rows are irrelevant in considering the row space of a matrix, and we already know from (3.9.9) that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ if and only if $\mathbf{E_A} = \mathbf{E_B}$.

Problem: Determine whether or not the following sets span the same subspace:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 1 \\ 4 \end{pmatrix} \right\}, \qquad \mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Solution: Place the vectors as *rows* in matrices $\mathbf{A}$ and $\mathbf{B}$, and compute

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E_A}$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \mathbf{E_B}.$$

Hence $span\{\mathcal{A}\} = span\{\mathcal{B}\}$ because the nonzero rows in $\mathbf{E_A}$ and $\mathbf{E_B}$ agree.

We already know that the rows of $\mathbf{A}$ span $R(\mathbf{A}^T)$, and the columns of $\mathbf{A}$ span $R(\mathbf{A})$, but it's often possible to span these spaces with fewer vectors than the full set of rows and columns.

Spanning the Row Space and Range

Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{U}$ be any row echelon form derived from $\mathbf{A}$. Spanning sets for the row and column spaces are as follows:

- The nonzero rows of $\mathbf{U}$ span $R(\mathbf{A}^T)$. (4.2.7)
- The basic columns in $\mathbf{A}$ span $R(\mathbf{A})$. (4.2.8)

Proof. Statement (4.2.7) is an immediate consequence of (4.2.5). To prove (4.2.8), suppose that the basic columns in $\mathbf{A}$ are in positions $b_1, b_2, \ldots, b_r$, and the nonbasic columns occupy positions $n_1, n_2, \ldots, n_t$, and let $\mathbf{Q}_1$ be the permutation matrix that permutes all of the basic columns in $\mathbf{A}$ to the left-hand side so that $\mathbf{AQ}_1 = \begin{pmatrix} \mathbf{B}_{m\times r} & \mathbf{N}_{m\times t} \end{pmatrix}$, where $\mathbf{B}$ contains the basic columns and $\mathbf{N}$ contains the nonbasic columns. Since the nonbasic columns are linear combinations of the basic columns—recall (2.2.3)—we can annihilate the nonbasic columns in $\mathbf{N}$ using elementary column operations. In other words, there is a nonsingular matrix $\mathbf{Q}_2$ such that $\begin{pmatrix} \mathbf{B} & \mathbf{N} \end{pmatrix}\mathbf{Q}_2 = \begin{pmatrix} \mathbf{B} & \mathbf{0} \end{pmatrix}$. Thus $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$ is a nonsingular matrix such that $\mathbf{AQ} = \mathbf{AQ}_1\mathbf{Q}_2 = \begin{pmatrix} \mathbf{B} & \mathbf{N} \end{pmatrix}\mathbf{Q}_2 = \begin{pmatrix} \mathbf{B} & \mathbf{0} \end{pmatrix}$, and hence $\mathbf{A} \overset{\text{col}}{\sim} \begin{pmatrix} \mathbf{B} & \mathbf{0} \end{pmatrix}$. The conclusion (4.2.8) now follows from (4.2.6). ∎

Example 4.2.3

Problem: Determine spanning sets for $R(\mathbf{A})$ and $R(\mathbf{A}^T)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Solution: Reducing $\mathbf{A}$ to any row echelon form $\mathbf{U}$ provides the solution—the basic columns in $\mathbf{A}$ correspond to the pivotal positions in $\mathbf{U}$, and the nonzero rows of $\mathbf{U}$ span the row space of $\mathbf{A}$. Using $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ produces

$$R(\mathbf{A}) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad R\left(\mathbf{A}^T\right) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

So far, only two of the four fundamental subspaces associated with each matrix $\mathbf{A} \in \Re^{m\times n}$ have been discussed, namely, $R(\mathbf{A})$ and $R\left(\mathbf{A}^T\right)$. To see where the other two fundamental subspaces come from, consider again a general linear function f mapping $\Re^m$ into $\Re^n$, and focus on $\mathcal{N}(f) = \{\mathbf{x} \,|\, f(\mathbf{x}) = \mathbf{0}\}$ (the set of vectors that are mapped to $\mathbf{0}$). $\mathcal{N}(f)$ is called the ***nullspace*** of f (some texts call it the ***kernel*** of f), and it's easy to see that $\mathcal{N}(f)$ is a subspace of $\Re^n$ because the closure properties **(A1)** and **(M1)** are satisfied. Indeed, if $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(f)$, then $f(\mathbf{x}_1) = \mathbf{0}$ and $f(\mathbf{x}_2) = \mathbf{0}$, so the linearity of f produces

$$f(\mathbf{x}_1 + \mathbf{x}_2) = f(\mathbf{x}_1) + f(\mathbf{x}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(f). \qquad \textbf{(A1)}$$

Similarly, if $\alpha \in \Re$, and if $\mathbf{x} \in \mathcal{N}(f)$, then $f(\mathbf{x}) = \mathbf{0}$ and linearity implies

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0} \implies \alpha\mathbf{x} \in \mathcal{N}(f). \qquad \textbf{(M1)}$$

By considering the linear functions $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$, the other two fundamental subspaces defined by $\mathbf{A} \in \Re^{m\times n}$ are obtained. They are $\mathcal{N}(f) = \{\mathbf{x}_{n\times 1} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$ and $\mathcal{N}(g) = \left\{\mathbf{y}_{m\times 1} \mid \mathbf{A^T y} = \mathbf{0}\right\} \subseteq \Re^m$.

Nullspace

- For an $m \times n$ matrix $\mathbf{A}$, the set $N(\mathbf{A}) = \{\mathbf{x}_{n\times 1} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \subseteq \Re^n$ is called the ***nullspace*** of $\mathbf{A}$. In other words, $N(\mathbf{A})$ is simply the set of all solutions to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.
- The set $N\left(\mathbf{A}^T\right) = \left\{\mathbf{y}_{m\times 1} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\right\} \subseteq \Re^m$ is called the ***left-hand nullspace*** of $\mathbf{A}$ because $N\left(\mathbf{A}^T\right)$ is the set of all solutions to the left-hand homogeneous system $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$.

Example 4.2.4

Problem: Determine a spanning set for $N(\mathbf{A})$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

Solution: $N(\mathbf{A})$ is merely the general solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$, and this is determined by reducing $\mathbf{A}$ to a row echelon form $\mathbf{U}$. As discussed in §2.4, any such $\mathbf{U}$ will suffice, so we will use $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. Consequently, $x_1 = -2x_2 - 3x_3$, where x_2 and x_3 are free, so the general solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

In other words, $N(\mathbf{A})$ is the set of all possible linear combinations of the vectors

$$\mathbf{h}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix},$$

and therefore $span\{\mathbf{h}_1, \mathbf{h}_2\} = N(\mathbf{A})$. For this example, $N(\mathbf{A})$ is the plane in $\Re^3$ that passes through the origin and the two points $\mathbf{h}_1$ and $\mathbf{h}_2$.

Example 4.2.4 indicates the general technique for determining a spanning set for $N(\mathbf{A})$. Below is a formal statement of this procedure.

Spanning the Nullspace

To determine a spanning set for $N(\mathbf{A})$, where $rank(\mathbf{A}_{m\times n}) = r$, row reduce $\mathbf{A}$ to a row echelon form $\mathbf{U}$, and solve $\mathbf{Ux} = \mathbf{0}$ for the basic variables in terms of the free variables to produce the general solution of $\mathbf{Ax} = \mathbf{0}$ in the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}. \tag{4.2.9}$$

By definition, the set $\mathcal{H} = \{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}\}$ spans $N(\mathbf{A})$. Moreover, it can be proven that $\mathcal{H}$ is unique in the sense that $\mathcal{H}$ is independent of the row echelon form $\mathbf{U}$.

It was established in §2.4 that a homogeneous system $\mathbf{Ax} = \mathbf{0}$ possesses a unique solution (i.e., only the trivial solution $\mathbf{x} = \mathbf{0}$) if and only if the rank of the coefficient matrix equals the number of unknowns. This may now be restated using vector space terminology.

Zero Nullspace

If $\mathbf{A}$ is an $m \times n$ matrix, then

- $N(\mathbf{A}) = \{\mathbf{0}\}$ if and only if $rank(\mathbf{A}) = n$; (4.2.10)
- $N\left(\mathbf{A}^T\right) = \{\mathbf{0}\}$ if and only if $rank(\mathbf{A}) = m$. (4.2.11)

Proof. We already know that the trivial solution $\mathbf{x} = \mathbf{0}$ is the only solution to $\mathbf{Ax} = \mathbf{0}$ if and only if the rank of $\mathbf{A}$ is the number of unknowns, and this is what (4.2.10) says. Similarly, $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ has only the trivial solution $\mathbf{y} = \mathbf{0}$ if and only if $rank\left(\mathbf{A}^T\right) = m$. Recall from (3.9.11) that $rank\left(\mathbf{A}^T\right) = rank(\mathbf{A})$ in order to conclude that (4.2.11) holds. ∎

Finally, let's think about how to determine a spanning set for $N\left(\mathbf{A}^T\right)$. Of course, we can proceed in the same manner as described in Example 4.2.4 by reducing $\mathbf{A}^T$ to a row echelon form to extract the general solution for $\mathbf{A}^T\mathbf{x} = \mathbf{0}$. However, the other three fundamental subspaces are derivable directly from $\mathbf{E}_\mathbf{A}$ (or any other row echelon form $\mathbf{U} \overset{\text{row}}{\sim} \mathbf{A}$), so it's rather awkward to have to start from scratch and compute a new echelon form just to get a spanning set for $N\left(\mathbf{A}^T\right)$. It would be better if a single reduction to echelon form could produce all four of the fundamental subspaces. Note that $\mathbf{E}_{\mathbf{A}^T} \neq \mathbf{E}_\mathbf{A}^T$, so $\mathbf{E}_\mathbf{A}^T$ won't easily lead to $N\left(\mathbf{A}^T\right)$. The following theorem helps resolve this issue.

Left-Hand Nullspace

If $rank\,(\mathbf{A}_{m\times n}) = r$, and if $\mathbf{PA} = \mathbf{U}$, where $\mathbf{P}$ is nonsingular and $\mathbf{U}$ is in row echelon form, then the last $m-r$ rows in $\mathbf{P}$ span the left-hand nullspace of $\mathbf{A}$. In other words, if $\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}$, where $\mathbf{P}_2$ is $(m-r)\times m$, then

$$N\left(\mathbf{A}^T\right) = R\left(\mathbf{P}_2^T\right). \tag{4.2.12}$$

Proof. If $\mathbf{U} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{C}_{r\times n}$, then $\mathbf{PA} = \mathbf{U}$ implies $\mathbf{P}_2\mathbf{A} = \mathbf{0}$, and this says $R\left(\mathbf{P}_2^T\right) \subseteq N\left(\mathbf{A}^T\right)$. To show equality, demonstrate containment in the opposite direction by arguing that every vector in $N\left(\mathbf{A}^T\right)$ must also be in $R\left(\mathbf{P}_2^T\right)$. Suppose $\mathbf{y}^T \in N\left(\mathbf{A}^T\right)$, and let $\mathbf{P}^{-1} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{pmatrix}$ to conclude that

$$\mathbf{0} = \mathbf{y}^T\mathbf{A} = \mathbf{y}^T\mathbf{P}^{-1}\mathbf{U} = \mathbf{y}^T\mathbf{Q}_1\mathbf{C} \implies \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1$$

because $N\left(\mathbf{C}^T\right) = \{\mathbf{0}\}$ by (4.2.11). Now observe that $\mathbf{PP}^{-1} = \mathbf{I} = \mathbf{P}^{-1}\mathbf{P}$ insures $\mathbf{P}_1\mathbf{Q}_1 = \mathbf{I}_r$ and $\mathbf{Q}_1\mathbf{P}_1 = \mathbf{I}_m - \mathbf{Q}_2\mathbf{P}_2$, so

$$\begin{aligned} \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1 &\implies \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1\mathbf{P}_1 = \mathbf{y}^T\left(\mathbf{I} - \mathbf{Q}_2\mathbf{P}_2\right) \\ &\implies \mathbf{y}^T = \mathbf{y}^T\mathbf{Q}_2\mathbf{P}_2 = \left(\mathbf{y}^T\mathbf{Q}_2\right)\mathbf{P}_2 \\ &\implies \mathbf{y} \in R\left(\mathbf{P}_2^T\right) \implies \mathbf{y}^T \in R\left(\mathbf{P}_2^T\right). \quad \blacksquare \end{aligned}$$

Example 4.2.5

Problem: Determine a spanning set for $N\left(\mathbf{A}^T\right)$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$.

Solution: To find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, proceed as described in Exercise 3.9.1 and row reduce the augmented matrix $\left(\mathbf{A} \mid \mathbf{I}\right)$ to $\left(\mathbf{U} \mid \mathbf{P}\right)$. It must be the case that $\mathbf{PA} = \mathbf{U}$ because $\mathbf{P}$ is the product of the elementary matrices corresponding to the elementary row operations used. Since any row echelon form will suffice, we may use Gauss–Jordan reduction to reduce $\mathbf{A}$ to $\mathbf{E_A}$ as shown below:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 \\ 3 & 6 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 0 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -5/3 & 1 \end{array}\right)$$

$$\mathbf{P} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 2/3 & -1/3 & 0 \\ 1/3 & -5/3 & 1 \end{pmatrix}, \text{ so (4.2.12) implies } N\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix} 1/3 \\ -5/3 \\ 1 \end{pmatrix}\right\}.$$

Example 4.2.6

Problem: Suppose $rank(\mathbf{A}_{m\times n}) = r$, and let $\mathbf{P} = \begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U} = \begin{pmatrix}\mathbf{C}_{r\times n}\\ \mathbf{0}\end{pmatrix}$, where $\mathbf{U}$ is in row echelon form. Prove

$$R(\mathbf{A}) = N(\mathbf{P}_2). \tag{4.2.13}$$

Solution: The strategy is to first prove $R(\mathbf{A}) \subseteq N(\mathbf{P}_2)$ and then show the reverse inclusion $N(\mathbf{P}_2) \subseteq R(\mathbf{A})$. The equation $\mathbf{PA} = \mathbf{U}$ implies $\mathbf{P}_2\mathbf{A} = \mathbf{0}$, so all columns of $\mathbf{A}$ are in $N(\mathbf{P}_2)$, and thus $R(\mathbf{A}) \subseteq N(\mathbf{P}_2)$. To show inclusion in the opposite direction, suppose $\mathbf{b} \in N(\mathbf{P}_2)$, so that

$$\mathbf{Pb} = \begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}\mathbf{b} = \begin{pmatrix}\mathbf{P}_1\mathbf{b}\\ \mathbf{P}_2\mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{d}_{r\times 1}\\ \mathbf{0}\end{pmatrix}.$$

Consequently, $\mathbf{P}\big(\mathbf{A}\,|\,\mathbf{b}\big) = \big(\mathbf{PA} \mid \mathbf{Pb}\big) = \begin{pmatrix}\mathbf{C} & \mathbf{d}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$, and this implies

$$rank[\mathbf{A}|\mathbf{b}] = r = rank(\mathbf{A}).$$

Recall from (2.3.4) that this means the system $\mathbf{Ax} = \mathbf{b}$ is consistent, and thus $\mathbf{b} \in R(\mathbf{A})$ by (4.2.3). Therefore, $N(\mathbf{P}_2) \subseteq R(\mathbf{A})$, and we may conclude that $N(\mathbf{P}_2) = R(\mathbf{A})$.

It's often important to know when two matrices have the same nullspace (or left-hand nullspace). Below is one test for determining this.

Equal Nullspaces

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same shape:

- $N(\mathbf{A}) = N(\mathbf{B})$ if and only if $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. (4.2.14)
- $N\big(\mathbf{A}^T\big) = N\big(\mathbf{B}^T\big)$ if and only if $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$. (4.2.15)

Proof. We will prove (4.2.15). If $N\big(\mathbf{A}^T\big) = N\big(\mathbf{B}^T\big)$, then (4.2.12) guarantees $R\big(\mathbf{P}_2^T\big) = N\big(\mathbf{B}^T\big)$, and hence $\mathbf{P}_2\mathbf{B} = \mathbf{0}$. But this means the columns of $\mathbf{B}$ are in $N(\mathbf{P}_2)$. That is, $R(\mathbf{B}) \subseteq N(\mathbf{P}_2) = R(\mathbf{A})$ by using (4.2.13). If $\mathbf{A}$ is replaced by $\mathbf{B}$ in the preceding argument—and in (4.2.13)— the result is that $R(\mathbf{A}) \subseteq R(\mathbf{B})$, and consequently we may conclude that $R(\mathbf{A}) = R(\mathbf{B})$. The desired conclusion (4.2.15) follows from (4.2.6). Statement (4.2.14) now follows by replacing $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A}^T$ and $\mathbf{B}^T$ in (4.2.15). ∎

Summary

The four fundamental subspaces associated with $\mathbf{A}_{m\times n}$ are as follows.

- The range or column space: $R(\mathbf{A}) = \{\mathbf{Ax}\} \subseteq \Re^m$.
- The row space or left-hand range: $R(\mathbf{A}^T) = \{\mathbf{A}^T\mathbf{y}\} \subseteq \Re^n$.
- The nullspace: $N(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$.
- The left-hand nullspace: $N(\mathbf{A}^T) = \{\mathbf{y} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\} \subseteq \Re^m$.

Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$, where $\mathbf{U}$ is in row echelon form, and suppose $rank(\mathbf{A}) = r$.

- Spanning set for $R(\mathbf{A}) =$ the basic columns in $\mathbf{A}$.
- Spanning set for $R(\mathbf{A}^T) =$ the nonzero rows in $\mathbf{U}$.
- Spanning set for $N(\mathbf{A}) =$the $\mathbf{h}_i$'s in the general solution of $\mathbf{Ax} = \mathbf{0}$.
- Spanning set for $N(\mathbf{A}^T) =$ the last $m - r$ rows of $\mathbf{P}$.

If $\mathbf{A}$ and $\mathbf{B}$ have the same shape, then

- $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \Longleftrightarrow N(\mathbf{A}) = N(\mathbf{B}) \Longleftrightarrow R(\mathbf{A}^T) = R(\mathbf{B}^T)$.
- $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \Longleftrightarrow R(\mathbf{A}) = R(\mathbf{B}) \Longleftrightarrow N(\mathbf{A}^T) = N(\mathbf{B}^T)$.

Exercises for section 4.2

4.2.1. Determine spanning sets for each of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}.$$

4.2.2. Consider a linear system of equations $\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{b}$.

(a) Explain why $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in R(\mathbf{A})$.

(b) Explain why a consistent system $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $N(\mathbf{A}) = \{\mathbf{0}\}$.

4.2.3. Suppose that $\mathbf{A}$ is a 3×3 matrix such that

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{N} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

span $R(\mathbf{A})$ and $N(\mathbf{A})$, respectively, and consider a linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ -7 \\ 0 \end{pmatrix}$.

(a) Explain why $\mathbf{Ax} = \mathbf{b}$ must be consistent.
(b) Explain why $\mathbf{Ax} = \mathbf{b}$ cannot have a unique solution.

4.2.4. If $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -5 \\ -6 \\ -7 \\ -7 \end{pmatrix}$, is $\mathbf{b} \in R(\mathbf{A})$?

4.2.5. Suppose that $\mathbf{A}$ is an $n \times n$ matrix.

(a) If $R(\mathbf{A}) = \Re^n$, explain why $\mathbf{A}$ must be nonsingular.
(b) If $\mathbf{A}$ is nonsingular, describe its four fundamental subspaces.

4.2.6. Consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{pmatrix}$.

(a) Do $\mathbf{A}$ and $\mathbf{B}$ have the same row space?
(b) Do $\mathbf{A}$ and $\mathbf{B}$ have the same column space?
(c) Do $\mathbf{A}$ and $\mathbf{B}$ have the same nullspace?
(d) Do $\mathbf{A}$ and $\mathbf{B}$ have the same left-hand nullspace?

4.2.7. If $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$ is a square matrix such that $N(\mathbf{A}_1) = R\left(\mathbf{A}_2^T\right)$, prove that $\mathbf{A}$ must be nonsingular.

4.2.8. Consider a linear system of equations $\mathbf{Ax} = \mathbf{b}$ for which $\mathbf{y}^T\mathbf{b} = 0$ for every $\mathbf{y} \in N\left(\mathbf{A}^T\right)$. Explain why this means the system must be consistent.

4.2.9. For matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times p}$, prove that

$$R(\mathbf{A} \mid \mathbf{B}) = R(\mathbf{A}) + R(\mathbf{B}).$$

4.2.10. Let $\mathbf{p}$ be one particular solution of a linear system $\mathbf{Ax} = \mathbf{b}$.

(a) Explain the significance of the set

$$\mathbf{p} + N(\mathbf{A}) = \{\mathbf{p} + \mathbf{h} \mid \mathbf{h} \in N(\mathbf{A})\}.$$

(b) If $rank(\mathbf{A}_{3\times 3}) = 1$, sketch a picture of $\mathbf{p} + N(\mathbf{A})$ in $\Re^3$.

(c) Repeat part (b) for the case when $rank(\mathbf{A}_{3\times 3}) = 2$.

4.2.11. Suppose that $\mathbf{Ax} = \mathbf{b}$ is a consistent system of linear equations, and let $\mathbf{a} \in R\left(\mathbf{A}^T\right)$. Prove that the inner product $\mathbf{a}^T\mathbf{x}$ is constant for all solutions to $\mathbf{Ax} = \mathbf{b}$.

4.2.12. For matrices such that the product $\mathbf{AB}$ is defined, explain why each of the following statements is true.

(a) $R(\mathbf{AB}) \subseteq R(\mathbf{A})$.

(b) $N(\mathbf{AB}) \supseteq N(\mathbf{B})$.

4.2.13. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a spanning set for $R(\mathbf{B})$. Prove that $\mathbf{A}(\mathcal{B}) = \{\mathbf{Ab}_1, \mathbf{Ab}_2, \ldots, \mathbf{Ab}_n\}$ is a spanning set for $R(\mathbf{AB})$.

4.3 LINEAR INDEPENDENCE

For a given set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ there may or may not exist dependency relationships in the sense that it may or may not be possible to express one vector as a linear combination of the others. For example, in the set

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 9 \\ -3 \\ 4 \end{pmatrix} \right\},$$

the third vector is a linear combination of the first two—i.e., $\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2$. Such a dependency always can be expressed in terms of a homogeneous equation by writing

$$3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

On the other hand, it is evident that there are no dependency relationships in the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

because no vector can be expressed as a combination of the others. Another way to say this is to state that there are no solutions for $\alpha_1, \alpha_2,$ and α_3 in the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$$

other than the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. These observations are the basis for the following definitions.

Linear Independence

A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be a ***linearly independent set*** whenever the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \tag{4.3.1}$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Whenever there is a nontrivial solution for the α 's (i.e., at least one $\alpha_i \neq 0$) in (4.3.1), the set $\mathcal{S}$ is said to be a ***linearly dependent set.*** In other words, linearly independent sets are those that contain no dependency relationships, and linearly dependent sets are those in which at least one vector is a combination of the others. We will agree that the empty set is always linearly independent.

It is important to realize that the concepts of linear independence and dependence are defined only for sets—individual vectors are neither linearly independent nor dependent. For example consider the following sets:

$$\mathcal{S}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{S}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{S}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

It should be clear that $\mathcal{S}_1$ and $\mathcal{S}_2$ are linearly independent sets while $\mathcal{S}_3$ is linearly dependent. This shows that individual vectors can simultaneously belong to linearly independent sets as well as linearly dependent sets. Consequently, it makes no sense to speak of "linearly independent vectors" or "linearly dependent vectors."

Example 4.3.1

Problem: Determine whether or not the set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}$$

is linearly independent.

Solution: Simply determine whether or not there exists a nontrivial solution for the α's in the homogeneous equation

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, equivalently, if there is a nontrivial solution to the homogeneous system

$$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}$, then $\mathbf{E_A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, and therefore there exist nontrivial solutions. Consequently, $\mathcal{S}$ is a linearly dependent set. Notice that one particular dependence relationship in $\mathcal{S}$ is revealed by $\mathbf{E_A}$ because it guarantees that $\mathbf{A}_{*3} = 3\mathbf{A}_{*1} + 2\mathbf{A}_{*2}$. This example indicates why the question of whether or not a subset of $\Re^m$ is linearly independent is really a question about whether or not the nullspace of an associated matrix is trivial. The following is a more formal statement of this fact.

Linear Independence and Matrices

Let $\mathbf{A}$ be an $m \times n$ matrix.

- Each of the following statements is equivalent to saying that the columns of $\mathbf{A}$ form a linearly independent set.
  - ▷ $N(\mathbf{A}) = \{\mathbf{0}\}$. (4.3.2)
  - ▷ $rank(\mathbf{A}) = n$. (4.3.3)
- Each of the following statements is equivalent to saying that the rows of $\mathbf{A}$ form a linearly independent set.
  - ▷ $N\left(\mathbf{A}^T\right) = \{\mathbf{0}\}$. (4.3.4)
  - ▷ $rank(\mathbf{A}) = m$. (4.3.5)
- When $\mathbf{A}$ is a square matrix, each of the following statements is equivalent to saying that $\mathbf{A}$ is nonsingular.
  - ▷ The columns of $\mathbf{A}$ form a linearly independent set. (4.3.6)
  - ▷ The rows of $\mathbf{A}$ form a linearly independent set. (4.3.7)

Proof. By definition, the columns of $\mathbf{A}$ are a linearly independent set when the only set of α's satisfying the homogeneous equation

$$\mathbf{0} = \alpha_1 \mathbf{A}_{*1} + \alpha_2 \mathbf{A}_{*2} + \cdots + \alpha_n \mathbf{A}_{*n} = \left(\mathbf{A}_{*1} \,|\, \mathbf{A}_{*2} \,|\, \cdots \,|\, \mathbf{A}_{*n}\right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, which is equivalent to saying $N(\mathbf{A}) = \{\mathbf{0}\}$. The fact that $N(\mathbf{A}) = \{\mathbf{0}\}$ is equivalent to $rank(\mathbf{A}) = n$ was demonstrated in (4.2.10). Statements (4.3.4) and (4.3.5) follow by replacing $\mathbf{A}$ by $\mathbf{A}^T$ in (4.3.2) and (4.3.3) and by using the fact that $rank(\mathbf{A}) = rank\left(\mathbf{A}^T\right)$. Statements (4.3.6) and (4.3.7) are simply special cases of (4.3.3) and (4.3.5). ∎

Example 4.3.2

Any set $\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}\}$ consisting of distinct unit vectors is a linearly independent set because $rank\left(\mathbf{e}_{i_1} \,|\, \mathbf{e}_{i_2} \,|\, \cdots \,|\, \mathbf{e}_{i_n}\right) = n$. For example, the set of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$ in $\Re^4$ is linearly independent because $rank \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$.

Example 4.3.3

Diagonal Dominance. A matrix $\mathbf{A}_{n\times n}$ is said to be ***diagonally dominant*** whenever

$$|a_{ii}| > \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| \quad \text{for each } i = 1, 2, \ldots, n.$$

That is, the magnitude of each diagonal entry exceeds the sum of the magnitudes of the off-diagonal entries in the corresponding row. Diagonally dominant matrices occur naturally in a wide variety of practical applications, and when solving a diagonally dominant system by Gaussian elimination, partial pivoting is never required—you are asked to provide the details in Exercise 4.3.15.

Problem: In 1900, Minkowski (p. 278) discovered that all diagonally dominant matrices are nonsingular. Establish the validity of Minkowski's result.

Solution: The strategy is to prove that if $\mathbf{A}$ is diagonally dominant, then $N(\mathbf{A}) = \{\mathbf{0}\}$, so that (4.3.2) together with (4.3.6) will provide the desired conclusion. Use an indirect argument—suppose there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$, and assume that x_k is the entry of maximum magnitude in $\mathbf{x}$. Focus on the k^{th} component of $\mathbf{A}\mathbf{x}$, and write the equation $\mathbf{A}_{k*}\mathbf{x} = 0$ as

$$a_{kk}x_k = -\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_j.$$

Taking absolute values of both sides and using the triangle inequality together with the fact that $|x_j| \leq |x_k|$ for each j produces

$$|a_{kk}|\,|x_k| = \left|\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_j\right| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}x_j| = \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\,|x_j| \leq \left(\sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\right)|x_k|.$$

But this implies that

$$|a_{kk}| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|,$$

which violates the hypothesis that $\mathbf{A}$ is diagonally dominant. Therefore, the assumption that there exists a nonzero vector in $N(\mathbf{A})$ must be false, so we may conclude that $N(\mathbf{A}) = \{\mathbf{0}\}$, and hence $\mathbf{A}$ is nonsingular.

Note: An alternate solution is given in Example 7.1.6 on p. 499.

Example 4.3.4

Vandermonde Matrices. Matrices of the form

$$\mathbf{V}_{m\times n} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix}$$

in which $x_i \neq x_j$ for all $i \neq j$ are called ***Vandermonde***[26] ***matrices.***

Problem: Explain why the columns in $\mathbf{V}$ constitute a linearly independent set whenever $n \leq m$.

Solution: According to (4.3.2), the columns of $\mathbf{V}$ form a linearly independent set if and only if $N(\mathbf{V}) = \{\mathbf{0}\}$. If

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.3.8}$$

then for each $i = 1, 2, \ldots, m$,

$$\alpha_0 + x_i\alpha_1 + x_i^2\alpha_2 + \cdots + x_i^{n-1}\alpha_{n-1} = 0.$$

This implies that the polynomial

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n-1}x^{n-1}$$

has m distinct roots—namely, the x_i 's. However, $\deg p(x) \leq n-1$ and the fundamental theorem of algebra guarantees that if $p(x)$ is not the zero polynomial, then $p(x)$ can have at most $n-1$ distinct roots. Therefore, (4.3.8) holds if and only if $\alpha_i = 0$ for all i, and thus (4.3.2) insures that the columns of $\mathbf{V}$ form a linearly independent set.

[26] This is named in honor of the French mathematician Alexandre-Theophile Vandermonde (1735–1796). He made a variety of contributions to mathematics, but he is best known perhaps for being the first European to give a logically complete exposition of the theory of determinants. He is regarded by many as being the founder of that theory. However, the matrix $\mathbf{V}$ (and an associated determinant) named after him, by Lebesgue, does not appear in Vandermonde's published work. Vandermonde's first love was music, and he took up mathematics only after he was 35 years old. He advocated the theory that all art and music rested upon a general principle that could be expressed mathematically, and he claimed that almost anyone could become a composer with the aid of mathematics.

Example 4.3.5

Problem: Given a set of m points $\mathcal{S} = \{(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_m, y_m)\}$ in which the x_i 's are distinct, explain why there is a unique polynomial

$$\ell(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{m-1} t^{m-1} \tag{4.3.9}$$

of degree $m-1$ that passes through each point in $\mathcal{S}$.

Solution: The coefficients α_i must satisfy the equations

$$\begin{aligned}
\alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots + \alpha_{m-1} x_1^{m-1} &= \ell(x_1) = y_1,\\
\alpha_0 + \alpha_1 x_2 + \alpha_2 x_2^2 + \cdots + \alpha_{m-1} x_2^{m-1} &= \ell(x_2) = y_2,\\
&\ \vdots\\
\alpha_0 + \alpha_1 x_m + \alpha_2 x_m^2 + \cdots + \alpha_{m-1} x_m^{m-1} &= \ell(x_m) = y_m.
\end{aligned}$$

Writing this $m \times m$ system as

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{m-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

reveals that the coefficient matrix is a square Vandermonde matrix, so the result of Example 4.3.4 guarantees that it is nonsingular. Consequently, the system has a unique solution, and thus there is one and only one possible set of coefficients for the polynomial $\ell(t)$ in (4.3.9). In fact, $\ell(t)$ must be given by

$$\ell(t) = \sum_{i=1}^{m} \left(y_i \frac{\prod_{j \neq i}^{m} (t - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)} \right).$$

Verify this by showing that the right-hand side is indeed a polynomial of degree $m-1$ that passes through the points in $\mathcal{S}$. The polynomial $\ell(t)$ is known as the ***Lagrange***[27] ***interpolation polynomial*** of degree $m-1$.

If $rank\,(\mathbf{A}_{m\times n}) < n$, then the columns of $\mathbf{A}$ must be a dependent set—recall (4.3.3). For such matrices we often wish to extract a ***maximal linearly independent subset*** of columns—i.e., a linearly independent set containing as many columns from $\mathbf{A}$ as possible. Although there can be several ways to make such a selection, the basic columns in $\mathbf{A}$ always constitute one solution.

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27 Joseph Louis Lagrange (1736–1813), born in Turin, Italy, is considered by many to be one of the two greatest mathematicians of the eighteenth century—Euler is the other. Lagrange occupied Euler's vacated position in 1766 in Berlin at the court of Frederick the Great who wrote that "the greatest king in Europe" wishes to have at his court "the greatest mathematician of Europe." After 20 years, Lagrange left Berlin and eventually moved to France. Lagrange's mathematical contributions are extremely wide and deep, but he had a particularly strong influence on the way mathematical research evolved. He was the first of the top-class mathematicians to recognize the weaknesses in the foundations of calculus, and he was among the first to attempt a rigorous development.

Maximal Independent Subsets

If $rank\,(\mathbf{A}_{m\times n}) = r$, then the following statements hold.

- Any maximal independent subset of columns from $\mathbf{A}$ contains exactly r columns. (4.3.10)
- Any maximal independent subset of rows from $\mathbf{A}$ contains exactly r rows. (4.3.11)
- In particular, the r basic columns in $\mathbf{A}$ constitute one maximal independent subset of columns from $\mathbf{A}$. (4.3.12)

Proof. Exactly the same linear relationships that exist among the columns of $\mathbf{A}$ must also hold among the columns of $\mathbf{E_A}$—by (3.9.6). This guarantees that a subset of columns from $\mathbf{A}$ is linearly independent if and only if the columns in the corresponding positions in $\mathbf{E_A}$ are an independent set. Let

$$\mathbf{C} = \big(\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \cdots \,|\, \mathbf{c}_k\big)$$

be a matrix that contains an independent subset of columns from $\mathbf{E_A}$ so that $rank\,(\mathbf{C}) = k$—recall (4.3.3). Since each column in $\mathbf{E_A}$ is a combination of the r basic (unit) columns in $\mathbf{E_A}$, there are scalars β_{ij} such that $\mathbf{c}_j = \sum_{i=1}^{r} \beta_{ij}\mathbf{e}_i$ for $j = 1, 2, \ldots, k$. These equations can be written as the single matrix equation

$$\big(\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \cdots \,|\, \mathbf{c}_k\big) = \big(\mathbf{e}_1 \,|\, \mathbf{e}_2 \,|\, \cdots \,|\, \mathbf{e}_r\big) \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{rk} \end{pmatrix}$$

or

$$\mathbf{C}_{m\times k} = \begin{pmatrix} \mathbf{I}_r \\ \mathbf{0} \end{pmatrix} \mathbf{B}_{r\times k} = \begin{pmatrix} \mathbf{B}_{r\times k} \\ \mathbf{0} \end{pmatrix}, \qquad \text{where} \qquad \mathbf{B} = [\beta_{ij}].$$

Consequently, $r \geq rank\,(\mathbf{C}) = k$, and therefore any independent subset of columns from $\mathbf{E_A}$—and hence any independent set of columns from $\mathbf{A}$—cannot contain more than r vectors. Because the r basic (unit) columns in $\mathbf{E_A}$ form an independent set, the r basic columns in $\mathbf{A}$ constitute an independent set. This proves (4.3.10) and (4.3.12). The proof of (4.3.11) follows from the fact that $rank\,(\mathbf{A}) = rank\,(\mathbf{A}^T)$—recall (3.9.11). ■

Basic Facts of Independence

For a nonempty set of vectors $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ in a space $\mathcal{V}$, the following statements are true.

- If $\mathcal{S}$ contains a linearly dependent subset, then $\mathcal{S}$ itself must be linearly dependent. (4.3.13)
- If $\mathcal{S}$ is linearly independent, then every subset of $\mathcal{S}$ is also linearly independent. (4.3.14)
- If $\mathcal{S}$ is linearly independent and if $\mathbf{v} \in \mathcal{V}$, then the ***extension set*** $\mathcal{S}_{ext} = \mathcal{S} \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin span(\mathcal{S})$. (4.3.15)
- If $\mathcal{S} \subseteq \Re^m$ and if $n > m$, then $\mathcal{S}$ must be linearly dependent. (4.3.16)

Proof of (4.3.13). Suppose that $\mathcal{S}$ contains a linearly dependent subset, and, for the sake of convenience, suppose that the vectors in $\mathcal{S}$ have been permuted so that this dependent subset is $\mathcal{S}_{dep} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. According to the definition of dependence, there must be scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all of which are zero, such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$. This means that we can write

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + 0\mathbf{u}_{k+1} + \cdots + 0\mathbf{u}_n = \mathbf{0},$$

where not all of the scalars are zero, and hence $\mathcal{S}$ is linearly dependent.

Proof of (4.3.14). This is an immediate consequence of (4.3.13).

Proof of (4.3.15). If $\mathcal{S}_{ext}$ is linearly independent, then $\mathbf{v} \notin span(\mathcal{S})$, for otherwise $\mathbf{v}$ would be a combination of vectors from $\mathcal{S}$ thus forcing $\mathcal{S}_{ext}$ to be a dependent set. Conversely, suppose $\mathbf{v} \notin span(\mathcal{S})$. To prove that $\mathcal{S}_{ext}$ is linearly independent, consider a linear combination

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n + \alpha_{n+1}\mathbf{v} = \mathbf{0}. \tag{4.3.17}$$

It must be the case that $\alpha_{n+1} = 0$, for otherwise $\mathbf{v}$ would be a combination of vectors from $\mathcal{S}$. Consequently,

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}.$$

But this implies that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

because $\mathcal{S}$ is linearly independent. Therefore, the only solution for the α's in (4.3.17) is the trivial set, and hence $\mathcal{S}_{ext}$ must be linearly independent.

Proof of (4.3.16). This follows from (4.3.3) because if the $\mathbf{u}_i$'s are placed as columns in a matrix $\mathbf{A}_{m\times n}$, then $rank(\mathbf{A}) \le m < n$. ■

Example 4.3.6

Let $\mathcal{V}$ be the vector space of real-valued functions of a real variable, and let $\mathcal{S} = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a set of functions that are $n-1$ times differentiable. The ***Wronski***[28] ***matrix*** is defined to be

$$\mathbf{W}(x) = \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}.$$

Problem: If there is at least one point $x = x_0$ such that $\mathbf{W}(x_0)$ is nonsingular, prove that $\mathcal{S}$ must be a linearly independent set.

Solution: Suppose that

$$0 = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x) \tag{4.3.18}$$

for all values of x. When $x = x_0$, it follows that

$$\begin{aligned} 0 &= \alpha_1 f_1(x_0) + \alpha_2 f_2(x_0) + \cdots + \alpha_n f_n(x_0), \\ 0 &= \alpha_1 f_1'(x_0) + \alpha_2 f_2'(x_0) + \cdots + \alpha_n f_n'(x_0), \\ &\vdots \\ 0 &= \alpha_1 f_1^{(n-1)}(x_0) + \alpha_2 f_2^{(n-1)}(x_0) + \cdots + \alpha_n f_n^{(n-1)}(x_0), \end{aligned}$$

which means that $\mathbf{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in N\big(\mathbf{W}(x_0)\big)$. But $N\big(\mathbf{W}(x_0)\big) = \{\mathbf{0}\}$ because $\mathbf{W}(x_0)$ is nonsingular, and hence $\mathbf{v} = \mathbf{0}$. Therefore, the only solution for the α's in (4.3.18) is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ thereby insuring that $\mathcal{S}$ is linearly independent.

[28] This matrix is named in honor of the Polish mathematician Jozef Maria Höené Wronski (1778–1853), who studied four special forms of determinants, one of which was the determinant of the matrix that bears his name. Wronski was born to a poor family near Poznan, Poland, but he studied in Germany and spent most of his life in France. He is reported to have been an egotistical person who wrote in an exhaustively wearisome style. Consequently, almost no one read his work. Had it not been for his lone follower, Ferdinand Schweins (1780–1856) of Heidelberg, Wronski would probably be unknown today. Schweins preserved and extended Wronski's results in his own writings, which in turn received attention from others. Wronski also wrote on philosophy. While trying to reconcile Kant's metaphysics with Leibniz's calculus, Wronski developed a social philosophy called "Messianism" that was based on the belief that absolute truth could be achieved through mathematics.

For example, to verify that the set of polynomials $\mathcal{P} = \{1, x, x^2, \ldots, x^n\}$ is linearly independent, observe that the associated Wronski matrix

$$\mathbf{W}(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & \cdots & nx^{n-1} \\ 0 & 0 & 2 & \cdots & n(n-1)x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n! \end{pmatrix}$$

is triangular with nonzero diagonal entries. Consequently, $\mathbf{W}(x)$ is nonsingular for every value of x, and hence $\mathcal{P}$ must be an independent set.

Exercises for section 4.3

4.3.1. Determine which of the following sets are linearly independent. For those sets that are linearly dependent, write one of the vectors as a linear combination of the others.

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \right\}$,

(b) $\{(1\ \ 2\ \ 3), (0\ \ 4\ \ 5), (0\ \ 0\ \ 6), (1\ \ 1\ \ 1)\}$,

(c) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$,

(d) $\{(2\ \ 2\ \ 2\ \ 2), (2\ \ 2\ \ 0\ \ 2), (2\ \ 0\ \ 2\ \ 2)\}$,

(e) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$.

4.3.2. Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 2 \\ 6 & 3 & 2 & 2 \end{pmatrix}$.

(a) Determine a maximal linearly independent subset of columns from $\mathbf{A}$.

(b) Determine the total number of linearly independent subsets that can be constructed using the columns of $\mathbf{A}$.

4.3.3. Suppose that in a population of a million children the height of each one is measured at ages 1 year, 2 years, and 3 years, and accumulate this data in a matrix

$$\begin{array}{c} \\ \#1 \\ \#2 \\ \vdots \\ \#i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} 1\text{ yr} & 2\text{ yr} & 3\text{ yr} \end{array} \\ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ \vdots & \vdots & \vdots \\ h_{i1} & h_{i2} & h_{i3} \\ \vdots & \vdots & \vdots \end{pmatrix} \end{array} = \mathbf{H}.$$

Explain why there are at most three "independent children" in the sense that the heights of all the other children must be a combination of these "independent" ones.

4.3.4. Consider a particular species of wildflower in which each plant has several stems, leaves, and flowers, and for each plant let the following hold.

$S =$ the average stem length (in inches).
$L =$ the average leaf width (in inches).
$F =$ the number of flowers.

Four particular plants are examined, and the information is tabulated in the following matrix:

$$\mathbf{A} = \begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{ccc} S & L & F \end{array} \\ \begin{pmatrix} 1 & 1 & 10 \\ 2 & 1 & 12 \\ 2 & 2 & 15 \\ 3 & 2 & 17 \end{pmatrix} \end{array}.$$

For these four plants, determine whether or not there exists a linear relationship between S, L, and F. In other words, do there exist constants $\alpha_0, \alpha_1, \alpha_2,$ and α_3 such that $\alpha_0 + \alpha_1 S + \alpha_2 L + \alpha_3 F = 0$?

4.3.5. Let $\mathcal{S} = \{\mathbf{0}\}$ be the set containing only the zero vector.

(a) Explain why $\mathcal{S}$ must be linearly dependent.
(b) Explain why any set containing a zero vector must be linearly dependent.

4.3.6. If $\mathbf{T}$ is a triangular matrix in which each $t_{ii} \neq 0$, explain why the rows and columns of $\mathbf{T}$ must each be linearly independent sets.

4.3.7. Determine whether or not the following set of matrices is a linearly independent set:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

4.3.8. Without doing any computation, determine whether the following matrix is singular or nonsingular:

$$\mathbf{A} = \begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & n & 1 & \cdots & 1 \\ 1 & 1 & n & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}_{n \times n}.$$

4.3.9. In theory, determining whether or not a given set is linearly independent is a well-defined problem with a straightforward solution. In practice, however, this problem is often not so well defined because it becomes clouded by the fact that we usually cannot use exact arithmetic, and contradictory conclusions may be produced depending upon the precision of the arithmetic. For example, let

$$\mathcal{S} = \left\{ \begin{pmatrix} .1 \\ .4 \\ .7 \end{pmatrix}, \begin{pmatrix} .2 \\ .5 \\ .8 \end{pmatrix}, \begin{pmatrix} .3 \\ .6 \\ .901 \end{pmatrix} \right\}.$$

(a) Use exact arithmetic to determine whether or not $\mathcal{S}$ is linearly independent.
(b) Use 3-digit arithmetic (without pivoting or scaling) to determine whether or not $\mathcal{S}$ is linearly independent.

4.3.10. If $\mathbf{A}_{m \times n}$ is a matrix such that $\sum_{j=1}^{n} a_{ij} = 0$ for each $i = 1, 2, \ldots, m$ (i.e., each row sum is 0), explain why the columns of $\mathbf{A}$ are a linearly dependent set, and hence $rank\,(\mathbf{A}) < n$.

4.3.11. If $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a linearly independent subset of $\Re^{m \times 1}$, and if $\mathbf{P}_{m \times m}$ is a nonsingular matrix, explain why the set

$$\mathbf{P}(\mathcal{S}) = \{\mathbf{Pu}_1, \mathbf{Pu}_2, \ldots, \mathbf{Pu}_n\}$$

must also be a linearly independent set. Is this result still true if $\mathbf{P}$ is singular?

4.3.12. Suppose that $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\Re^m$. Prove that $\mathcal{S}$ is linearly independent if and only if the set

$$\mathcal{S}' = \left\{ \mathbf{u}_1,\ \sum_{i=1}^{2} \mathbf{u}_i,\ \sum_{i=1}^{3} \mathbf{u}_i,\ \ldots,\ \sum_{i=1}^{n} \mathbf{u}_i \right\}$$

is linearly independent.

4.3.13. Which of the following sets of functions are linearly independent?

(a) $\{\sin x,\ \cos x,\ x \sin x\}$.

(b) $\{e^x,\ xe^x,\ x^2 e^x\}$.

(c) $\{\sin^2 x,\ \cos^2 x,\ \cos 2x\}$.

4.3.14. Prove that the converse of the statement given in Example 4.3.6 is false by showing that $\mathcal{S} = \{x^3,\ |x|^3\}$ is a linearly independent set, but the associated Wronski matrix $\mathbf{W}(x)$ is singular for all values of x.

4.3.15. If $\mathbf{A}^T$ is diagonally dominant, explain why partial pivoting is not needed when solving $\mathbf{Ax} = \mathbf{b}$ by Gaussian elimination. **Hint:** If after one step of Gaussian elimination we have

$$\mathbf{A} = \begin{pmatrix} \alpha & \mathbf{d}^T \\ \mathbf{c} & \mathbf{B} \end{pmatrix} \xrightarrow{\textit{one step}} \begin{pmatrix} \alpha & \mathbf{d}^T \\ \mathbf{0} & \mathbf{B} - \frac{\mathbf{c}\mathbf{d}^T}{\alpha} \end{pmatrix},$$

show that $\mathbf{A}^T$ being diagonally dominant implies $\mathbf{X} = \left(\mathbf{B} - \frac{\mathbf{c}\mathbf{d}^T}{\alpha}\right)^T$ must also be diagonally dominant.

4.4 BASIS AND DIMENSION

Recall from §4.1 that $\mathcal{S}$ is a spanning set for a space $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ is a linear combination of vectors in $\mathcal{S}$. However, spanning sets can contain redundant vectors. For example, a subspace $\mathcal{L}$ defined by a line through the origin in $\Re^2$ may be spanned by any number of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathcal{L}$, but any one of the vectors $\{\mathbf{v}_i\}$ by itself will suffice. Similarly, a plane $\mathcal{P}$ through the origin in $\Re^3$ can be spanned in many different ways, but the parallelogram law indicates that a minimal spanning set need only be an independent set of two vectors from $\mathcal{P}$. These considerations motivate the following definition.

Basis

A linearly independent spanning set for a vector space $\mathcal{V}$ is called a ***basis*** for $\mathcal{V}$.

It can be proven that every vector space $\mathcal{V}$ possesses a basis—details for the case when $\mathcal{V} \subseteq \Re^m$ are asked for in the exercises. Just as in the case of spanning sets, a space can possess many different bases.

Example 4.4.1

- The unit vectors $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\Re^n$ are a basis for $\Re^n$. This is called the ***standard basis*** for $\Re^n$.

- If $\mathbf{A}$ is an $n \times n$ nonsingular matrix, then the set of rows in $\mathbf{A}$ as well as the set of columns from $\mathbf{A}$ constitute a basis for $\Re^n$. For example, (4.3.3) insures that the columns of $\mathbf{A}$ are linearly independent, and we know they span $\Re^n$ because $R(\mathbf{A}) = \Re^n$—recall Exercise 4.2.5(b).

- For the trivial vector space $\mathcal{Z} = \{\mathbf{0}\}$, there is no nonempty linearly independent spanning set. Consequently, the empty set is considered to be a basis for $\mathcal{Z}$.

- The set $\{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space of polynomials having degree n or less.

- The infinite set $\{1, x, x^2, \ldots\}$ is a basis for the vector space of all polynomials. It should be clear that no finite basis is possible.

Spaces that possess a basis containing an infinite number of vectors are referred to as ***infinite-dimensional spaces,*** and those that have a finite basis are called ***finite-dimensional spaces.*** This is often a line of demarcation in the study of vector spaces. A complete theoretical treatment would include the analysis of infinite-dimensional spaces, but this text is primarily concerned with finite-dimensional spaces over the real or complex numbers. It can be shown that, in effect, this amounts to analyzing $\Re^n$ or $\mathcal{C}^n$ and their subspaces.

The original concern of this section was to try to eliminate redundancies from spanning sets so as to provide spanning sets containing a minimal number of vectors. The following theorem shows that a basis is indeed such a set.

Characterizations of a Basis

Let $\mathcal{V}$ be a subspace of $\Re^m$, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq \mathcal{V}$. The following statements are equivalent.

- $\mathcal{B}$ is a basis for $\mathcal{V}$. (4.4.1)
- $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$. (4.4.2)
- $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$. (4.4.3)

Proof. First argue that (4.4.1) $\Longrightarrow$ (4.4.2) $\Longrightarrow$ (4.4.1), and then show (4.4.1) is equivalent to (4.4.3).

Proof of (4.4.1) $\Longrightarrow$ (4.4.2). First suppose that $\mathcal{B}$ is a basis for $\mathcal{V}$, and prove that $\mathcal{B}$ is a minimal spanning set by using an indirect argument—i.e., assume that $\mathcal{B}$ is *not* minimal, and show that this leads to a contradiction. If $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a basis for $\mathcal{V}$ in which $k < n$, then each $\mathbf{b}_j$ can be written as a combination of the $\mathbf{x}_i$'s. That is, there are scalars α_{ij} such that

$$\mathbf{b}_j = \sum_{i=1}^{k} \alpha_{ij}\mathbf{x}_i \quad \text{for } j = 1, 2, \ldots, n. \tag{4.4.4}$$

If the $\mathbf{b}$'s and $\mathbf{x}$'s are placed as columns in matrices

$$\mathbf{B}_{m\times n} = \big(\mathbf{b}_1 \,|\, \mathbf{b}_2 \,|\, \cdots \,|\, \mathbf{b}_n\big) \quad \text{and} \quad \mathbf{X}_{m\times k} = \big(\mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\, \cdots \,|\, \mathbf{x}_k\big),$$

then (4.4.4) can be expressed as the matrix equation

$$\mathbf{B} = \mathbf{XA}, \quad \text{where,} \quad \mathbf{A}_{k\times n} = [\alpha_{ij}].$$

Since the rank of a matrix cannot exceed either of its size dimensions, and since $k < n$, we have that $rank(\mathbf{A}) \le k < n$, so that $N(\mathbf{A}) \ne \{\mathbf{0}\}$—recall (4.2.10). If $\mathbf{z} \ne \mathbf{0}$ is such that $\mathbf{Az} = \mathbf{0}$, then $\mathbf{Bz} = \mathbf{0}$. But this is impossible because the columns of $\mathbf{B}$ are linearly independent, and hence $N(\mathbf{B}) = \{\mathbf{0}\}$—recall (4.3.2). Therefore, the supposition that there exists a basis for $\mathcal{V}$ containing fewer than n vectors must be false, and we may conclude that $\mathcal{B}$ is indeed a minimal spanning set.

Proof of (4.4.2) $\Longrightarrow$ (4.4.1). If $\mathcal{B}$ is a minimal spanning set, then $\mathcal{B}$ must be a *linearly independent* spanning set. Otherwise, some $\mathbf{b}_i$ would be a linear combination of the other $\mathbf{b}$'s, and the set

$$\mathcal{B}' = \{\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_n\}$$

would still span $\mathcal{V}$, but $\mathcal{B}'$ would contain fewer vectors than $\mathcal{B}$, which is impossible because $\mathcal{B}$ is a *minimal* spanning set.

Proof of (4.4.3) $\Longrightarrow$ (4.4.1). If $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$, but not a basis for $\mathcal{V}$, then there exists a vector $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v} \notin span(\mathcal{B})$. This means that the extension set

$$\mathcal{B} \cup \{\mathbf{v}\} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n, \mathbf{v}\}$$

is linearly independent—recall (4.3.15). But this is impossible because $\mathcal{B}$ is a *maximal* linearly independent subset of $\mathcal{V}$. Therefore, $\mathcal{B}$ is a basis for $\mathcal{V}$.

Proof of (4.4.1) $\Longrightarrow$ (4.4.3). Suppose that $\mathcal{B}$ is a basis for $\mathcal{V}$, but not a maximal linearly independent subset of $\mathcal{V}$, and let

$$\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\} \subseteq \mathcal{V}, \quad \text{where} \quad k > n$$

be a maximal linearly independent subset—recall that (4.3.16) insures the existence of such a set. The previous argument shows that $\mathcal{Y}$ must be a basis for $\mathcal{V}$. But this is impossible because we already know that a basis must be a minimal spanning set, and $\mathcal{B}$ is a spanning set containing fewer vectors than $\mathcal{Y}$. Therefore, $\mathcal{B}$ must be a maximal linearly independent subset of $\mathcal{V}$. ∎

Although a space $\mathcal{V}$ can have many different bases, the preceding result guarantees that all bases for $\mathcal{V}$ contain the same number of vectors. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are each a basis for $\mathcal{V}$, then each is a minimal spanning set, and thus they must contain the same number of vectors. As we are about to see, this number is quite important.

Dimension

The ***dimension*** of a vector space $\mathcal{V}$ is defined to be

$$\begin{aligned}\dim \mathcal{V} &= \text{number of vectors in any basis for } \mathcal{V} \\ &= \text{number of vectors in any minimal spanning set for } \mathcal{V} \\ &= \text{number of vectors in any maximal independent subset of } \mathcal{V}.\end{aligned}$$

Example 4.4.2

- If $\mathcal{Z} = \{\mathbf{0}\}$ is the trivial subspace, then $\dim \mathcal{Z} = 0$ because the basis for this space is the empty set.

- If $\mathcal{L}$ is a line through the origin in $\Re^3$, then $\dim \mathcal{L} = 1$ because a basis for $\mathcal{L}$ consists of any nonzero vector lying along $\mathcal{L}$.
- If $\mathcal{P}$ is a plane through the origin in $\Re^3$, then $\dim \mathcal{P} = 2$ because a minimal spanning set for $\mathcal{P}$ must contain two vectors from $\mathcal{P}$.
- $\dim \Re^3 = 3$ because the three unit vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ constitute a basis for $\Re^3$.
- $\dim \Re^n = n$ because the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\Re^n$ form a basis.

Example 4.4.3

Problem: If $\mathcal{V}$ is an n-dimensional space, explain why every independent subset $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathcal{V}$ containing n vectors must be a basis for $\mathcal{V}$.

Solution: $\dim \mathcal{V} = n$ means that every subset of $\mathcal{V}$ that contains more than n vectors must be linearly dependent. Consequently, $\mathcal{S}$ is a maximal independent subset of $\mathcal{V}$, and hence $\mathcal{S}$ is a basis for $\mathcal{V}$.

Example 4.4.2 shows that in a loose sense the dimension of a space is a measure of the amount of "stuff" in the space—a plane $\mathcal{P}$ in $\Re^3$ has more "stuff" in it than a line $\mathcal{L}$, but $\mathcal{P}$ contains less "stuff" than the entire space $\Re^3$. Recall from the discussion in §4.1 that subspaces of $\Re^n$ are generalized versions of flat surfaces through the origin. The concept of dimension gives us a way to distinguish between these "flat" objects according to how much "stuff" they contain—much the same way we distinguish between lines and planes in $\Re^3$. Another way to think about dimension is in terms of "degrees of freedom." In the trivial space $\mathcal{Z}$, there are no degrees of freedom—you can move nowhere—whereas on a line there is one degree of freedom—length; in a plane there are two degrees of freedom—length and width; in $\Re^3$ there are three degrees of freedom—length, width, and height; etc.

It is important not to confuse the dimension of a vector space $\mathcal{V}$ with the number of components contained in the individual vectors from $\mathcal{V}$. For example, if $\mathcal{P}$ is a plane through the origin in $\Re^3$, then $\dim \mathcal{P} = 2$, but the individual vectors in $\mathcal{P}$ each have three components. Although the dimension of a space $\mathcal{V}$ and the number of components contained in the individual vectors from $\mathcal{V}$ need not be the same, they are nevertheless related. For example, if $\mathcal{V}$ is a subspace of $\Re^n$, then (4.3.16) insures that no linearly independent subset in $\mathcal{V}$ can contain more than n vectors and, consequently, $\dim \mathcal{V} \leq n$. This observation generalizes to produce the following theorem.

Subspace Dimension

For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$, the following statements are true.

- $\dim \mathcal{M} \leq \dim \mathcal{N}$. (4.4.5)
- If $\dim \mathcal{M} = \dim \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$. (4.4.6)

Proof. Let $\dim \mathcal{M} = m$ and $\dim \mathcal{N} = n$, and use an indirect argument to prove (4.4.5). If it were the case that $m > n$, then there would exist a linearly independent subset of $\mathcal{N}$ (namely, a basis for $\mathcal{M}$) containing more than n vectors. But this is impossible because $\dim \mathcal{N}$ is the size of a maximal independent subset of $\mathcal{N}$. Thus $m \leq n$. Now prove (4.4.6). If $m = n$ but $\mathcal{M} \neq \mathcal{N}$, then there exists a vector $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{N}$ but $\mathbf{x} \notin \mathcal{M}$. If $\mathcal{B}$ is a basis for $\mathcal{M}$, then $\mathbf{x} \notin span(\mathcal{B})$, and the extension set $\mathcal{E} = \mathcal{B} \cup \{\mathbf{x}\}$ is a linearly independent subset of $\mathcal{N}$—recall (4.3.15). But $\mathcal{E}$ contains $m + 1 = n + 1$ vectors, which is impossible because $\dim \mathcal{N} = n$ is the size of a maximal independent subset of $\mathcal{N}$. Hence $\mathcal{M} = \mathcal{N}$. ∎

Let's now find bases and dimensions for the four fundamental subspaces of an $m \times n$ matrix $\mathbf{A}$ of rank r, and let's start with $R(\mathbf{A})$. The entire set of columns in $\mathbf{A}$ spans $R(\mathbf{A})$, but they won't form a basis when there are dependencies among some of the columns. However, the set of *basic* columns in $\mathbf{A}$ is also a spanning set—recall (4.2.8)—and the basic columns always constitute a linearly independent set because no basic column can be a combination of other basic columns (otherwise it wouldn't be basic). So, the set of basic columns is a basis for $R(\mathbf{A})$, and, since there are r of them, $\dim R(\mathbf{A}) = r = rank(\mathbf{A})$.

Similarly, the entire set of rows in $\mathbf{A}$ spans $R(\mathbf{A}^T)$, but the set of all rows is not a basis when dependencies exist. Recall from (4.2.7) that if $\mathbf{U} = \begin{pmatrix} \mathbf{C}_{r \times n} \\ \mathbf{0} \end{pmatrix}$ is any row echelon form that is row equivalent to $\mathbf{A}$, then the rows of $\mathbf{C}$ span $R(\mathbf{A}^T)$. Since $rank(\mathbf{C}) = r$, (4.3.5) insures that the rows of $\mathbf{C}$ are linearly independent. Consequently, the rows in $\mathbf{C}$ are a basis for $R(\mathbf{A}^T)$, and, since there are r of them, $\dim R(\mathbf{A}^T) = r = rank(\mathbf{A})$. Older texts referred to $\dim R(\mathbf{A}^T)$ as the *row rank* of $\mathbf{A}$, while $\dim R(\mathbf{A})$ was called the *column rank* of $\mathbf{A}$, and it was a major task to prove that the row rank always agrees with the column rank. Notice that this is a consequence of the discussion above where it was observed that $\dim R(\mathbf{A}^T) = r = \dim R(\mathbf{A})$.

Turning to the nullspaces, let's first examine $N(\mathbf{A}^T)$. We know from (4.2.12) that if $\mathbf{P}$ is a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, then the last $m - r$ rows in $\mathbf{P}$ span $N(\mathbf{A}^T)$. Because the set of rows in a nonsingular matrix is a linearly independent set, and because any subset of an independent set is again independent—see (4.3.7) and (4.3.14)—it follows that the last $m-r$ rows in $\mathbf{P}$ are linearly independent, and hence they constitute a basis for $N\left(\mathbf{A}^T\right)$. And this implies $\dim N\left(\mathbf{A}^T\right) = m-r$ (i.e., the number of rows in $\mathbf{A}$ minus the rank of $\mathbf{A}$). Replacing $\mathbf{A}$ by $\mathbf{A}^T$ shows that $\dim N\left(\mathbf{A}^{T^T}\right) = \dim N(\mathbf{A})$ is the number of rows in $\mathbf{A}^T$ minus $rank\left(\mathbf{A}^T\right)$. But $rank\left(\mathbf{A}^T\right) = rank(\mathbf{A}) = r$, so $\dim N(\mathbf{A}) = n-r$. We deduced $\dim N(\mathbf{A})$ without exhibiting a specific basis, but a basis for $N(\mathbf{A})$ is easy to describe. Recall that the set $\mathcal{H}$ containing the $\mathbf{h}_i$'s appearing in the general solution (4.2.9) of $\mathbf{Ax} = \mathbf{0}$ spans $N(\mathbf{A})$. Since there are exactly $n-r$ vectors in $\mathcal{H}$, and since $\dim N(\mathbf{A}) = n-r$, $\mathcal{H}$ is a minimal spanning set, so, by (4.4.2), $\mathcal{H}$ must be a basis for $N(\mathbf{A})$. Below is a summary of facts uncovered above.

Fundamental Subspaces—Dimension and Bases

For an $m \times n$ matrix of real numbers such that $rank(\mathbf{A}) = r$,

- $\dim R(\mathbf{A}) = r,$ (4.4.7)
- $\dim N(\mathbf{A}) = n-r,$ (4.4.8)
- $\dim R\left(\mathbf{A}^T\right) = r,$ (4.4.9)
- $\dim N\left(\mathbf{A}^T\right) = m-r.$ (4.4.10)

Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, and let $\mathcal{H}$ be the set of $\mathbf{h}_i$'s appearing in the general solution (4.2.9) of $\mathbf{Ax} = \mathbf{0}$.

- The basic columns of $\mathbf{A}$ form a basis for $R(\mathbf{A})$. (4.4.11)
- The nonzero rows of $\mathbf{U}$ form a basis for $R\left(\mathbf{A}^T\right)$. (4.4.12)
- The set $\mathcal{H}$ is a basis for $N(\mathbf{A})$. (4.4.13)
- The last $m-r$ rows of $\mathbf{P}$ form a basis for $N\left(\mathbf{A}^T\right)$. (4.4.14)

For matrices with complex entries, the above statements remain valid provided that $\mathbf{A}^T$ is replaced with $\mathbf{A}^*$.

Statements (4.4.7) and (4.4.8) combine to produce the following theorem.

Rank Plus Nullity Theorem

- $\dim R(\mathbf{A}) + \dim N(\mathbf{A}) = n$ for all $m \times n$ matrices. (4.4.15)

In loose terms, this is a kind of conservation law—it says that as the amount of "stuff" in $R(\mathbf{A})$ increases, the amount of "stuff" in $N(\mathbf{A})$ must decrease, and vice versa. The phrase *rank plus nullity* is used because $\dim R(\mathbf{A})$ is the rank of $\mathbf{A}$, and $\dim N(\mathbf{A})$ was traditionally known as the ***nullity of*** $\mathbf{A}$.

Example 4.4.4

Problem: Determine the dimension as well as a basis for the space spanned by

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}.$$

Solution 1: Place the vectors as columns in a matrix $\mathbf{A}$, and reduce

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix} \longrightarrow \mathbf{E_A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $span(\mathcal{S}) = R(\mathbf{A})$, we have

$$\dim\big(span(\mathcal{S})\big) = \dim R(\mathbf{A}) = rank(\mathbf{A}) = 2.$$

The basic columns $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$ are a basis for $R(\mathbf{A}) = span(\mathcal{S})$. Other bases are also possible. Examining $\mathbf{E_A}$ reveals that any two vectors in $\mathcal{S}$ form an independent set, and therefore any pair of vectors from $\mathcal{S}$ constitutes a basis for $span(\mathcal{S})$.

Solution 2: Place the vectors from $\mathcal{S}$ as rows in a matrix $\mathbf{B}$, and reduce $\mathbf{B}$ to row echelon form:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 5 & 6 & 7 \end{pmatrix} \longrightarrow \mathbf{U} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This time we have $span(\mathcal{S}) = R\left(\mathbf{B}^T\right)$, so that

$$\dim\big(span(\mathcal{S})\big) = \dim R\left(\mathbf{B}^T\right) = rank(\mathbf{B}) = rank(\mathbf{U}) = 2,$$

and a basis for $span(\mathcal{S}) = R\left(\mathbf{B}^T\right)$ is given by the nonzero rows in $\mathbf{U}$.

Example 4.4.5

Problem: If $\mathcal{S}_r = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent subset of an n-dimensional space $\mathcal{V}$, where $r < n$, explain why it must be possible to find extension vectors $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ from $\mathcal{V}$ such that

$$\mathcal{S}_n = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$$

is a basis for $\mathcal{V}$.

Solution 1: $r < n$ means that $span(\mathcal{S}_r) \neq \mathcal{V}$, and hence there exists a vector $\mathbf{v}_{r+1} \in \mathcal{V}$ such that $\mathbf{v}_{r+1} \notin span(\mathcal{S}_r)$. The extension set $\mathcal{S}_{r+1} = \mathcal{S}_r \cup \{\mathbf{v}_{r+1}\}$ is an independent subset of $\mathcal{V}$ containing $r+1$ vectors—recall (4.3.15). Repeating this process generates independent subsets $\mathcal{S}_{r+2}, \mathcal{S}_{r+3}, \ldots,$ and eventually leads to a maximal independent subset $\mathcal{S}_n \subset \mathcal{V}$ containing n vectors.

Solution 2: The first solution shows that it is theoretically possible to find extension vectors, but the argument given is not much help in actually computing them. It is easy to remedy this situation. Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be any basis for $\mathcal{V}$, and place the given $\mathbf{v}_i$'s along with the $\mathbf{b}_i$'s as columns in a matrix

$$\mathbf{A} = \big(\mathbf{v}_1 \,|\, \cdots \,|\, \mathbf{v}_r \,|\, \mathbf{b}_1 \,|\, \cdots \,|\, \mathbf{b}_n\big).$$

Clearly, $R(\mathbf{A}) = \mathcal{V}$ so that the set of basic columns from $\mathbf{A}$ is a basis for $\mathcal{V}$. Observe that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ are basic columns in $\mathbf{A}$ because no one of these is a combination of preceding ones. Therefore, the remaining $n-r$ basic columns must be a subset of $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$—say they are $\left\{\mathbf{b}_{j_1}, \mathbf{b}_{j_2}, \ldots, \mathbf{b}_{j_{n-r}}\right\}$. The complete set of basic columns from $\mathbf{A}$, and a basis for $\mathcal{V}$, is the set

$$\mathcal{B} = \left\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{b}_{j_1}, \ldots, \mathbf{b}_{j_{n-r}}\right\}.$$

For example, to extend the independent set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

to a basis for $\Re^4$, append the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ to the vectors in $\mathcal{S}$, and perform the reduction

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \mathbf{E_A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \end{pmatrix}.$$

This reveals that $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*4}, \mathbf{A}_{*5}\}$ are the basic columns in $\mathbf{A}$, and therefore

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\Re^4$ that contains $\mathcal{S}$.

Example 4.4.6

Rank and Connectivity. A set of points (or *nodes*), $\{N_1, N_2, \ldots, N_m\}$, together with a set of paths (or *edges*), $\{E_1, E_2, \ldots, E_n\}$, between the nodes is called a ***graph***. A *connected graph* is one in which there is a sequence of edges linking any pair of nodes, and a *directed graph* is one in which each edge has been assigned a direction. For example, the graph in Figure 4.4.1 is both connected and directed.

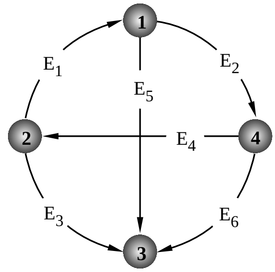


FIGURE 4.4.1

The connectivity of a directed graph is independent of the directions assigned to the edges—i.e., changing the direction of an edge doesn't change the connectivity. (Exercise 4.4.20 presents another type of connectivity in which direction matters.) On the surface, the concepts of graph connectivity and matrix rank seem to have little to do with each other, but, in fact, there is a close relationship. The ***incidence matrix*** associated with a directed graph containing m nodes and n edges is defined to be the $m \times n$ matrix $\mathbf{E}$ whose (k, j)-entry is

$$e_{kj} = \begin{cases} 1 & \text{if edge } E_j \text{ is directed } \textit{toward} \text{ node } N_k. \\ -1 & \text{if edge } E_j \text{ is directed } \textit{away from} \text{ node } N_k. \\ 0 & \text{if edge } E_j \text{ neither begins nor ends at node } N_k. \end{cases}$$

For example, the incidence matrix associated with the graph in Figure 4.4.1 is

$$\mathbf{E} = \begin{matrix} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{matrix} \begin{matrix} \begin{matrix} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{matrix} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix} \end{matrix}. \tag{4.4.16}$$

Each edge in a directed graph is associated with two nodes—the nose and the tail of the edge—so each column in $\mathbf{E}$ must contain exactly two nonzero entries—a $(+1)$ and a (-1). Consequently, all column sums are zero. In other words, if $\mathbf{e}^T = (1 \quad 1 \quad \cdots \quad 1)$, then $\mathbf{e}^T\mathbf{E} = \mathbf{0}$, so $\mathbf{e} \in N\left(\mathbf{E}^T\right)$, and

$$rank\,(\mathbf{E}) = rank\left(\mathbf{E}^T\right) = m - \dim N\left(\mathbf{E}^T\right) \leq m - 1. \tag{4.4.17}$$

This inequality holds regardless of the connectivity of the associated graph, but marvelously, equality is attained if and only if the graph is connected.

Rank and Connectivity

Let $\mathcal{G}$ be a graph containing m nodes. If $\mathcal{G}$ is undirected, arbitrarily assign directions to the edges to make $\mathcal{G}$ directed, and let $\mathbf{E}$ be the corresponding incidence matrix.

- $\mathcal{G}$ is connected if and only if $rank(\mathbf{E}) = m - 1$. (4.4.18)

Proof. Suppose $\mathcal{G}$ is connected. Prove $rank(\mathbf{E}) = m - 1$ by arguing that $\dim N(\mathbf{E}^T) = 1$, and do so by showing $\mathbf{e} = (1 \quad 1 \quad \cdots \quad 1)^T$ is a basis $N(\mathbf{E}^T)$. To see that $\mathbf{e}$ spans $N(\mathbf{E}^T)$, consider an arbitrary $\mathbf{x} \in N(\mathbf{E}^T)$, and focus on any two components x_i and x_k in $\mathbf{x}$ along with the corresponding nodes N_i and N_k in $\mathcal{G}$. Since $\mathcal{G}$ is connected, there must exist a subset of r nodes,

$$\{N_{j_1}, N_{j_2}, \ldots, N_{j_r}\}, \quad \text{where} \quad i = j_1 \quad \text{and} \quad k = j_r,$$

such that there is an edge between N_{j_p} and $N_{j_{p+1}}$ for each $p = 1, 2, \ldots, r-1$. Therefore, corresponding to each of the $r-1$ pairs $(N_{j_p}, N_{j_{p+1}})$, there must exist a column $\mathbf{c}_p$ in $\mathbf{E}$ (not necessarily the p^{th} column) such that components j_p and j_{p+1} in $\mathbf{c}_p$ are complementary in the sense that one is $(+1)$ while the other is (-1) (all other components are zero). Because $\mathbf{x}^T\mathbf{E} = \mathbf{0}$, it follows that $\mathbf{x}^T\mathbf{c}_p = 0$, and hence $x_{j_p} = x_{j_{p+1}}$. But this holds for every $p = 1, 2, \ldots, r-1$, so $x_i = x_k$ for each i and k, and hence $\mathbf{x} = \alpha\mathbf{e}$ for some scalar α. Thus $\{\mathbf{e}\}$ spans $N(\mathbf{E}^T)$. Clearly, $\{\mathbf{e}\}$ is linearly independent, so it is a basis $N(\mathbf{E}^T)$, and, therefore, $\dim N(\mathbf{E}^T) = 1$ or, equivalently, $rank(\mathbf{E}) = m-1$. Conversely, suppose $rank(\mathbf{E}) = m-1$, and prove $\mathcal{G}$ is connected with an indirect argument. If $\mathcal{G}$ is not connected, then $\mathcal{G}$ is decomposable into two nonempty subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in which there are no edges between nodes in $\mathcal{G}_1$ and nodes in $\mathcal{G}_2$. This means that the nodes in $\mathcal{G}$ can be ordered so as to make $\mathbf{E}$ have the form

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix},$$

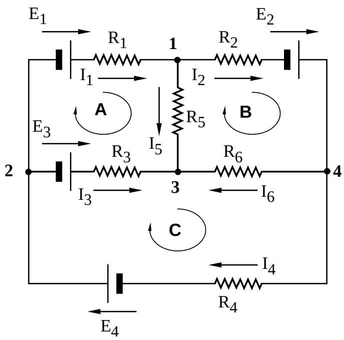
where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the incidence matrices for $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. If $\mathcal{G}_1$ and $\mathcal{G}_2$ contain m_1 and m_2 nodes, respectively, then (4.4.17) insures that

$$rank(\mathbf{E}) = rank\begin{pmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix} = rank(\mathbf{E}_1) + rank(\mathbf{E}_1) \leq (m_1 - 1) + (m_2 - 1) = m - 2.$$

But this contradicts the hypothesis that $rank(\mathbf{E}) = m-1$, so the supposition that $\mathcal{G}$ is not connected must be false. ∎

Example 4.4.7

An Application to Electrical Circuits. Recall from the discussion on p. 73 that applying Kirchhoff's node rule to an electrical circuit containing m nodes and n branches produces m homogeneous linear equations in n unknowns (the branch currents), and Kirchhoff's loop rule provides a nonhomogeneous equation for each simple loop in the circuit. For example, consider the circuit in Figure 4.4.2 along with its four nodal equations and three loop equations—this is the same circuit appearing on p. 73, and the equations are derived there.



$$\begin{aligned}
\text{Node 1:}\quad & I_1 - I_2 - I_5 = 0\\
\text{Node 2:}\quad & -I_1 - I_3 + I_4 = 0\\
\text{Node 3:}\quad & I_3 + I_5 + I_6 = 0\\
\text{Node 4:}\quad & I_2 - I_4 - I_6 = 0\\
\text{Loop A:}\quad & I_1R_1 - I_3R_3 + I_5R_5 = E_1 - E_3\\
\text{Loop B:}\quad & I_2R_2 - I_5R_5 + I_6R_6 = E_2\\
\text{Loop C:}\quad & I_3R_3 + I_4R_4 - I_6R_6 = E_3 + E_4
\end{aligned}$$

FIGURE 4.4.2

The directed graph and associated incidence matrix $\mathbf{E}$ defined by this circuit are the same as those appearing in Example 4.4.6 in Figure 4.4.1 and equation (4.4.16), so it's apparent that the 4×3 homogeneous system of nodal equations is precisely the system $\mathbf{Ex} = \mathbf{0}$. This observation holds for general circuits. The goal is to compute the six currents $I_1, I_2, \ldots, I_6$ by selecting six independent equations from the entire set of node and loop equations. In general, if a circuit containing m nodes is connected in the graph sense, then (4.4.18) insures that $rank\,(\mathbf{E}) = m - 1$, so there are m independent nodal equations. But Example 4.4.6 also shows that $\mathbf{0} = \mathbf{e}^T\mathbf{E} = \mathbf{E}_{1*} + \mathbf{E}_{2*} + \cdots + \mathbf{E}_{m*}$, which means that any row can be written in terms of the others, and this in turn implies that *every* subset of $m - 1$ rows in $\mathbf{E}$ must be independent (see Exercise 4.4.13). Consequently, when any nodal equation is discarded, the remaining ones are guaranteed to be independent. To determine an $n \times n$ nonsingular system that has the n branch currents as its unique solution, it's therefore necessary to find $n - m + 1$ additional independent equations, and, as shown in §2.6, these are the loop equations. A simple loop in a circuit is now seen to be a connected subgraph that does not properly contain other connected subgraphs. Physics dictates that the currents must be uniquely determined, so there must always be $n - m + 1$ simple loops, and the combination of these loop equations together with any subset of $m - 1$ nodal equations will be a nonsingular $n \times n$ system that yields the branch currents as its unique solution. For example, any three of the nodal equations in Figure 4.4.2 can be coupled with the three simple loop equations to produce a 6×6 nonsingular system whose solution is the six branch currents.

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then the sum of $\mathcal{X}$ and $\mathcal{Y}$ was defined in §4.1 to be

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{y} \in \mathcal{Y}\},$$

and it was demonstrated in (4.1.1) that $\mathcal{X} + \mathcal{Y}$ is again a subspace of $\mathcal{V}$. You were asked in Exercise 4.1.8 to prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$. We are now in a position to exhibit an important relationship between $\dim(\mathcal{X} + \mathcal{Y})$ and $\dim(\mathcal{X} \cap \mathcal{Y})$.

Dimension of a Sum

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then

$$\dim(\mathcal{X} + \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y}). \tag{4.4.19}$$

Proof. The strategy is to construct a basis for $\mathcal{X} + \mathcal{Y}$ and count the number of vectors it contains. Let $\mathcal{S} = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t\}$ be a basis for $\mathcal{X} \cap \mathcal{Y}$. Since $\mathcal{S} \subseteq \mathcal{X}$ and $\mathcal{S} \subseteq \mathcal{Y}$, there must exist extension vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ such that

$$\mathcal{B}_X = \{\mathbf{z}_1, \ldots, \mathbf{z}_t, \mathbf{x}_1, \ldots, \mathbf{x}_m\} = \text{ a basis for } \mathcal{X}$$

and

$$\mathcal{B}_Y = \{\mathbf{z}_1, \ldots, \mathbf{z}_t, \mathbf{y}_1, \ldots, \mathbf{y}_n\} = \text{ a basis for } \mathcal{Y}.$$

We know from (4.1.2) that $\mathcal{B} = \mathcal{B}_X \cup \mathcal{B}_Y$ spans $\mathcal{X} + \mathcal{Y}$, and we wish show that $\mathcal{B}$ is linearly independent. If

$$\sum_{i=1}^{t} \alpha_i \mathbf{z}_i + \sum_{j=1}^{m} \beta_j \mathbf{x}_j + \sum_{k=1}^{n} \gamma_k \mathbf{y}_k = \mathbf{0}, \tag{4.4.20}$$

then

$$\sum_{k=1}^{n} \gamma_k \mathbf{y}_k = -\left(\sum_{i=1}^{t} \alpha_i \mathbf{z}_i + \sum_{j=1}^{m} \beta_j \mathbf{x}_j \right) \in \mathcal{X}.$$

Since it is also true that $\sum_k \gamma_k \mathbf{y}_k \in \mathcal{Y}$, we have that $\sum_k \gamma_k \mathbf{y}_k \in \mathcal{X} \cap \mathcal{Y}$, and hence there must exist scalars δ_i such that

$$\sum_{k=1}^{n} \gamma_k \mathbf{y}_k = \sum_{i=1}^{t} \delta_i \mathbf{z}_i \quad \text{or, equivalently,} \quad \sum_{k=1}^{n} \gamma_k \mathbf{y}_k - \sum_{i=1}^{t} \delta_i \mathbf{z}_i = \mathbf{0}.$$

Since $\mathcal{B}_Y$ is an independent set, it follows that all of the γ_k's (as well as all δ_i's) are zero, and (4.4.20) reduces to $\sum_{i=1}^{t} \alpha_i \mathbf{z}_i + \sum_{j=1}^{m} \beta_j \mathbf{x}_j = \mathbf{0}$. But $\mathcal{B}_X$ is also an independent set, so the only way this can hold is for all of the α_i's as well as all of the β_j's to be zero. Therefore, the only possible solution for the α's, β's, and γ's in the homogeneous equation (4.4.20) is the trivial solution, and thus $\mathcal{B}$ is linearly independent. Since $\mathcal{B}$ is an independent spanning set, it is a basis for $\mathcal{X} + \mathcal{Y}$ and, consequently,

$$\dim(\mathcal{X} + \mathcal{Y}) = t+m+n = (t+m)+(t+n)-t = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y}). \quad \blacksquare$$

Example 4.4.8

Problem: Show that $rank\,(\mathbf{A}+\mathbf{B}) \le rank\,(\mathbf{A}) + rank\,(\mathbf{B})$.

Solution: Observe that

$$R\,(\mathbf{A}+\mathbf{B}) \subseteq R\,(\mathbf{A}) + R\,(\mathbf{B})$$

because if $\mathbf{b} \in R\,(\mathbf{A}+\mathbf{B})$, then there is a vector $\mathbf{x}$ such that

$$\mathbf{b} = (\mathbf{A}+\mathbf{B})\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \in R\,(\mathbf{A}) + R\,(\mathbf{B}).$$

Recall from (4.4.5) that if $\mathcal{M}$ and $\mathcal{N}$ are vector spaces such that $\mathcal{M} \subseteq \mathcal{N}$, then $\dim \mathcal{M} \le \dim \mathcal{N}$. Use this together with formula (4.4.19) for the dimension of a sum to conclude that

$$\begin{aligned} rank\,(\mathbf{A}+\mathbf{B}) = \dim R\,(\mathbf{A}+\mathbf{B}) &\le \dim \Big(R\,(\mathbf{A}) + R\,(\mathbf{B})\Big) \\ &= \dim R\,(\mathbf{A}) + \dim R\,(\mathbf{B}) - \dim \Big(R\,(\mathbf{A}) \cap R\,(\mathbf{B})\Big) \\ &\le \dim R\,(\mathbf{A}) + \dim R\,(\mathbf{B}) = rank\,(\mathbf{A}) + rank\,(\mathbf{B}). \end{aligned}$$

Exercises for section 4.4

4.4.1. Find the dimensions of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

4.4.2. Find a basis for each of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 3 & 6 & 1 & 9 & 6 \\ 2 & 4 & 1 & 7 & 5 \end{pmatrix}.$$

4.4.3. Determine the dimension of the space spanned by the set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ -4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 6 \end{pmatrix} \right\}.$$

4.4.4. Determine the dimensions of each of the following vector spaces:

(a) The space of polynomials having degree n or less.
(b) The space $\Re^{m\times n}$ of $m \times n$ matrices.
(c) The space of $n \times n$ symmetric matrices.

4.4.5. Consider the following matrix and column vector:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 0 & 5 \\ 2 & 4 & 3 & 1 & 8 \\ 3 & 6 & 1 & 5 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -8 \\ 1 \\ 3 \\ 3 \\ 0 \end{pmatrix}.$$

Verify that $\mathbf{v} \in N(\mathbf{A})$, and then extend $\{\mathbf{v}\}$ to a basis for $N(\mathbf{A})$.

4.4.6. Determine whether or not the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a basis for the space spanned by the set

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}.$$

4.4.7. Construct a 4×4 homogeneous system of equations that has no zero coefficients and three linearly independent solutions.

4.4.8. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $\mathcal{V}$. Prove that each $\mathbf{v} \in \mathcal{V}$ can be expressed as a linear combination of the $\mathbf{b}_i$'s

$$\mathbf{v} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_n\mathbf{b}_n,$$

in only one way—i.e., the *coordinates* α_i are unique.

4.4.9. For $\mathbf{A} \in \Re^{m\times n}$ and a subspace $\mathcal{S}$ of $\Re^{n\times 1}$, the image

$$\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \mathcal{S}\}$$

of $\mathcal{S}$ under $\mathbf{A}$ is a subspace of $\Re^{m\times 1}$—recall Exercise 4.1.9. Prove that if $\mathcal{S} \cap N(\mathbf{A}) = \mathbf{0}$, then $\dim \mathbf{A}(\mathcal{S}) = \dim(\mathcal{S})$. **Hint:** Use a basis $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ for $\mathcal{S}$ to determine a basis for $\mathbf{A}(\mathcal{S})$.

4.4.10. Explain why $\big|rank(\mathbf{A}) - rank(\mathbf{B})\big| \le rank(\mathbf{A} - \mathbf{B})$.

4.4.11. If $rank(\mathbf{A}_{m\times n}) = r$ and $rank(\mathbf{E}_{m\times n}) = k \le r$, explain why

$$r - k \le rank(\mathbf{A} + \mathbf{E}) \le r + k.$$

In words, this says that a perturbation of rank k can change the rank by at most k.

4.4.12. Explain why every nonzero subspace $\mathcal{V} \subseteq \Re^n$ must possess a basis.

4.4.13. Explain why *every* set of $m-1$ rows in the incidence matrix $\mathbf{E}$ of a connected directed graph containing m nodes is linearly independent.

4.4.14. For the incidence matrix $\mathbf{E}$ of a directed graph, explain why

$$\left[\mathbf{E}\mathbf{E}^T\right]_{ij} = \begin{cases} \text{number of edges at node } i & \text{when } i = j, \\ -(\text{number of edges between nodes } i \text{ and } j) & \text{when } i \ne j. \end{cases}$$

4.4.15. If $\mathcal{M}$ and $\mathcal{N}$ are subsets of a space $\mathcal{V}$, explain why

$$\dim\big(span(\mathcal{M} \cup \mathcal{N})\big) = \dim\big(span(\mathcal{M})\big) + \dim\big(span(\mathcal{N})\big) - \dim\big(span(\mathcal{M}) \cap span(\mathcal{N})\big).$$

4.4.16. Consider two matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times k}$.

(a) Explain why

$$rank(\mathbf{A} \mid \mathbf{B}) = rank(\mathbf{A}) + rank(\mathbf{B}) - \dim\big(R(\mathbf{A}) \cap R(\mathbf{B})\big).$$

Hint: Recall Exercise 4.2.9.

(b) Now explain why

$$\dim N(\mathbf{A} \mid \mathbf{B}) = \dim N(\mathbf{A}) + \dim N(\mathbf{B}) + \dim\big(R(\mathbf{A}) \cap R(\mathbf{B})\big).$$

(c) Determine $\dim\big(R(\mathbf{C}) \cap N(\mathbf{C})\big)$ and $\dim\big(R(\mathbf{C}) + N(\mathbf{C})\big)$ for

$$\mathbf{C} = \begin{pmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{pmatrix}.$$

4.4.17. Suppose that $\mathbf{A}$ is a matrix with m rows such that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \Re^m$. Explain why this means that $\mathbf{A}$ must be square and nonsingular.

4.4.18. Let $\mathcal{S}$ be the solution set for a consistent system of linear equations $\mathbf{Ax} = \mathbf{b}$.

(a) If $\mathcal{S}_{max} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_t\}$ is a maximal independent subset of $\mathcal{S}$, and if $\mathbf{p}$ is any particular solution, prove that

$$span(\mathcal{S}_{max}) = span\{\mathbf{p}\} + N(\mathbf{A}).$$

Hint: First show that $\mathbf{x} \in \mathcal{S}$ implies $\mathbf{x} \in span(\mathcal{S}_{max})$, and then demonstrate set inclusion in both directions with the aid of Exercise 4.2.10.

(b) If $\mathbf{b} \neq \mathbf{0}$ and $rank(\mathbf{A}_{m\times n}) = r$, explain why $\mathbf{Ax} = \mathbf{b}$ has $n - r + 1$ "independent solutions."

4.4.19. Let $rank(\mathbf{A}_{m\times n}) = r$, and suppose $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ is a consistent system. If $\mathcal{H} = \{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}\}$ is a basis for $N(\mathbf{A})$, and if $\mathbf{p}$ is a particular solution to $\mathbf{Ax} = \mathbf{b}$, show that

$$\mathcal{S}_{max} = \{\mathbf{p},\ \mathbf{p}+\mathbf{h}_1,\ \mathbf{p}+\mathbf{h}_2,\ \ldots,\ \mathbf{p}+\mathbf{h}_{n-r}\}$$

is a maximal independent set of solutions.

4.4.20. **Strongly Connected Graphs.** In Example 4.4.6 we started with a graph to construct a matrix, but it's also possible to reverse the situation by starting with a matrix to build an associated graph. The graph of $\mathbf{A}_{n\times n}$ (denoted by $\mathcal{G}(\mathbf{A})$) is defined to be the directed graph on n nodes $\{N_1, N_2, \ldots, N_n\}$ in which there is a directed edge leading *from* N_i *to* N_j if and only if $a_{ij} \neq 0$. The directed graph $\mathcal{G}(\mathbf{A})$ is said to be ***strongly connected*** provided that for each pair of nodes (N_i, N_k) there is a sequence of directed edges leading *from* N_i *to* N_k. The matrix $\mathbf{A}$ is said to be ***reducible*** if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{P}^T\mathbf{AP} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}$, where $\mathbf{X}$ and $\mathbf{Z}$ are both square matrices. Otherwise, $\mathbf{A}$ is said to be ***irreducible.*** Prove that $\mathcal{G}(\mathbf{A})$ is strongly connected if and only if $\mathbf{A}$ is irreducible. **Hint:** Prove the contrapositive: $\mathcal{G}(\mathbf{A})$ is *not* strongly connected if and only if $\mathbf{A}$ is reducible.

4.5 MORE ABOUT RANK

Since equivalent matrices have the same rank, it follows that if $\mathbf{P}$ and $\mathbf{Q}$ are nonsingular matrices such that the product $\mathbf{PAQ}$ is defined, then

$$rank\,(\mathbf{A}) = rank\,(\mathbf{PAQ}) = rank\,(\mathbf{PA}) = rank\,(\mathbf{AQ}).$$

In other words, rank is invariant under multiplication by a nonsingular matrix. However, multiplication by rectangular or singular matrices can alter the rank, and the following formula shows exactly how much alteration occurs.

Rank of a Product

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then

$$rank\,(\mathbf{AB}) = rank\,(\mathbf{B}) - \dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}). \tag{4.5.1}$$

Proof. Start with a basis $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ for $N\,(\mathbf{A}) \cap R\,(\mathbf{B})$, and notice $N\,(\mathbf{A}) \cap R\,(\mathbf{B}) \subseteq R\,(\mathbf{B})$. If $\dim R\,(\mathbf{B}) = s + t$, then, as discussed in Example 4.4.5, there exists an extension set $\mathcal{S}_{ext} = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t\}$ such that $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_s, \mathbf{z}_1, \ldots, \mathbf{z}_t\}$ is a basis for $R\,(\mathbf{B})$. The goal is to prove that $\dim R\,(\mathbf{AB}) = t$, and this is done by showing $\mathcal{T} = \{\mathbf{Az}_1, \mathbf{Az}_2, \ldots, \mathbf{Az}_t\}$ is a basis for $R\,(\mathbf{AB})$. $\mathcal{T}$ spans $R\,(\mathbf{AB})$ because if $\mathbf{b} \in R\,(\mathbf{AB})$, then $\mathbf{b} = \mathbf{ABy}$ for some $\mathbf{y}$, but $\mathbf{By} \in R\,(\mathbf{B})$ implies $\mathbf{By} = \sum_{i=1}^{s} \xi_i \mathbf{x}_i + \sum_{i=1}^{t} \eta_i \mathbf{z}_i$, so

$$\mathbf{b} = \mathbf{A}\left(\sum_{i=1}^{s} \xi_i \mathbf{x}_i + \sum_{i=1}^{t} \eta_i \mathbf{z}_i\right) = \sum_{i=1}^{s} \xi_i \mathbf{Ax}_i + \sum_{i=1}^{t} \eta_i \mathbf{Az}_i = \sum_{i=1}^{t} \eta_i \mathbf{Az}_i.$$

$\mathcal{T}$ is linearly independent because if $\mathbf{0} = \sum_{i=1}^{t} \alpha_i \mathbf{Az}_i = \mathbf{A} \sum_{i=1}^{t} \alpha_i \mathbf{z}_i$, then $\sum_{i=1}^{t} \alpha_i \mathbf{z}_i \in N\,(\mathbf{A}) \cap R\,(\mathbf{B})$, so there are scalars β_j such that

$$\sum_{i=1}^{t} \alpha_i \mathbf{z}_i = \sum_{j=1}^{s} \beta_j \mathbf{x}_j \quad \text{or, equivalently,} \quad \sum_{i=1}^{t} \alpha_i \mathbf{z}_i - \sum_{j=1}^{s} \beta_j \mathbf{x}_j = \mathbf{0},$$

and hence the only solution for the α_i 's and β_i 's is the trivial solution because $\mathcal{B}$ is an independent set. Thus $\mathcal{T}$ is a basis for $R\,(\mathbf{AB})$, so $t = \dim R\,(\mathbf{AB}) = rank\,(\mathbf{AB})$, and hence

$$rank\,(\mathbf{B}) = \dim R\,(\mathbf{B}) = s + t = \dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}) + rank\,(\mathbf{AB}). \quad \blacksquare$$

It's sometimes necessary to determine an explicit basis for $N\,(\mathbf{A}) \cap R\,(\mathbf{B})$. In particular, such a basis is needed to construct the Jordan chains that are associated with the Jordan form that is discussed on pp. 582 and 594. The following example outlines a procedure for finding such a basis.

Basis for an Intersection

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then a basis for $N(\mathbf{A}) \cap R(\mathbf{B})$ can be constructed by the following procedure.

- ▷ Find a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ for $R(\mathbf{B})$.
- ▷ Set $\mathbf{X}_{n\times r} = \big(\mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\, \cdots \,|\, \mathbf{x}_r\big)$.
- ▷ Find a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ for $N(\mathbf{AX})$.
- ▷ $\mathcal{B} = \{\mathbf{Xv}_1, \mathbf{Xv}_2, \ldots, \mathbf{Xv}_s\}$ is a basis for $N(\mathbf{A}) \cap R(\mathbf{B})$.

Proof. The strategy is to argue that $\mathcal{B}$ is a maximal linear independent subset of $N(\mathbf{A}) \cap R(\mathbf{B})$. Since each $\mathbf{Xv}_j$ belongs to $R(\mathbf{X}) = R(\mathbf{B})$, and since $\mathbf{AXv}_j = \mathbf{0}$, it's clear that $\mathcal{B} \subset N(\mathbf{A}) \cap R(\mathbf{B})$. Let $\mathbf{V}_{r\times s} = \big(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \cdots \,|\, \mathbf{v}_s\big)$, and notice that $\mathbf{V}$ and $\mathbf{X}$ each have full column rank. Consequently, $N(\mathbf{X}) = \mathbf{0}$ so, by (4.5.1),

$$rank\,(\mathbf{XV})_{n\times s} = rank\,(\mathbf{V}) - \dim N(\mathbf{X}) \cap R(\mathbf{V}) = rank\,(\mathbf{V}) = s,$$

which insures that $\mathcal{B}$ is linearly independent. $\mathcal{B}$ is a *maximal* independent subset of $N(\mathbf{A}) \cap R(\mathbf{B})$ because (4.5.1) also guarantees that

$$\begin{aligned} s &= \dim N(\mathbf{AX}) = \dim N(\mathbf{X}) + \dim N(\mathbf{A}) \cap R(\mathbf{X}) \quad \text{(see Exercise 4.5.10)} \\ &= \dim N(\mathbf{A}) \cap R(\mathbf{B}). \quad \blacksquare \end{aligned}$$

The utility of (4.5.1) is mitigated by the fact that although $rank\,(\mathbf{A})$ and $rank\,(\mathbf{B})$ are frequently known or can be estimated, the term $\dim N(\mathbf{A}) \cap R(\mathbf{B})$ can be costly to obtain. In such cases (4.5.1) still provides us with useful upper and lower bounds for $rank\,(\mathbf{AB})$ that depend only on $rank\,(\mathbf{A})$ and $rank\,(\mathbf{B})$.

Bounds on the Rank of a Product

If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then

- $rank\,(\mathbf{AB}) \le \min\{rank\,(\mathbf{A}),\ rank\,(\mathbf{B})\}$, (4.5.2)
- $rank\,(\mathbf{A}) + rank\,(\mathbf{B}) - n \le rank\,(\mathbf{AB})$. (4.5.3)

Proof. In words, (4.5.2) says that the rank of a product cannot exceed the rank of either factor. To prove $rank\,(\mathbf{AB}) \le rank\,(\mathbf{B})$, use (4.5.1) and write

$$rank\,(\mathbf{AB}) = rank\,(\mathbf{B}) - \dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}) \le rank\,(\mathbf{B}).$$

This says that the rank of a product cannot exceed the rank of the right-hand factor. To show that $rank\,(\mathbf{AB}) \le rank\,(\mathbf{A})$, remember that transposition does not alter rank, and use the reverse order law for transposes together with the previous statement to write

$$rank\,(\mathbf{AB}) = rank\,(\mathbf{AB})^T = rank\left(\mathbf{B}^T\mathbf{A}^T\right) \le rank\left(\mathbf{A}^T\right) = rank\,(\mathbf{A}).$$

To prove (4.5.3), notice that $N\,(\mathbf{A}) \cap R\,(\mathbf{B}) \subseteq N\,(\mathbf{A})$, and recall from (4.4.5) that if $\mathcal{M}$ and $\mathcal{N}$ are spaces such that $\mathcal{M} \subseteq \mathcal{N}$, then $\dim \mathcal{M} \le \dim \mathcal{N}$. Therefore,

$$\dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}) \le \dim N\,(\mathbf{A}) = n - rank\,(\mathbf{A}),$$

and the lower bound on $rank\,(\mathbf{AB})$ is obtained from (4.5.1) by writing

$$rank\,(\mathbf{AB}) = rank\,(\mathbf{B}) - \dim N\,(\mathbf{A}) \cap R\,(\mathbf{B}) \ge rank\,(\mathbf{B}) + rank\,(\mathbf{A}) - n. \quad \blacksquare$$

The products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ and their complex counterparts $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ deserve special attention because they naturally appear in a wide variety of applications.

Products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$

For $\mathbf{A} \in \Re^{m\times n}$, the following statements are true.

- $rank\left(\mathbf{A}^T\mathbf{A}\right) = rank\,(\mathbf{A}) = rank\left(\mathbf{A}\mathbf{A}^T\right).$ (4.5.4)
- $R\left(\mathbf{A}^T\mathbf{A}\right) = R\left(\mathbf{A}^T\right)$ and $R\left(\mathbf{A}\mathbf{A}^T\right) = R\,(\mathbf{A}).$ (4.5.5)
- $N\left(\mathbf{A}^T\mathbf{A}\right) = N\,(\mathbf{A})$ and $N\left(\mathbf{A}\mathbf{A}^T\right) = N\left(\mathbf{A}^T\right).$ (4.5.6)

For $\mathbf{A} \in \mathcal{C}^{m\times n}$, the transpose operation $(\star)^T$ must be replaced by the conjugate transpose operation $(\star)^*$.

Proof. First observe that $N\left(\mathbf{A}^T\right) \cap R(\mathbf{A}) = \{\mathbf{0}\}$ because

$$\begin{aligned}
\mathbf{x} \in N\left(\mathbf{A}^T\right) \cap R(\mathbf{A}) &\implies \mathbf{A}^T\mathbf{x} = \mathbf{0} \text{ and } \mathbf{x} = \mathbf{Ay} \text{ for some } \mathbf{y} \\
&\implies \mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}^T\mathbf{x} = 0 \implies \sum x_i^2 = 0 \\
&\implies \mathbf{x} = \mathbf{0}.
\end{aligned}$$

Formula (4.5.1) for the rank of a product now guarantees that

$$rank\left(\mathbf{A}^T\mathbf{A}\right) = rank(\mathbf{A}) - \dim N\left(\mathbf{A}^T\right) \cap R(\mathbf{A}) = rank(\mathbf{A}),$$

which is half of (4.5.4)—the other half is obtained by reversing the roles of $\mathbf{A}$ and $\mathbf{A}^T$. To prove (4.5.5) and (4.5.6), use the facts $R(\mathbf{AB}) \subseteq R(\mathbf{A})$ and $N(\mathbf{B}) \subseteq N(\mathbf{AB})$ (see Exercise 4.2.12) to write $R\left(\mathbf{A}^T\mathbf{A}\right) \subseteq R\left(\mathbf{A}^T\right)$ and $N(\mathbf{A}) \subseteq N\left(\mathbf{A}^T\mathbf{A}\right)$. The first half of (4.5.5) and (4.5.6) now follows because

$$\dim R\left(\mathbf{A}^T\mathbf{A}\right) = rank\left(\mathbf{A}^T\mathbf{A}\right) = rank(\mathbf{A}) = rank\left(\mathbf{A}^T\right) = \dim R\left(\mathbf{A}^T\right),$$

$$\dim N(\mathbf{A}) = n - rank(\mathbf{A}) = n - rank\left(\mathbf{A}^T\mathbf{A}\right) = \dim N\left(\mathbf{A}^T\mathbf{A}\right).$$

Reverse the roles of $\mathbf{A}$ and $\mathbf{A}^T$ to get the second half of (4.5.5) and (4.5.6). ∎

To see why (4.5.4)—(4.5.6) might be important, consider an $m \times n$ system of equations $\mathbf{Ax} = \mathbf{b}$ that may or may not be consistent. Multiplying on the left-hand side by $\mathbf{A}^T$ produces the $n \times n$ system

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

called ***the associated system of normal equations***, which has some extremely interesting properties. First, notice that the normal equations are always consistent, regardless of whether or not the original system is consistent because (4.5.5) guarantees that $\mathbf{A}^T\mathbf{b} \in R\left(\mathbf{A}^T\right) = R\left(\mathbf{A}^T\mathbf{A}\right)$ (i.e., the right-hand side is in the range of the coefficient matrix), so (4.2.3) insures consistency. However, if $\mathbf{Ax} = \mathbf{b}$ happens to be consistent, then $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ have the same solution set because if $\mathbf{p}$ is a particular solution of the original system, then $\mathbf{Ap} = \mathbf{b}$ implies $\mathbf{A}^T\mathbf{Ap} = \mathbf{A}^T\mathbf{b}$ (i.e., $\mathbf{p}$ is also a particular solution of the normal equations), so the general solution of $\mathbf{Ax} = \mathbf{b}$ is $\mathcal{S} = \mathbf{p} + N(\mathbf{A})$, and the general solution of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ is

$$\mathbf{p} + N\left(\mathbf{A}^T\mathbf{A}\right) = \mathbf{p} + N(\mathbf{A}) = \mathcal{S}.$$

Furthermore, if $\mathbf{Ax} = \mathbf{b}$ is consistent and has a unique solution, then the same is true for $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$, and the unique solution common to both systems is

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}. \tag{4.5.7}$$

This follows because a unique solution (to either system) exists if and only if $\mathbf{0} = N(\mathbf{A}) = N\left(\mathbf{A}^T\mathbf{A}\right)$, and this insures $(\mathbf{A}^T\mathbf{A})_{n\times n}$ must be nonsingular (by (4.2.11)), so (4.5.7) is the unique solution to both systems. **Caution!** When $\mathbf{A}$ is not square, $\mathbf{A}^{-1}$ does not exist, and the reverse order law for inversion doesn't apply to $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}$, so (4.5.7) cannot be further simplified.

There is one outstanding question—what do the solutions of the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ represent when the original system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is *not* consistent? The answer, which is of fundamental importance, will have to wait until §4.6, but let's summarize what has been said so far.

Normal Equations

- For an $m \times n$ system $\mathbf{A}\mathbf{x} = \mathbf{b}$, the associated system of ***normal equations*** is defined to be the $n \times n$ system $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$.
- $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ is always consistent, even when $\mathbf{A}\mathbf{x} = \mathbf{b}$ is not consistent.
- When $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent, its solution set agrees with that of $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. As discussed in §4.6, the normal equations provide least squares solutions to $\mathbf{A}\mathbf{x} = \mathbf{b}$ when $\mathbf{A}\mathbf{x} = \mathbf{b}$ is inconsistent.
- $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ has a unique solution if and only if $rank(\mathbf{A}) = n$, in which case the unique solution is $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.
- When $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent and has a unique solution, then the same is true for $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$, and the unique solution to both systems is given by $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.

Example 4.5.1

Caution! Use of the product $\mathbf{A}^T\mathbf{A}$ or the normal equations is not recommended for numerical computation. Any sensitivity to small perturbations that is present in the underlying matrix $\mathbf{A}$ is magnified by forming the product $\mathbf{A}^T\mathbf{A}$. In other words, if $\mathbf{A}\mathbf{x} = \mathbf{b}$ is somewhat ill-conditioned, then the associated system of normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ will be ill-conditioned to an even greater extent, and the theoretical properties surrounding $\mathbf{A}^T\mathbf{A}$ and the normal equations may be lost in practical applications. For example, consider the nonsingular system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2.01 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 9 \\ 3.01 \end{pmatrix}.$$

If Gaussian elimination with 3-digit floating-point arithmetic is used to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$, then the 3-digit solution is $(1,\ 1)$, and this agrees with the exact solution. However if 3-digit arithmetic is used to form the associated system of normal equations, the result is

$$\begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 60.1 \end{pmatrix}.$$

The 3-digit representation of $\mathbf{A}^T\mathbf{A}$ is singular, and the associated system of normal equations is inconsistent. For these reasons, the normal equations are often avoided in numerical computations. Nevertheless, the normal equations are an important theoretical idea that leads to practical tools of fundamental importance such as the method of least squares developed in §4.6 and §5.13.

Because the concept of rank is at the heart of our subject, it's important to understand rank from a variety of different viewpoints. The statement below is one more way to think about rank.[29]

Rank and the Largest Nonsingular Submatrix

The rank of a matrix $\mathbf{A}_{m\times n}$ is precisely the order of a maximal square nonsingular submatrix of $\mathbf{A}$. In other words, to say $rank\,(\mathbf{A}) = r$ means that there is at least one $r \times r$ nonsingular submatrix in $\mathbf{A}$, and there are no nonsingular submatrices of larger order.

Proof. First demonstrate that there exists an $r \times r$ nonsingular submatrix in $\mathbf{A}$, and then show there can be no nonsingular submatrix of larger order. Begin with the fact that there must be a maximal linearly independent set of r rows in $\mathbf{A}$ as well as a maximal independent set of r columns, and prove that the submatrix $\mathbf{M}_{r\times r}$ lying on the intersection of these r rows and r columns is nonsingular. The r independent rows can be permuted to the top, and the remaining rows can be annihilated using row operations, so

$$\mathbf{A} \overset{\text{row}}{\sim} \begin{pmatrix} \mathbf{U}_{r\times n} \\ \mathbf{0} \end{pmatrix}.$$

Now permute the r independent columns containing $\mathbf{M}$ to the left-hand side, and use column operations to annihilate the remaining columns to conclude that

$$\mathbf{A} \overset{\text{row}}{\sim} \begin{pmatrix} \mathbf{U}_{r\times n} \\ \mathbf{0} \end{pmatrix} \overset{\text{col}}{\sim} \begin{pmatrix} \mathbf{M}_{r\times r} & \mathbf{N} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \overset{\text{col}}{\sim} \begin{pmatrix} \mathbf{M}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

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[29] This is the last characterization of rank presented in this text, but historically this was the essence of the first definition (p. 44) of rank given by Georg Frobenius (p. 662) in 1879.

Rank isn't changed by row or column operations, so $r = rank(\mathbf{A}) = rank(\mathbf{M})$, and thus $\mathbf{M}$ is nonsingular. Now suppose that $\mathbf{W}$ is any other nonsingular submatrix of $\mathbf{A}$, and let $\mathbf{P}$ and $\mathbf{Q}$ be permutation matrices such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix}$. If

$$\mathbf{E} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{YW}^{-1} & \mathbf{I} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{I} & -\mathbf{W}^{-1}\mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \text{and} \quad \mathbf{S} = \mathbf{Z} - \mathbf{YW}^{-1}\mathbf{X},$$

then

$$\mathbf{EPAQF} = \begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix} \implies \mathbf{A} \sim \begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix}, \tag{4.5.8}$$

and hence $r = rank(\mathbf{A}) = rank(\mathbf{W}) + rank(\mathbf{S}) \geq rank(\mathbf{W})$ (recall Example 3.9.3). This guarantees that no nonsingular submatrix of $\mathbf{A}$ can have order greater than $r = rank(\mathbf{A})$. ∎

Example 4.5.2

Problem: Determine the rank of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{pmatrix}$.

Solution: $rank(\mathbf{A}) = 2$ because there is at least one 2×2 nonsingular submatrix (e.g., there is one lying on the intersection of rows 1 and 2 with columns 2 and 3), and there is no larger nonsingular submatrix (the entire matrix is singular). Notice that not all 2×2 matrices are nonsingular (e.g., consider the one lying on the intersection of rows 1 and 2 with columns 1 and 2).

Earlier in this section we saw that it is impossible to *increase* the rank by means of matrix multiplication—i.e., (4.5.2) says $rank(\mathbf{AE}) \leq rank(\mathbf{A})$. In a certain sense there is a dual statement for matrix addition that says that it is impossible to *decrease* the rank by means of a "small" matrix addition—i.e., $rank(\mathbf{A} + \mathbf{E}) \geq rank(\mathbf{A})$ whenever $\mathbf{E}$ has entries of small magnitude.

Small Perturbations Can't Reduce Rank

If $\mathbf{A}$ and $\mathbf{E}$ are $m \times n$ matrices such that $\mathbf{E}$ has entries of sufficiently small magnitude, then

$$rank(\mathbf{A} + \mathbf{E}) \geq rank(\mathbf{A}). \tag{4.5.9}$$

The term "sufficiently small" is further clarified in Exercise 5.12.4.

Proof. Suppose $rank(\mathbf{A}) = r$, and let $\mathbf{P}$ and $\mathbf{Q}$ be nonsingular matrices that reduce $\mathbf{A}$ to rank normal form—i.e., $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. If $\mathbf{P}$ and $\mathbf{Q}$ are applied to $\mathbf{E}$ to form $\mathbf{PEQ} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix}$, where $\mathbf{E}_{11}$ is $r \times r$, then

$$\mathbf{P}(\mathbf{A}+\mathbf{E})\mathbf{Q} = \begin{pmatrix} \mathbf{I}_r + \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix}. \tag{4.5.10}$$

If the magnitude of the entries in $\mathbf{E}$ are small enough to insure that $\mathbf{E}_{11}^k \to \mathbf{0}$ as $k \to \infty$, then the discussion of the Neumann series on p. 126 insures that $\mathbf{I} + \mathbf{E}_{11}$ is nonsingular. (Exercise 4.5.14 gives another condition on the size of $\mathbf{E}_{11}$ to insure this.) This allows the right-hand side of (4.5.10) to be further reduced by writing

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{E}_{21}(\mathbf{I}+\mathbf{E}_{11})^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I}+\mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -(\mathbf{I}+\mathbf{E}_{11})^{-1}\mathbf{E}_{12} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{I}-\mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix},$$

where $\mathbf{S} = \mathbf{E}_{22} - \mathbf{E}_{21}\left(\mathbf{I} + \mathbf{E}_{11}\right)^{-1}\mathbf{E}_{12}$. In other words,

$$\mathbf{A} + \mathbf{E} \sim \begin{pmatrix} \mathbf{I}-\mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix},$$

and therefore

$$\begin{aligned} rank(\mathbf{A}+\mathbf{E}) &= rank(\mathbf{I}_r + \mathbf{E}_{11}) + rank(\mathbf{S}) \quad \text{(recall Example 3.9.3)} \\ &= rank(\mathbf{A}) + rank(\mathbf{S}) \qquad (4.5.11) \\ &\geq rank(\mathbf{A}). \quad \blacksquare \end{aligned}$$

Example 4.5.3

A Pitfall in Solving Singular Systems. Solving $\mathbf{Ax} = \mathbf{b}$ with floating-point arithmetic produces the exact solution of a perturbed system whose coefficient matrix is $\mathbf{A}+\mathbf{E}$. If $\mathbf{A}$ is nonsingular, and if we are using a stable algorithm (an algorithm that insures that the entries in $\mathbf{E}$ have small magnitudes), then (4.5.9) guarantees that we are finding the exact solution to a nearby system that is also nonsingular. On the other hand, if $\mathbf{A}$ is singular, then perturbations of even the slightest magnitude can increase the rank, thereby producing a system with fewer free variables than the original system theoretically demands, so even a stable algorithm can result in a significant loss of information. But what are the chances that this will actually occur in practice? To answer this, recall from (4.5.11) that

$$rank(\mathbf{A}+\mathbf{E}) = rank(\mathbf{A}) + rank(\mathbf{S}), \quad \text{where} \quad \mathbf{S} = \mathbf{E}_{22} - \mathbf{E}_{21}\left(\mathbf{I}+\mathbf{E}_{11}\right)^{-1}\mathbf{E}_{12}.$$

If the rank is not to jump, then the perturbation $\mathbf{E}$ must be such that $\mathbf{S} = \mathbf{0}$, which is equivalent to saying $\mathbf{E}_{22} = \mathbf{E}_{21}(\mathbf{I} + \mathbf{E}_{11})^{-1}\mathbf{E}_{12}$. Clearly, this requires the existence of a very specific (and quite special) relationship among the entries of $\mathbf{E}$, and a random perturbation will almost never produce such a relationship. Although rounding errors cannot be considered to be truly random, they are random enough so as to make the possibility that $\mathbf{S} = \mathbf{0}$ very unlikely. Consequently, when $\mathbf{A}$ is singular, the small perturbation $\mathbf{E}$ due to roundoff makes the possibility that $rank(\mathbf{A} + \mathbf{E}) > rank(\mathbf{A})$ very likely. The moral is to avoid floating-point solutions of singular systems. Singular problems can often be distilled down to a nonsingular core or to nonsingular pieces, and these are the components you should be dealing with.

Since no more significant characterizations of rank will be given, it is appropriate to conclude this section with a summary of all of the different ways we have developed to say "rank."

Summary of Rank

For $\mathbf{A} \in \Re^{m\times n}$, each of the following statements is true.

- $rank(\mathbf{A}) =$ The number of nonzero rows in any row echelon form that is row equivalent to $\mathbf{A}$.
- $rank(\mathbf{A}) =$ The number of pivots obtained in reducing $\mathbf{A}$ to a row echelon form with row operations.
- $rank(\mathbf{A}) =$ The number of basic columns in $\mathbf{A}$ (as well as the number of basic columns in any matrix that is row equivalent to $\mathbf{A}$).
- $rank(\mathbf{A}) =$ The number of independent columns in $\mathbf{A}$—i.e., the size of a maximal independent set of columns from $\mathbf{A}$.
- $rank(\mathbf{A}) =$ The number of independent rows in $\mathbf{A}$—i.e., the size of a maximal independent set of rows from $\mathbf{A}$.
- $rank(\mathbf{A}) = \dim R(\mathbf{A})$.
- $rank(\mathbf{A}) = \dim R(\mathbf{A}^T)$.
- $rank(\mathbf{A}) = n - \dim N(\mathbf{A})$.
- $rank(\mathbf{A}) = m - \dim N(\mathbf{A}^T)$.
- $rank(\mathbf{A}) =$ The size of the largest nonsingular submatrix in $\mathbf{A}$.

For $\mathbf{A} \in \mathcal{C}^{m\times n}$, replace $(\star)^T$ with $(\star)^*$.

Exercises for section 4.5

4.5.1. Verify that $rank\left(\mathbf{A}^T\mathbf{A}\right) = rank\,(\mathbf{A}) = rank\left(\mathbf{A}\mathbf{A}^T\right)$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}.$$

4.5.2. Determine $\dim N(\mathbf{A}) \cap R(\mathbf{B})$ for

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}.$$

4.5.3. For the matrices given in Exercise 4.5.2, use the procedure described on p. 211 to determine a basis for $N(\mathbf{A}) \cap R(\mathbf{B})$.

4.5.4. If $\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k$ is a product of square matrices such that some $\mathbf{A}_i$ is singular, explain why the entire product must be singular.

4.5.5. For $\mathbf{A} \in \Re^{m\times n}$, explain why $\mathbf{A}^T\mathbf{A} = \mathbf{0}$ implies $\mathbf{A} = \mathbf{0}$.

4.5.6. Find $rank\,(\mathbf{A})$ and all nonsingular submatrices of maximal order in

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -2 & 1 \\ 8 & -4 & 1 \end{pmatrix}.$$

4.5.7. Is it possible that $rank\,(\mathbf{AB}) < rank\,(\mathbf{A})$ and $rank\,(\mathbf{AB}) < rank\,(\mathbf{B})$ for the same pair of matrices?

4.5.8. Is $rank\,(\mathbf{AB}) = rank\,(\mathbf{BA})$ when both products are defined? Why?

4.5.9. Explain why $rank\,(\mathbf{AB}) = rank\,(\mathbf{A}) - \dim N\left(\mathbf{B}^T\right) \cap R\left(\mathbf{A}^T\right)$.

4.5.10. Explain why $\dim N(\mathbf{A}_{m\times n}\mathbf{B}_{n\times p}) = \dim N(\mathbf{B}) + \dim R(\mathbf{B}) \cap N(\mathbf{A})$.

4.5.11. ***Sylvester's law of nullity,*** given by James J. Sylvester in 1884, states that for *square matrices* $\mathbf{A}$ and $\mathbf{B}$,

$$\max\{\nu(\mathbf{A}),\ \nu(\mathbf{B})\} \le \nu(\mathbf{AB}) \le \nu(\mathbf{A}) + \nu(\mathbf{B}),$$

where $\nu(\star) = \dim N(\star)$ denotes the nullity.

(a) Establish the validity of Sylvester's law.

(b) Show Sylvester's law is not valid for rectangular matrices because $\nu(\mathbf{A}) > \nu(\mathbf{AB})$ is possible. Is $\nu(\mathbf{B}) > \nu(\mathbf{AB})$ possible?

4.5.12. For matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times p}$, prove each of the following statements:

(a) $rank(\mathbf{AB}) = rank(\mathbf{A})$ and $R(\mathbf{AB}) = R(\mathbf{A})$ if $rank(\mathbf{B}) = n$.

(b) $rank(\mathbf{AB}) = rank(\mathbf{B})$ and $N(\mathbf{AB}) = N(\mathbf{B})$ if $rank(\mathbf{A}) = n$.

4.5.13. Perform the following calculations using the matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2.01 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1.01 \end{pmatrix}.$$

(a) Find $rank(\mathbf{A})$, and solve $\mathbf{Ax} = \mathbf{b}$ using exact arithmetic.

(b) Find $rank\left(\mathbf{A}^T\mathbf{A}\right)$, and solve $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ exactly.

(c) Find $rank(\mathbf{A})$, and solve $\mathbf{Ax} = \mathbf{b}$ with 3-digit arithmetic.

(d) Find $\mathbf{A}^T\mathbf{A}$, $\mathbf{A}^T\mathbf{b}$, and the solution of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ with 3-digit arithmetic.

4.5.14. Prove that if the entries of $\mathbf{F}_{r\times r}$ satisfy $\sum_{j=1}^{r} |f_{ij}| < 1$ for each i (i.e., each absolute row sum < 1), then $\mathbf{I} + \mathbf{F}$ is nonsingular. **Hint:** Use the triangle inequality for scalars $|\alpha + \beta| \le |\alpha| + |\beta|$ to show $N(\mathbf{I} + \mathbf{F}) = \mathbf{0}$.

4.5.15. If $\mathbf{A} = \begin{pmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix}$, where $rank(\mathbf{A}) = r = rank(\mathbf{W}_{r\times r})$, show that there are matrices $\mathbf{B}$ and $\mathbf{C}$ such that

$$\mathbf{A} = \begin{pmatrix} \mathbf{W} & \mathbf{WC} \\ \mathbf{BW} & \mathbf{BWC} \end{pmatrix} = \begin{pmatrix} \mathbf{I} \\ \mathbf{B} \end{pmatrix} \mathbf{W} \left(\mathbf{I} \mid \mathbf{C}\right).$$

4.5.16. For a convergent sequence $\{\mathbf{A}_k\}_{k=1}^{\infty}$ of matrices, let $\mathbf{A} = \lim_{k\to\infty} \mathbf{A}_k$.

(a) Prove that if each $\mathbf{A}_k$ is singular, then $\mathbf{A}$ is singular.

(b) If each $\mathbf{A}_k$ is nonsingular, must $\mathbf{A}$ be nonsingular? Why?

4.5.17. The Frobenius Inequality. Establish the validity of Frobenius's 1911 result that states that if $\mathbf{ABC}$ exists, then

$$rank\,(\mathbf{AB}) + rank\,(\mathbf{BC}) \leq rank\,(\mathbf{B}) + rank\,(\mathbf{ABC}).$$

Hint: If $\mathcal{M} = R\,(\mathbf{BC}) \cap N\,(\mathbf{A})$ and $\mathcal{N} = R\,(\mathbf{B}) \cap N\,(\mathbf{A})$, then $\mathcal{M} \subseteq \mathcal{N}$.

4.5.18. If $\mathbf{A}$ is $n \times n$, prove that the following statements are equivalent:

(a) $N\,(\mathbf{A}) = N\left(\mathbf{A}^2\right)$.

(b) $R\,(\mathbf{A}) = R\left(\mathbf{A}^2\right)$.

(c) $R\,(\mathbf{A}) \cap N\,(\mathbf{A}) = \{\mathbf{0}\}$.

4.5.19. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices such that $\mathbf{A} = \mathbf{A}^2$, $\mathbf{B} = \mathbf{B}^2$, and $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$.

(a) Prove that $rank\,(\mathbf{A} + \mathbf{B}) = rank\,(\mathbf{A}) + rank\,(\mathbf{B})$. **Hint:** Consider $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} (\mathbf{A} + \mathbf{B})(\mathbf{A} \mid \mathbf{B})$.

(b) Prove that $rank\,(\mathbf{A}) + rank\,(\mathbf{I} - \mathbf{A}) = n$.

4.5.20. Moore–Penrose Inverse. For $\mathbf{A} \in \Re^{m \times n}$ such that $rank\,(\mathbf{A}) = r$, let $\mathbf{A} = \mathbf{BC}$ be the full rank factorization of $\mathbf{A}$ in which $\mathbf{B}_{m \times r}$ is the matrix of basic columns from $\mathbf{A}$ and $\mathbf{C}_{r \times n}$ is the matrix of nonzero rows from $\mathbf{E_A}$ (see Exercise 3.9.8). The matrix defined by

$$\mathbf{A}^{\dagger} = \mathbf{C}^T \left(\mathbf{B}^T \mathbf{A} \mathbf{C}^T\right)^{-1} \mathbf{B}^T$$

is called the ***Moore–Penrose***[30] ***inverse*** of $\mathbf{A}$. Some authors refer to $\mathbf{A}^{\dagger}$ as the *pseudoinverse* or the *generalized inverse* of $\mathbf{A}$. A more elegant treatment is given on p. 423, but it's worthwhile to introduce the idea here so that it can be used and viewed from different perspectives.

(a) Explain why the matrix $\mathbf{B}^T \mathbf{A} \mathbf{C}^T$ is nonsingular.

(b) Verify that $\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b}$ solves the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ (as well as $\mathbf{Ax} = \mathbf{b}$ when it is consistent).

(c) Show that the general solution for $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ (as well as $\mathbf{Ax} = \mathbf{b}$ when it is consistent) can be described as

$$\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b} + \left(\mathbf{I} - \mathbf{A}^{\dagger}\mathbf{A}\right)\mathbf{h},$$

---

[30] This is in honor of Eliakim H. Moore (1862–1932) and Roger Penrose (a famous contemporary English mathematical physicist). Each formulated a concept of generalized matrix inversion—Moore's work was published in 1922, and Penrose's work appeared in 1955. E. H. Moore is considered by many to be America's first great mathematician.

where $\mathbf{h}$ is a "free variable" vector in $\Re^{n\times 1}$.
Hint: Verify $\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}$, and then show $R\left(\mathbf{I} - \mathbf{A}^{\dagger}\mathbf{A}\right) = N(\mathbf{A})$.

(d) If $rank(\mathbf{A}) = n$, explain why $\mathbf{A}^{\dagger} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$.

(e) If $\mathbf{A}$ is square and nonsingular, explain why $\mathbf{A}^{\dagger} = \mathbf{A}^{-1}$.

(f) Verify that $\mathbf{A}^{\dagger} = \mathbf{C}^T\left(\mathbf{B}^T\mathbf{A}\mathbf{C}^T\right)^{-1}\mathbf{B}^T$ satisfies the Penrose equations:

$$\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}, \qquad \left(\mathbf{A}\mathbf{A}^{\dagger}\right)^T = \mathbf{A}\mathbf{A}^{\dagger},$$

$$\mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}, \qquad \left(\mathbf{A}^{\dagger}\mathbf{A}\right)^T = \mathbf{A}^{\dagger}\mathbf{A}.$$

Penrose originally defined $\mathbf{A}^{\dagger}$ to be the unique solution to these four equations.

4.6 CLASSICAL LEAST SQUARES

The following problem arises in almost all areas where mathematics is applied. At discrete points t_i (often points in time), observations b_i of some phenomenon are made, and the results are recorded as a set of ordered pairs

$$\mathcal{D} = \{(t_1, b_1), \quad (t_2, b_2), \; \ldots, \; (t_m, b_m)\}.$$

On the basis of these observations, the problem is to make estimations or predictions at points (times) $\hat{t}$ that are between or beyond the observation points t_i. A standard approach is to find the equation of a curve $y = f(t)$ that closely fits the points in $\mathcal{D}$ so that the phenomenon can be estimated at any nonobservation point $\hat{t}$ with the value $\hat{y} = f(\hat{t})$.

Let's begin by fitting a straight line to the points in $\mathcal{D}$. Once this is understood, it will be relatively easy to see how to fit the data with curved lines.

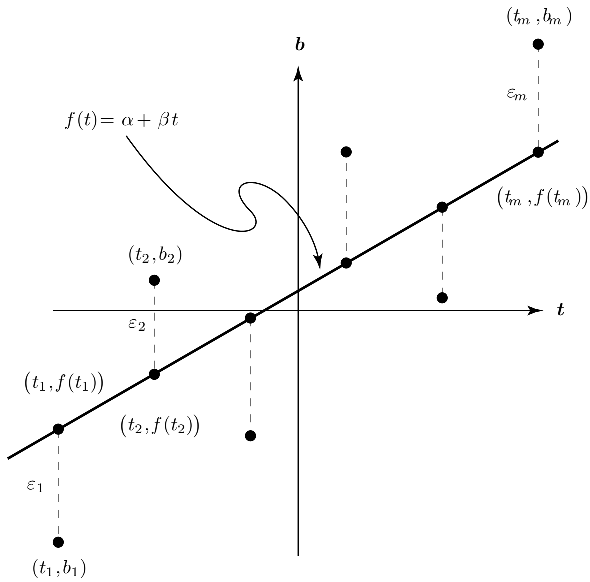

FIGURE 4.6.1

The strategy is to determine the coefficients α and β in the equation of the line $f(t) = \alpha + \beta t$ that best fits the points (t_i, b_i) in the sense that the sum of the squares of the vertical[31] errors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ indicated in Figure 4.6.1 is

[31] We consider only vertical errors because there is a tacit assumption that only the observations b_i are subject to error or variation. The t_i 's are assumed to be errorless constants—think of them as being exact points in time (as they often are). If the t_i 's are also subject to variation, then horizontal as well as vertical errors have to be considered in Figure 4.6.1, and a more complicated theory known as *total least squares* (not considered in this text) emerges. The least squares line $\mathcal{L}$ obtained by minimizing only vertical deviations will not be the closest line to points in $\mathcal{D}$ in terms of perpendicular distance, but $\mathcal{L}$ is the best line for the purpose of linear estimation—see §5.14 (p. 446).

minimal. The distance from (t_i, b_i) to a line $f(t) = \alpha + \beta t$ is

$$\varepsilon_i = |f(t_i) - b_i| = |\alpha + \beta t_i - b_i|,$$

so that the objective is to find values for α and β such that

$$\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 \quad \text{is minimal.}$$

Minimization techniques from calculus tell us that the minimum value must occur at a solution to the two equations

$$0 = \frac{\partial \left(\sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 \right)}{\partial \alpha} = 2 \sum_{i=1}^{m} (\alpha + \beta t_i - b_i),$$

$$0 = \frac{\partial \left(\sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 \right)}{\partial \beta} = 2 \sum_{i=1}^{m} (\alpha + \beta t_i - b_i)\, t_i.$$

Rearranging terms produces two equations in the two unknowns α and β

$$\begin{aligned} \left(\sum_{i=1}^{m} 1 \right) \alpha + \left(\sum_{i=1}^{m} t_i \right) \beta &= \sum_{i=1}^{m} b_i, \\ \left(\sum_{i=1}^{m} t_i \right) \alpha + \left(\sum_{i=1}^{m} t_i^2 \right) \beta &= \sum_{i=1}^{m} t_i b_i. \end{aligned} \tag{4.6.1}$$

By setting

$$\mathbf{A} = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

we see that the two equations (4.6.1) have the matrix form $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. In other words, (4.6.1) is the system of normal equations associated with the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ (see p. 213). The t_i 's are assumed to be distinct numbers, so $rank\,(\mathbf{A}) = 2$, and (4.5.7) insures that the normal equations have a unique solution given by

$$\begin{aligned} \mathbf{x} &= \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} \\ &= \frac{1}{m\sum t_i^2 - \left(\sum t_i\right)^2} \begin{pmatrix} \sum t_i^2 & -\sum t_i \\ -\sum t_i & m \end{pmatrix} \begin{pmatrix} \sum b_i \\ \sum t_i b_i \end{pmatrix} \\ &= \frac{1}{m\sum t_i^2 - \left(\sum t_i\right)^2} \begin{pmatrix} \sum t_i^2 \sum b_i - \sum t_i \sum t_i b_i \\ m\sum t_i b_i - \sum t_i \sum b_i \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \end{aligned}$$

Finally, notice that the total sum of squares of the errors is given by

$$\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 = (\mathbf{A}\mathbf{x} - \mathbf{b})^T(\mathbf{A}\mathbf{x} - \mathbf{b}).$$

Example 4.6.1

Problem: A small company has been in business for four years and has recorded annual sales (in tens of thousands of dollars) as follows.

Year	1	2	3	4
Sales	23	27	30	34

When this data is plotted as shown in Figure 4.6.2, we see that although the points do not exactly lie on a straight line, there nevertheless appears to be a linear trend. Predict the sales for any future year if this trend continues.

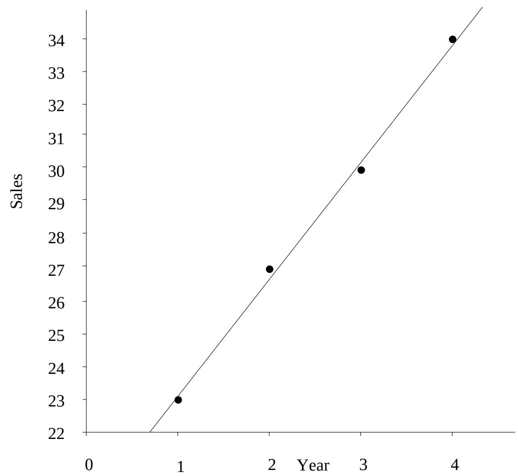

Figure 4.6.2

Solution: Determine the line $f(t) = \alpha + \beta t$ that best fits the data in the sense of least squares. If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 23 \\ 27 \\ 30 \\ 34 \end{pmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

then the previous discussion guarantees that $\mathbf{x}$ is the solution of the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. That is,

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 114 \\ 303 \end{pmatrix}.$$

The solution is easily found to be $\alpha = 19.5$ and $\beta = 3.6$, so we predict that the sales in year t will be $f(t) = 19.5 + 3.6t$. For example, the estimated sales for year five is \$375,000. To get a feel for how close the least squares line comes to passing through the data points, let $\boldsymbol{\varepsilon} = \mathbf{Ax} - \mathbf{b}$, and compute the sum of the squares of the errors to be

$$\sum_{i=1}^{m} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = (\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}) = .2.$$

General Least Squares Problem

For $\mathbf{A} \in \Re^{m\times n}$ and $\mathbf{b} \in \Re^m$, let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{Ax} - \mathbf{b}$. The general least squares problem is to find a vector $\mathbf{x}$ that minimizes the quantity

$$\sum_{i=1}^{m} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = (\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}).$$

Any vector that provides a minimum value for this expression is called a ***least squares solution.***

- The set of all least squares solutions is precisely the set of solutions to the system of normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.
- There is a unique least squares solution if and only if $rank(\mathbf{A}) = n$, in which case it is given by $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.
- If $\mathbf{Ax} = \mathbf{b}$ is consistent, then the solution set for $\mathbf{Ax} = \mathbf{b}$ is the same as the set of least squares solutions.

Proof.[32] First prove that if $\mathbf{x}$ minimizes $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}$, then $\mathbf{x}$ must satisfy the normal equations. Begin by using $\mathbf{x}^T\mathbf{A}^T\mathbf{b} = \mathbf{b}^T\mathbf{Ax}$ (scalars are symmetric) to write

$$\sum_{i=1}^{m} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = (\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}) = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}. \tag{4.6.2}$$

To determine vectors $\mathbf{x}$ that minimize the expression (4.6.2), we will again use minimization techniques from calculus and differentiate the function

$$f(x_1, x_2, \ldots, x_n) = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b} \tag{4.6.3}$$

with respect to each x_i. Differentiating matrix functions is similar to differentiating scalar functions (see Exercise 3.5.9) in the sense that if $\mathbf{U} = [u_{ij}]$, then

$$\left[\frac{\partial\mathbf{U}}{\partial x}\right]_{ij} = \frac{\partial u_{ij}}{\partial x}, \quad \frac{\partial[\mathbf{U}+\mathbf{V}]}{\partial x} = \frac{\partial\mathbf{U}}{\partial x} + \frac{\partial\mathbf{V}}{\partial x}, \quad \text{and} \quad \frac{\partial[\mathbf{UV}]}{\partial x} = \frac{\partial\mathbf{U}}{\partial x}\mathbf{V} + \mathbf{U}\frac{\partial\mathbf{V}}{\partial x}.$$

---

[32] A more modern development not relying on calculus is given in §5.13 on p. 437, but the more traditional approach is given here because it's worthwhile to view least squares from both perspectives.

Applying these rules to the function in (4.6.3) produces

$$\frac{\partial f}{\partial x_i} = \frac{\partial \mathbf{x}^T}{\partial x_i}\mathbf{A}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{A}^T\mathbf{A}\frac{\partial \mathbf{x}}{\partial x_i} - 2\frac{\partial \mathbf{x}^T}{\partial x_i}\mathbf{A}^T\mathbf{b}.$$

Since $\partial\mathbf{x}/\partial x_i = \mathbf{e}_i$ (the i^{th} unit vector), we have

$$\frac{\partial f}{\partial x_i} = \mathbf{e}_i^T\mathbf{A}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{e}_i - 2\mathbf{e}_i^T\mathbf{A}^T\mathbf{b} = 2\mathbf{e}_i^T\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{e}_i^T\mathbf{A}^T\mathbf{b}.$$

Using $\mathbf{e}_i^T\mathbf{A}^T = \left(\mathbf{A}^T\right)_{i*}$ and setting $\partial f/\partial x_i = 0$ produces the n equations

$$\left(\mathbf{A}^T\right)_{i*}\mathbf{A}\mathbf{x} = \left(\mathbf{A}^T\right)_{i*}\mathbf{b} \quad \text{for } i = 1, 2, \ldots, n,$$

which can be written as the single matrix equation $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. Calculus guarantees that the minimum value of f occurs at *some* solution of this system. But this is not enough—we want to know that *every* solution of $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ is a least squares solution. So we must show that the function f in (4.6.3) attains its minimum value at each solution to $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. Observe that if $\mathbf{z}$ is a solution to the normal equations, then $f(\mathbf{z}) = \mathbf{b}^T\mathbf{b} - \mathbf{z}^T\mathbf{A}^T\mathbf{b}$. For any other $\mathbf{y} \in \Re^{n\times 1}$, let $\mathbf{u} = \mathbf{y} - \mathbf{z}$, so $\mathbf{y} = \mathbf{z} + \mathbf{u}$, and observe that

$$f(\mathbf{y}) = f(\mathbf{z}) + \mathbf{v}^T\mathbf{v}, \quad \text{where} \quad \mathbf{v} = \mathbf{A}\mathbf{u}.$$

Since $\mathbf{v}^T\mathbf{v} = \sum_i \mathbf{v}_i^2 \geq 0$, it follows that $f(\mathbf{z}) \leq f(\mathbf{y})$ for all $\mathbf{y} \in \Re^{n\times 1}$, and thus f attains its minimum value at each solution of the normal equations. The remaining statements in the theorem follow from the properties established on p. 213. ∎

The classical least squares problem discussed at the beginning of this section and illustrated in Example 4.6.1 is part of a broader topic known as *linear regression,* which is the study of situations where attempts are made to express one variable y as a linear combination of other variables $t_1, t_2, \ldots, t_n$. In practice, hypothesizing that y is linearly related to $t_1, t_2, \ldots, t_n$ means that one assumes the existence of a set of constants $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ (called *parameters*) such that

$$y = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n + \varepsilon,$$

where ε is a "random function" whose values "average out" to zero in some sense. Practical problems almost always involve more variables than we wish to consider, but it is frequently fair to assume that the effect of variables of lesser significance will indeed "average out" to zero. The random function ε accounts for this assumption. In other words, a linear hypothesis is the supposition that the expected (or mean) value of y at each point where the phenomenon can be observed is given by a linear equation

$$E(y) = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n.$$

To help seat these ideas, consider the problem of predicting the amount of weight that a pint of ice cream loses when it is stored at very low temperatures. There are many factors that may contribute to weight loss—e.g., storage temperature, storage time, humidity, atmospheric pressure, butterfat content, the amount of corn syrup, the amounts of various gums (guar gum, carob bean gum, locust bean gum, cellulose gum), and the never-ending list of other additives and preservatives. It is reasonable to believe that storage time and temperature are the primary factors, so to predict weight loss we will make a linear hypothesis of the form

$$y = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2 + \varepsilon,$$

where $y =$ weight loss (grams), $t_1 =$ storage time (weeks), $t_2 =$ storage temperature (${}^oF$), and ε is a random function to account for all other factors. The assumption is that all other factors "average out" to zero, so the expected (or mean) weight loss at each point (t_1, t_2) is

$$E(y) = \alpha_0 + \alpha_1 t_1 + \alpha_2 t_2. \tag{4.6.4}$$

Suppose that we conduct an experiment in which values for weight loss are measured for various values of storage time and temperature as shown below.

Time (weeks)	1	1	1	2	2	2	3	3	3
Temp (${}^oF$)	−10	−5	0	−10	−5	0	−10	−5	0
Loss (grams)	.15	.18	.20	.17	.19	.22	.20	.23	.25

If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -10 \\ 1 & 1 & -5 \\ 1 & 1 & 0 \\ 1 & 2 & -10 \\ 1 & 2 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & -10 \\ 1 & 3 & -5 \\ 1 & 3 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} .15 \\ .18 \\ .20 \\ .17 \\ .19 \\ .22 \\ .20 \\ .23 \\ .25 \end{pmatrix},$$

and if we were lucky enough to exactly observe the mean weight loss each time (i.e., if $\mathbf{b}_i = E(y_i)$), then equation (4.6.4) would insure that $\mathbf{Ax} = \mathbf{b}$ is a consistent system, so we could solve for the unknown parameters $\alpha_0, \alpha_1,$ and α_2. However, it is virtually impossible to observe the *exact* value of the mean weight loss for a given storage time and temperature, and almost certainly the system defined by $\mathbf{Ax} = \mathbf{b}$ will be inconsistent—especially when the number of observations greatly exceeds the number of parameters. Since we can't solve $\mathbf{Ax} = \mathbf{b}$ to find exact values for the α_i 's, the best we can hope for is a set of "good estimates" for these parameters.

The famous Gauss–Markov theorem (developed on p. 448) states that under certain reasonable assumptions concerning the random error function ε, the "best" estimates for the α_i 's are obtained by minimizing the sum of squares $(\mathbf{Ax}-\mathbf{b})^T(\mathbf{Ax}-\mathbf{b})$. In other words, the least squares estimates are the "best" way to estimate the α_i 's.

Returning to our ice cream example, it can be verified that $\mathbf{b}\notin R(\mathbf{A})$, so, as expected, the system $\mathbf{Ax}=\mathbf{b}$ is not consistent, and we cannot determine exact values for α_0,α_1, and α_2. The best we can do is to determine least squares estimates for the α_i 's by solving the associated normal equations $\mathbf{A}^T\mathbf{Ax}=\mathbf{A}^T\mathbf{b}$, which in this example are

$$\begin{pmatrix} 9 & 18 & -45 \\ 18 & 42 & -90 \\ -45 & -90 & 375 \end{pmatrix}\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix}=\begin{pmatrix}1.79\\ 3.73\\ -8.2\end{pmatrix}.$$

The solution is

$$\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix}=\begin{pmatrix}.174\\ .025\\ .005\end{pmatrix},$$

and the estimating equation for mean weight loss becomes

$$\hat{y}=.174+.025t_1+.005t_2.$$

For example, the mean weight loss of a pint of ice cream that is stored for nine weeks at a temperature of -35^oF is estimated to be

$$\hat{y}=.174+.025(9)+.005(-35)=.224 \text{ grams.}$$

Example 4.6.2

Least Squares Curve Fitting Problem: Find a polynomial

$$p(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_{n-1}t^{n-1}$$

with a specified degree that comes as close as possible in the sense of least squares to passing through a set of data points

$$\mathcal{D}=\{(t_1,b_1),\quad (t_2,b_2),\ \ldots,\ (t_m,b_m)\},$$

where the t_i 's are distinct numbers, and $n\le m$.

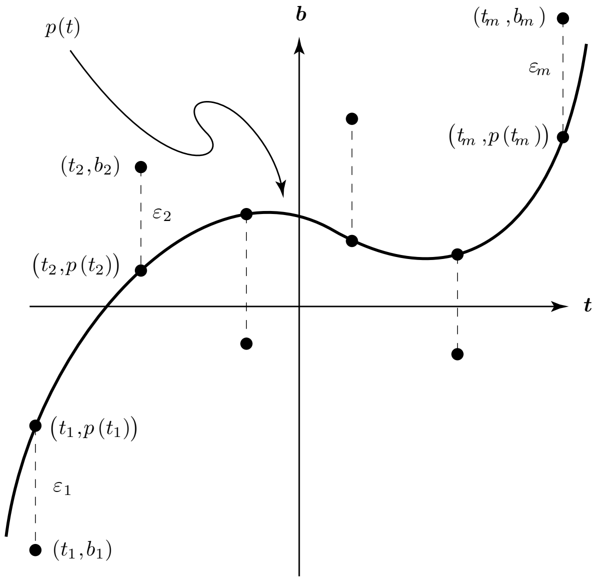

FIGURE 4.6.3

Solution: For the ε_i 's indicated in Figure 4.6.3, the objective is to minimize the sum of squares

$$\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (p(t_i) - b_i)^2 = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}),$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

In other words, the least squares polynomial of degree $n-1$ is obtained from the least squares solution associated with the system $\mathbf{Ax} = \mathbf{b}$. Furthermore, this least squares polynomial is unique because $\mathbf{A}_{m\times n}$ is the Vandermonde matrix of Example 4.3.4 with $n \le m$, so $rank(\mathbf{A}) = n$, and $\mathbf{Ax} = \mathbf{b}$ has a unique least squares solution given by $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.

Note: We know from Example 4.3.5 on p. 186 that the Lagrange interpolation polynomial $\ell(t)$ of degree $m-1$ will *exactly* fit the data—i.e., it passes through each point in $\mathcal{D}$. So why would one want to settle for a least squares fit when an exact fit is possible? One answer stems from the fact that in practical work the observations b_i are rarely exact due to small errors arising from imprecise measurements or from simplifying assumptions. For this reason, it is the *trend* of the observations that needs to be fitted and not the observations themselves. To hit the data points, the interpolation polynomial $\ell(t)$ is usually forced to oscillate between or beyond the data points, and as m becomes larger the oscillations can become more pronounced. Consequently, $\ell(t)$ is generally not useful in making estimations concerning the trend of the observations—Example 4.6.3 drives this point home. In addition to exactly hitting a prescribed set of data points, an interpolation polynomial called the *Hermite polynomial* (p. 607) can be constructed to have specified derivatives at each data point. While this helps, it still is not as good as least squares for making estimations on the basis of observations.

Example 4.6.3

A missile is fired from enemy territory, and its position in flight is observed by radar tracking devices at the following positions.

Position down range (miles)	0	250	500	750	1000
Height (miles)	0	8	15	19	20

Suppose our intelligence sources indicate that enemy missiles are programmed to follow a parabolic flight path—a fact that seems to be consistent with the diagram obtained by plotting the observations on the coordinate system shown in Figure 4.6.4.

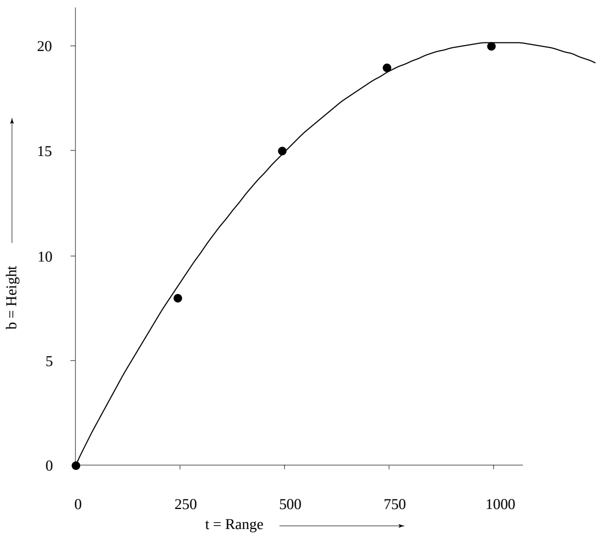

FIGURE 4.6.4

Problem: Predict how far down range the missile will land.

Solution: Determine the parabola $f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$ that best fits the observed data in the least squares sense. Then estimate where the missile will land by determining the roots of f (i.e., determine where the parabola crosses the horizontal axis). As it stands, the problem will involve numbers having relatively large magnitudes in conjunction with relatively small ones. Consequently, it is better to first scale the data by considering one unit to be 1000 miles. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & .25 & .0625 \\ 1 & .5 & .25 \\ 1 & .75 & .5625 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ .008 \\ .015 \\ .019 \\ .02 \end{pmatrix},$$

and if $\boldsymbol{\varepsilon} = \mathbf{Ax} - \mathbf{b}$, then the object is to find a least squares solution $\mathbf{x}$ that minimizes

$$\sum_{i=1}^{5} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}).$$

We know that such a least squares solution is given by the solution to the system of normal equations $\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}$, which in this case is

$$\begin{pmatrix} 5 & 2.5 & 1.875 \\ 2.5 & 1.875 & 1.5625 \\ 1.875 & 1.5625 & 1.3828125 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} .062 \\ .04375 \\ .0349375 \end{pmatrix}.$$

The solution (rounded to four significant digits) is

$$\mathbf{x} = \begin{pmatrix} -2.286 \times 10^{-4} \\ 3.983 \times 10^{-2} \\ -1.943 \times 10^{-2} \end{pmatrix},$$

and the least squares parabola is

$$f(t) = -.0002286 + .03983t - .01943t^2.$$

To estimate where the missile will land, determine where this parabola crosses the horizontal axis by applying the quadratic formula to find the roots of $f(t)$ to be $t = .005755$ and $t = 2.044$. Therefore, we estimate that the missile will land 2044 miles down range. The sum of the squares of the errors associated with the least squares solution is

$$\sum_{i=1}^{5} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = 4.571 \times 10^{-7}.$$

Least Squares vs. Lagrange Interpolation. Instead of using least squares, fit the observations exactly with the fourth-degree Lagrange interpolation polynomial

$$\ell(t) = \frac{11}{375}t + \frac{17}{750000}t^2 - \frac{1}{18750000}t^3 + \frac{1}{46875000000}t^4$$

described in Example 4.3.5 on p. 186 (you can verify that $\ell(t_i) = b_i$ for each observation). As the graph in Figure 4.6.5 indicates, $\ell(t)$ has only one real nonnegative root, so it is worthless for predicting where the missile will land. This is characteristic of Lagrange interpolation.

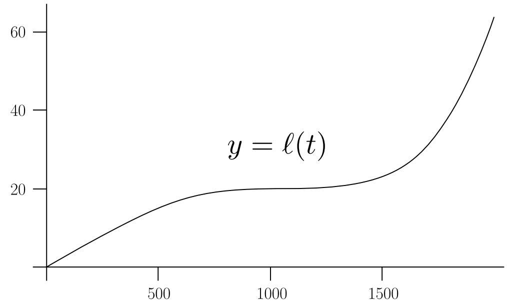

FIGURE 4.6.5

Computational Note: Theoretically, the least squares solutions of $\mathbf{Ax} = \mathbf{b}$ are exactly the solutions of the normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$, but forming and solving the normal equations to compute least squares solutions with floating-point arithmetic is not recommended. As pointed out in Example 4.5.1 on p. 214, any sensitivities to small perturbations that are present in the underlying problem are magnified by forming the normal equations. In other words, if the underlying problem is somewhat ill-conditioned, then the system of normal equations will be ill-conditioned to an even greater extent. Numerically stable techniques that avoid the normal equations are presented in Example 5.5.3 on p. 313 and Example 5.7.3 on p. 346.

Epilogue

While viewing a region in the Taurus constellation on January 1, 1801, Giuseppe Piazzi, an astronomer and director of the Palermo observatory, observed a small "star" that he had never seen before. As Piazzi and others continued to watch this new "star"—which was really an asteroid—they noticed that it was in fact moving, and they concluded that a new "planet" had been discovered. However, their new "planet" completely disappeared in the autumn of 1801. Well-known astronomers of the time joined the search to relocate the lost "planet," but all efforts were in vain.

In September of 1801 Carl F. Gauss decided to take up the challenge of finding this lost "planet." Gauss allowed for the possibility of an elliptical orbit rather than constraining it to be circular—which was an assumption of the others—and he proceeded to develop the method of least squares. By December the task was completed, and Gauss informed the scientific community not only where the lost "planet" was located, but he also predicted its position at future times. They looked, and it was exactly where Gauss had predicted it would be! The asteroid was named *Ceres*, and Gauss's contribution was recognized by naming another minor asteroid *Gaussia.*

This extraordinary feat of locating a tiny and distant heavenly body from apparently insufficient data astounded the scientific community. Furthermore, Gauss refused to reveal his methods, and there were those who even accused him of sorcery. These events led directly to Gauss's fame throughout the entire European community, and they helped to establish his reputation as a mathematical and scientific genius of the highest order.

Gauss waited until 1809, when he published his *Theoria Motus Corporum Coelestium In Sectionibus Conicis Solem Ambientium,* to systematically develop the theory of least squares and his methods of orbit calculation. This was in keeping with Gauss's philosophy to publish nothing but well-polished work of lasting significance. When criticized for not revealing more motivational aspects in his writings, Gauss remarked that architects of great cathedrals do not obscure the beauty of their work by leaving the scaffolds in place after the construction has been completed. Gauss's theory of least squares approximation has indeed proven to be a great mathematical cathedral of lasting beauty and significance.

Exercises for section 4.6

4.6.1. Hooke's law says that the displacement y of an ideal spring is proportional to the force x that is applied—i.e., $y = kx$ for some constant k. Consider a spring in which k is unknown. Various masses are attached, and the resulting displacements shown in Figure 4.6.6 are observed. Using these observations, determine the least squares estimate for k.

x (lb)	y (in)
5	11.1
7	15.4
8	17.5
10	22.0
12	26.3

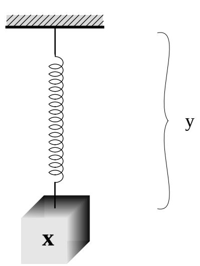

FIGURE 4.6.6

4.6.2. Show that the slope of the line that passes through the origin in $\Re^2$ and comes closest in the least squares sense to passing through the points $\{(x_1, y_1),\ (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $m = \sum_i x_i y_i / \sum_i x_i^2$.

4.6.3. A small company has been in business for three years and has recorded annual profits (in thousands of dollars) as follows.

Year	1	2	3
Sales	7	4	3

Assuming that there is a linear trend in the declining profits, predict the year and the month in which the company begins to lose money.

4.6.4. An economist hypothesizes that the change (in dollars) in the price of a loaf of bread is primarily a linear combination of the change in the price of a bushel of wheat and the change in the minimum wage. That is, if B is the change in bread prices, W is the change in wheat prices, and M is the change in the minimum wage, then $B = \alpha W + \beta M$. Suppose that for three consecutive years the change in bread prices, wheat prices, and the minimum wage are as shown below.

	Year 1	Year 2	Year 3
B	+\$1	+\$1	+\$1
W	+\$1	+\$2	0\$
M	+\$1	0\$	−\$1

Use the theory of least squares to estimate the change in the price of bread in Year 4 if wheat prices and the minimum wage each fall by \$1.

4.6.5. Suppose that a researcher hypothesizes that the weight loss of a pint of ice cream during storage is primarily a linear function of time. That is,

$$y = \alpha_0 + \alpha_1 t + \varepsilon,$$

where $y =$ the weight loss in grams, $t =$ the storage time in weeks, and ε is a random error function whose mean value is 0. Suppose that an experiment is conducted, and the following data is obtained.

Time (t)	1	2	3	4	5	6	7	8
Loss (y)	.15	.21	.30	.41	.49	.59	.72	.83

(a) Determine the least squares estimates for the parameters α_0 and α_1.

(b) Predict the mean weight loss for a pint of ice cream that is stored for 20 weeks.

4.6.6. After studying a certain type of cancer, a researcher hypothesizes that in the short run the number (y) of malignant cells in a particular tissue grows exponentially with time (t). That is, $y = \alpha_0 e^{\alpha_1 t}$. Determine least squares estimates for the parameters α_0 and α_1 from the researcher's observed data given below.

t (days)	1	2	3	4	5
y (cells)	16	27	45	74	122

Hint: What common transformation converts an exponential function into a linear function?

4.6.7. Using least squares techniques, fit the following data

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	2	7	9	12	13	14	14	13	10	8	4

with a line $y = \alpha_0 + \alpha_1 x$ and then fit the data with a quadratic $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$. Determine which of these two curves best fits the data by computing the sum of the squares of the errors in each case.

4.6.8. Consider the time (T) it takes for a runner to complete a marathon (26 miles and 385 yards). Many factors such as height, weight, age, previous training, etc. can influence an athlete's performance, but experience has shown that the following three factors are particularly important:

$$x_1 = \text{Ponderal index} = \frac{\text{height (in.)}}{[\text{weight (lbs.)}]^{\frac{1}{3}}},$$

$$x_2 = \text{Miles run the previous 8 weeks},$$

$$x_3 = \text{Age (years)}.$$

A linear model hypothesizes that the time T (in minutes) is given by $T = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \varepsilon$, where ε is a random function accounting for all other factors and whose mean value is assumed to be zero. On the basis of the five observations given below, estimate the expected marathon time for a 43-year-old runner of height 74 in., weight 180 lbs., who has run 450 miles during the previous eight weeks.

T	x_1	x_2	x_3
181	13.1	619	23
193	13.5	803	42
212	13.8	207	31
221	13.1	409	38
248	12.5	482	45

What is your personal predicted mean marathon time?

4.6.9. For $\mathbf{A} \in \Re^{m\times n}$ and $\mathbf{b} \in \Re^{m}$, prove that $\mathbf{x}_2$ is a least squares solution for $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{x}_2$ is part of a solution to the larger system

$$\begin{pmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n\times n} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}. \tag{4.6.5}$$

Note: It is not uncommon to encounter least squares problems in which $\mathbf{A}$ is extremely large but very sparse (mostly zero entries). For these situations, the system (4.6.5) will usually contain significantly fewer nonzero entries than the system of normal equations, thereby helping to overcome the memory requirements that plague these problems. Using (4.6.5) also eliminates the undesirable need to explicitly form the product $\mathbf{A}^T\mathbf{A}$—recall from Example 4.5.1 that forming $\mathbf{A}^T\mathbf{A}$ can cause loss of significant information.

4.6.10. In many least squares applications, the underlying data matrix $\mathbf{A}_{m\times n}$ does not have independent columns—i.e., $rank\,(\mathbf{A}) < n$—so the corresponding system of normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ will fail to have a unique solution. This means that in an associated linear estimation problem of the form

$$y = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n + \varepsilon$$

there will be infinitely many least squares estimates for the parameters α_i, and hence there will be infinitely many estimates for the mean value of y at any given point $(t_1, t_2, \ldots, t_n)$—which is clearly an undesirable situation. In order to remedy this problem, we restrict ourselves to making estimates only at those points $(t_1, t_2, \ldots, t_n)$ that are in the row space of $\mathbf{A}$. If

$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} \in R\left(\mathbf{A}^T\right), \quad \text{and if} \quad \mathbf{x} = \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_n \end{pmatrix}$$

is any least squares solution (i.e., $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$), prove that the estimate defined by

$$\hat{y} = \mathbf{t}^T\mathbf{x} = \sum_{i=1}^{n} t_i \hat{\alpha}_i$$

is unique in the sense that $\hat{y}$ is independent of which least squares solution $\mathbf{x}$ is used.

4.7 LINEAR TRANSFORMATIONS

The connection between linear functions and matrices is at the heart of our subject. As explained on p. 93, matrix algebra grew out of Cayley's observation that the composition of two linear functions can be represented by the multiplication of two matrices. It's now time to look deeper into such matters and to formalize the connections between matrices, vector spaces, and linear functions defined on vector spaces. This is the point at which linear algebra, as the study of linear functions on vector spaces, begins in earnest.

Linear Transformations

Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field $\mathcal{F}$ ($\Re$ or $\mathcal{C}$ for us).

- A ***linear transformation*** from $\mathcal{U}$ into $\mathcal{V}$ is defined to be a linear function $\mathbf{T}$ mapping $\mathcal{U}$ into $\mathcal{V}$. That is,

$$\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \quad \text{and} \quad \mathbf{T}(\alpha\mathbf{x}) = \alpha\mathbf{T}(\mathbf{x}) \qquad (4.7.1)$$

or, equivalently,

$$\mathbf{T}(\alpha\mathbf{x}+\mathbf{y}) = \alpha\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \ \text{ for all } \mathbf{x},\mathbf{y}\in\mathcal{U},\ \alpha\in\mathcal{F}. \qquad (4.7.2)$$

- A ***linear operator*** on $\mathcal{U}$ is defined to be a linear transformation from $\mathcal{U}$ into itself—i.e., a linear function mapping $\mathcal{U}$ back into $\mathcal{U}$.

Example 4.7.1

- The function $\mathbf{0}(\mathbf{x}) = \mathbf{0}$ that maps all vectors in a space $\mathcal{U}$ to the zero vector in another space $\mathcal{V}$ is a linear transformation from $\mathcal{U}$ into $\mathcal{V}$, and, not surprisingly, it is called the ***zero transformation.***
- The function $\mathbf{I}(\mathbf{x}) = \mathbf{x}$ that maps every vector from a space $\mathcal{U}$ back to itself is a linear operator on $\mathcal{U}$. $\mathbf{I}$ is called the ***identity operator*** on $\mathcal{U}$.
- For $\mathbf{A}\in\Re^{m\times n}$ and $\mathbf{x}\in\Re^{n\times 1}$, the function $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is a linear transformation from $\Re^n$ into $\Re^m$ because matrix multiplication satisfies $\mathbf{A}(\alpha\mathbf{x}+\mathbf{y}) = \alpha\mathbf{A}\mathbf{x}+\mathbf{A}\mathbf{y}$. $\mathbf{T}$ is a linear operator on $\Re^n$ if $\mathbf{A}$ is $n\times n$.
- If $\mathcal{W}$ is the vector space of all functions from $\Re$ to $\Re$, and if $\mathcal{V}$ is the space of all differentiable functions from $\Re$ to $\Re$, then the mapping $\mathbf{D}(f) = df/dx$ is a linear transformation from $\mathcal{V}$ into $\mathcal{W}$ because

$$\frac{d(\alpha f+g)}{dx} = \alpha\frac{df}{dx} + \frac{dg}{dx}.$$

- If $\mathcal{V}$ is the space of all continuous functions from $\Re$ into $\Re$, then the mapping defined by $\mathbf{T}(f) = \int_0^x f(t)dt$ is a linear operator on $\mathcal{V}$ because

$$\int_0^x [\alpha f(t)+g(t)]\,dt = \alpha\int_0^x f(t)dt + \int_0^x g(t)dt.$$

- The ***rotator*** $\mathbf{Q}$ that rotates vectors $\mathbf{u}$ in $\Re^2$ counterclockwise through an angle θ, as shown in Figure 4.7.1, is a linear operator on $\Re^2$ because the "action" of $\mathbf{Q}$ on $\mathbf{u}$ can be described by matrix multiplication in the sense that the coordinates of the rotated vector $\mathbf{Q}(\mathbf{u})$ are given by

$$\mathbf{Q}(\mathbf{u}) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- The ***projector*** $\mathbf{P}$ that maps each point $\mathbf{v} = (x, y, z) \in \Re^3$ to its orthogonal projection $(x, y, 0)$ in the xy-plane, as depicted in Figure 4.7.2, is a linear operator on $\Re^3$ because if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{P}(\alpha\mathbf{u} + \mathbf{v}) = (\alpha u_1 + v_1, \alpha u_2 + v_2, 0) = \alpha(u_1, u_2, 0) + (v_1, v_2, 0) = \alpha\mathbf{P}(\mathbf{u}) + \mathbf{P}(\mathbf{v}).$$

- The ***reflector*** $\mathbf{R}$ that maps each vector $\mathbf{v} = (x, y, z) \in \Re^3$ to its reflection $\mathbf{R}(\mathbf{v}) = (x, y, -z)$ about the xy-plane, as shown in Figure 4.7.3, is a linear operator on $\Re^3$.

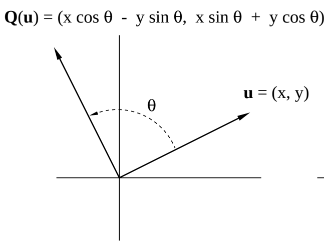

Figure 4.7.1

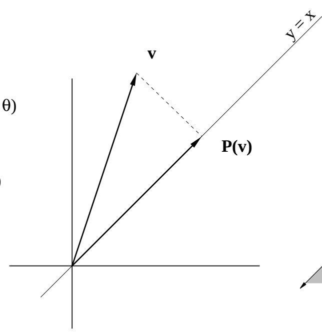

Figure 4.7.2

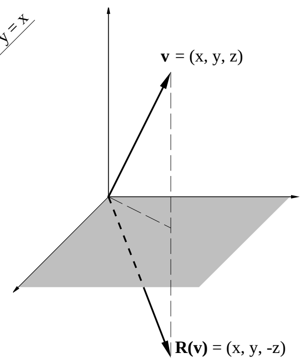

Figure 4.7.3

- Just as the rotator $\mathbf{Q}$ is represented by a matrix $[\mathbf{Q}] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, the projector $\mathbf{P}$ and the reflector $\mathbf{R}$ can be represented by matrices

$$[\mathbf{P}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad [\mathbf{R}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in the sense that the "action" of $\mathbf{P}$ and $\mathbf{R}$ on $\mathbf{v} = (x, y, z)$ can be accomplished with matrix multiplication using $[\mathbf{P}]$ and $[\mathbf{R}]$ by writing

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}.$$

It would be wrong to infer from Example 4.7.1 that all linear transformations can be represented by matrices (of finite size). For example, the differential and integral operators do not have matrix representations because they are defined on infinite-dimensional spaces. But linear transformations on *finite*-dimensional spaces will always have matrix representations. To see why, the concept of "coordinates" in higher dimensions must first be understood.

Recall that if $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for a vector space $\mathcal{U}$, then each $\mathbf{v} \in \mathcal{U}$ can be written as $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$. The α_i 's in this expansion are uniquely determined by $\mathbf{v}$ because if $\mathbf{v} = \sum_i \alpha_i\mathbf{u}_i = \sum_i \beta_i\mathbf{u}_i$, then $\mathbf{0} = \sum_i(\alpha_i - \beta_i)\mathbf{u}_i$, and this implies $\alpha_i - \beta_i = 0$ (i.e., $\alpha_i = \beta_i$) for each i because $\mathcal{B}$ is an independent set.

Coordinates of a Vector

Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for a vector space $\mathcal{U}$, and let $\mathbf{v} \in \mathcal{U}$. The coefficients α_i in the expansion $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$ are called the ***coordinates of* v *with respect to*** $\mathcal{B}$, and, from now on, $[\mathbf{v}]_{\mathcal{B}}$ will denote the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Caution! Order is important. If $\mathcal{B}'$ is a permutation of $\mathcal{B}$, then $[\mathbf{v}]_{\mathcal{B}'}$ is the corresponding permutation of $[\mathbf{v}]_{\mathcal{B}}$.

From now on, $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ will denote the standard basis of unit vectors (in natural order) for $\Re^n$ (or $\mathcal{C}^n$). If no other basis is explicitly mentioned, then the standard basis is assumed. For example, if no basis is mentioned, and if we write

$$\mathbf{v} = \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix},$$

then it is understood that this is the representation with respect to $\mathcal{S}$ in the sense that $\mathbf{v} = [\mathbf{v}]_{\mathcal{S}} = 8\mathbf{e}_1 + 7\mathbf{e}_2 + 4\mathbf{e}_3$. The ***standard coordinates*** of a vector are its coordinates with respect to $\mathcal{S}$. So, 8, 7, and 4 are the standard coordinates of $\mathbf{v}$ in the above example.

Example 4.7.2

Problem: If $\mathbf{v}$ is a vector in $\Re^3$ whose standard coordinates are

$$\mathbf{v} = \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix},$$

determine the coordinates of $\mathbf{v}$ with respect to the basis

$$\mathcal{B} = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Solution: The object is to find the three unknowns $\alpha_1, \alpha_2,$ and α_3 such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = \mathbf{v}$. This is simply a 3×3 system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix} \implies [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ -3 \end{pmatrix}.$$

The general rule for making a change of coordinates is given on p. 252.

Linear transformations possess coordinates in the same way vectors do because linear transformations from $\mathcal{U}$ to $\mathcal{V}$ also form a vector space.

Space of Linear Transformations

- For each pair of vector spaces $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{F}$, the set $\mathcal{L}(\mathcal{U},\mathcal{V})$ of all linear transformations from $\mathcal{U}$ to $\mathcal{V}$ is a vector space over $\mathcal{F}$.
- Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively, and let $\mathbf{B}_{ji}$ be the linear transformation from $\mathcal{U}$ into $\mathcal{V}$ defined by $\mathbf{B}_{ji}(\mathbf{u}) = \xi_j \mathbf{v}_i$, where $(\xi_1, \xi_2, \ldots, \xi_n)^T = [\mathbf{u}]_{\mathcal{B}}$. That is, pick off the j^{th} coordinate of $\mathbf{u}$, and attach it to $\mathbf{v}_i$.
  - ▷ $\mathcal{B}_{\mathcal{L}} = \{\mathbf{B}_{ji}\}_{j=1\ldots n}^{i=1\ldots m}$ is a basis for $\mathcal{L}(\mathcal{U},\mathcal{V})$.
  - ▷ $\dim \mathcal{L}(\mathcal{U},\mathcal{V}) = (\dim \mathcal{U})\,(\dim \mathcal{V})$.

Proof. $\mathcal{L}(\mathcal{U},\mathcal{V})$ is a vector space because the defining properties on p. 160 are satisfied—details are omitted. Prove $\mathcal{B}_{\mathcal{L}}$ is a basis by demonstrating that it is a linearly independent spanning set for $\mathcal{L}(\mathcal{U},\mathcal{V})$. To establish linear independence, suppose $\sum_{j,i} \eta_{ji}\mathbf{B}_{ji} = \mathbf{0}$ for scalars η_{ji}, and observe that for each $\mathbf{u}_k \in \mathcal{B}$,

$$\mathbf{B}_{ji}(\mathbf{u}_k) = \begin{cases} \mathbf{v}_i & \text{if } j = k \\ \mathbf{0} & \text{if } j \neq k \end{cases} \implies \mathbf{0} = \Big(\sum_{j,i} \eta_{ji}\mathbf{B}_{ji}\Big)(\mathbf{u}_k) = \sum_{j,i} \eta_{ji}\mathbf{B}_{ji}(\mathbf{u}_k) = \sum_{i=1}^{m} \eta_{ki}\mathbf{v}_i.$$

For each k, the independence of $\mathcal{B}'$ implies that $\eta_{ki} = 0$ for each i, and thus $\mathcal{B}_{\mathcal{L}}$ is linearly independent. To see that $\mathcal{B}_{\mathcal{L}}$ spans $\mathcal{L}(\mathcal{U},\mathcal{V})$, let $\mathbf{T} \in \mathcal{L}(\mathcal{U},\mathcal{V})$, and determine the action of $\mathbf{T}$ on any $\mathbf{u} \in \mathcal{U}$ by using $\mathbf{u} = \sum_{j=1}^{n} \xi_j \mathbf{u}_j$ and $\mathbf{T}(\mathbf{u}_j) = \sum_{i=1}^{m} \alpha_{ij} \mathbf{v}_i$ to write

$$\begin{aligned}\mathbf{T}(\mathbf{u}) &= \mathbf{T}\left(\sum_{j=1}^{n} \xi_j \mathbf{u}_j\right) = \sum_{j=1}^{n} \xi_j \mathbf{T}(\mathbf{u}_j) = \sum_{j=1}^{n} \xi_j \sum_{i=1}^{m} \alpha_{ij} \mathbf{v}_i \\ &= \sum_{i,j} \alpha_{ij} \xi_j \mathbf{v}_i = \sum_{i,j} \alpha_{ij} \mathbf{B}_{ji}(\mathbf{u}).\end{aligned} \tag{4.7.3}$$

This holds for all $\mathbf{u} \in \mathcal{U}$, so $\mathbf{T} = \sum_{i,j} \alpha_{ij} \mathbf{B}_{ji}$, and thus $\mathcal{B}_{\mathcal{L}}$ spans $\mathcal{L}(\mathcal{U}, \mathcal{V})$. ∎

It now makes sense to talk about the *coordinates* of $\mathbf{T} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with respect to the basis $\mathcal{B}_{\mathcal{L}}$. In fact, the rule for determining these coordinates is contained in the proof above, where it was demonstrated that $\mathbf{T} = \sum_{i,j} \alpha_{ij} \mathbf{B}_{ji}$ in which the coordinates α_{ij} are precisely the scalars in

$$\mathbf{T}(\mathbf{u}_j) = \sum_{i=1}^{m} \alpha_{ij} \mathbf{v}_i \text{ or, equivalently, } [\mathbf{T}(\mathbf{u}_j)]_{\mathcal{B}'} = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}, \quad j = 1, 2, \ldots, n.$$

This suggests that rather than listing all coordinates α_{ij} in a single column containing mn entries (as we did with coordinate vectors), it's more logical to arrange the α_{ij} 's as an $m \times n$ matrix in which the j^{th} column contains the coordinates of $\mathbf{T}(\mathbf{u}_j)$ with respect to $\mathcal{B}'$. These ideas are summarized below.

Coordinate Matrix Representations

Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. The ***coordinate matrix*** of $\mathbf{T} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with respect to the pair $(\mathcal{B}, \mathcal{B}')$ is defined to be the $m \times n$ matrix

$$[\mathbf{T}]_{\mathcal{B}\mathcal{B}'} = \Big([\mathbf{T}(\mathbf{u}_1)]_{\mathcal{B}'} \,\Big|\, [\mathbf{T}(\mathbf{u}_2)]_{\mathcal{B}'} \,\Big|\, \cdots \,\Big|\, [\mathbf{T}(\mathbf{u}_n)]_{\mathcal{B}'}\Big). \tag{4.7.4}$$

In other words, if $\mathbf{T}(\mathbf{u}_j) = \alpha_{1j}\mathbf{v}_1 + \alpha_{2j}\mathbf{v}_2 + \cdots + \alpha_{mj}\mathbf{v}_m$, then

$$[\mathbf{T}(\mathbf{u}_j)]_{\mathcal{B}'} = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{pmatrix} \text{ and } [\mathbf{T}]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}. \tag{4.7.5}$$

When $\mathbf{T}$ is a linear operator on $\mathcal{U}$, and when there is only one basis involved, $[\mathbf{T}]_{\mathcal{B}}$ is used in place of $[\mathbf{T}]_{\mathcal{B}\mathcal{B}}$ to denote the (necessarily square) coordinate matrix of $\mathbf{T}$ with respect to $\mathcal{B}$.

Example 4.7.3

Problem: If $\mathbf{P}$ is the projector defined in Example 4.7.1 that maps each point $\mathbf{v} = (x, y, z) \in \Re^3$ to its orthogonal projection $\mathbf{P}(\mathbf{v}) = (x, y, 0)$ in the xy-plane, determine the coordinate matrix $[\mathbf{P}]_{\mathcal{B}}$ with respect to the basis

$$\mathcal{B} = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Solution: According to (4.7.4), the j^{th} column in $[\mathbf{P}]_{\mathcal{B}}$ is $[\mathbf{P}(\mathbf{u}_j)]_{\mathcal{B}}$. Therefore,

$$\mathbf{P}(\mathbf{u}_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\mathbf{u}_1 + 1\mathbf{u}_2 - 1\mathbf{u}_3 \implies [\mathbf{P}(\mathbf{u}_1)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\mathbf{P}(\mathbf{u}_2) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0\mathbf{u}_1 + 3\mathbf{u}_2 - 2\mathbf{u}_3 \implies [\mathbf{P}(\mathbf{u}_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix},$$

$$\mathbf{P}(\mathbf{u}_3) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0\mathbf{u}_1 + 3\mathbf{u}_2 - 2\mathbf{u}_3 \implies [\mathbf{P}(\mathbf{u}_3)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix},$$

so that $[\mathbf{P}]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}.$

Example 4.7.4

Problem: Consider the same problem given in Example 4.7.3, but use different bases—say,

$$\mathcal{B} = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{B}' = \left\{ \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

For the projector defined by $\mathbf{P}(x, y, z) = (x, y, 0)$, determine $[\mathbf{P}]_{\mathcal{B}\mathcal{B}'}$.

Solution: Determine the coordinates of each $\mathbf{P}(\mathbf{u}_j)$ with respect to $\mathcal{B}'$, as shown below:

$$\mathbf{P}(\mathbf{u}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -1\mathbf{v_1} + 0\mathbf{v_2} + 0\mathbf{v_3} \implies [\mathbf{P}(\mathbf{u}_1)]_{\mathcal{B}'} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{P}(\mathbf{u}_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -1\mathbf{v_1} + 1\mathbf{v_2} + 0\mathbf{v_3} \implies [\mathbf{P}(\mathbf{u}_2)]_{\mathcal{B}'} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{P}(\mathbf{u}_3) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -1\mathbf{v_1} + 1\mathbf{v_2} + 0\mathbf{v_3} \implies [\mathbf{P}(\mathbf{u}_3)]_{\mathcal{B}'} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, according to (4.7.4), $[\mathbf{P}]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

At the heart of linear algebra is the realization that the theory of finite-dimensional linear transformations is essentially the same as the theory of matrices. This is due primarily to the fundamental fact that the action of a linear transformation $\mathbf{T}$ on a vector $\mathbf{u}$ is precisely matrix multiplication between the coordinates of $\mathbf{T}$ and the coordinates of $\mathbf{u}$.

Action as Matrix Multiplication

Let $\mathbf{T} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. For each $\mathbf{u} \in \mathcal{U}$, the action of $\mathbf{T}$ on $\mathbf{u}$ is given by matrix multiplication between their coordinates in the sense that

$$[\mathbf{T}(\mathbf{u})]_{\mathcal{B}'} = [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}[\mathbf{u}]_{\mathcal{B}}. \tag{4.7.6}$$

Proof. Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. If $\mathbf{u} = \sum_{j=1}^{n} \xi_j \mathbf{u}_j$ and $\mathbf{T}(\mathbf{u}_j) = \sum_{i=1}^{m} \alpha_{ij}\mathbf{v}_i$, then

$$[\mathbf{u}]_{\mathcal{B}} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \quad \text{and} \quad [\mathbf{T}]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix},$$

so, according to (4.7.3),

$$\mathbf{T}(\mathbf{u}) = \sum_{i,j} \alpha_{ij}\xi_j\mathbf{v}_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij}\xi_j \right) \mathbf{v}_i.$$

In other words, the coordinates of $\mathbf{T}(\mathbf{u})$ with respect to $\mathcal{B}'$ are the terms $\sum_{j=1}^{n} \alpha_{ij}\xi_j$ for $i = 1, 2, \ldots, m$, and therefore

$$[\mathbf{T}(\mathbf{u})]_{\mathcal{B}'} = \begin{pmatrix} \sum_j \alpha_{1j}\xi_j \\ \sum_j \alpha_{2j}\xi_j \\ \vdots \\ \sum_j \alpha_{mj}\xi_j \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}[\mathbf{u}]_{\mathcal{B}}. \quad \blacksquare$$

Example 4.7.5

Problem: Show how the action of the operator $\mathbf{D}\big(p(t)\big) = dp/dt$ on the space $\mathcal{P}_3$ of polynomials of degree three or less is given by matrix multiplication.

Solution: The coordinate matrix of $\mathbf{D}$ with respect to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ is

$$[\mathbf{D}]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\mathbf{p} = p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$, then $\mathbf{D}(\mathbf{p}) = \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2$ so that

$$[\mathbf{p}]_{\mathcal{B}} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad \text{and} \quad [\mathbf{D}(\mathbf{p})]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ 2\alpha_2 \\ 3\alpha_3 \\ 0 \end{pmatrix}.$$

The action of $\mathbf{D}$ is accomplished by means of matrix multiplication because

$$[\mathbf{D}(\mathbf{p})]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ 2\alpha_2 \\ 3\alpha_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = [\mathbf{D}]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}}.$$

For $\mathbf{T} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $\mathbf{L} \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, the ***composition*** of $\mathbf{L}$ with $\mathbf{T}$ is defined to be the function $\mathbf{C} : \mathcal{U} \to \mathcal{W}$ such that $\mathbf{C}(\mathbf{x}) = \mathbf{L}\big(\mathbf{T}(\mathbf{x})\big)$, and this composition, denoted by $\mathbf{C} = \mathbf{LT}$, is also a linear transformation because

$$\begin{aligned} \mathbf{C}(\alpha\mathbf{x} + \mathbf{y}) &= \mathbf{L}\big(\mathbf{T}(\alpha\mathbf{x} + \mathbf{y})\big) = \mathbf{L}\big(\alpha\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})\big) \\ &= \alpha\mathbf{L}\big(\mathbf{T}(\mathbf{x})\big) + \mathbf{L}\big(\mathbf{T}(\mathbf{y})\big) = \alpha\mathbf{C}(\mathbf{x}) + \mathbf{C}(\mathbf{y}). \end{aligned}$$

Consequently, if $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ are bases for $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, respectively, then $\mathbf{C}$ must have a coordinate matrix representation with respect to $(\mathcal{B}, \mathcal{B}'')$, so it's only natural to ask how $[\mathbf{C}]_{\mathcal{B}\mathcal{B}''}$ is related to $[\mathbf{L}]_{\mathcal{B}'\mathcal{B}''}$ and $[\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$. Recall that the motivation behind the definition of matrix multiplication given on p. 93 was based on the need to represent the composition of two linear transformations, so it should be no surprise to discover that $[\mathbf{C}]_{\mathcal{B}\mathcal{B}''} = [\mathbf{L}]_{\mathcal{B}'\mathcal{B}''}[\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$. This, along with the other properties given below, makes it clear that studying linear transformations on finite-dimensional spaces amounts to studying matrix algebra.

Connections with Matrix Algebra

- If $\mathbf{T}, \mathbf{L} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and if $\mathcal{B}$ and $\mathcal{B}'$ are bases for $\mathcal{U}$ and $\mathcal{V}$, then
  - ▷ $[\alpha \mathbf{T}]_{\mathcal{B}\mathcal{B}'} = \alpha [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$ for scalars α, (4.7.7)
  - ▷ $[\mathbf{T} + \mathbf{L}]_{\mathcal{B}\mathcal{B}'} = [\mathbf{T}]_{\mathcal{B}\mathcal{B}'} + [\mathbf{L}]_{\mathcal{B}\mathcal{B}'}$. (4.7.8)
- If $\mathbf{T} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $\mathbf{L} \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, and if $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ are bases for $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, respectively, then $\mathbf{LT} \in \mathcal{L}(\mathcal{U}, \mathcal{W})$, and
  - ▷ $[\mathbf{LT}]_{\mathcal{B}\mathcal{B}''} = [\mathbf{L}]_{\mathcal{B}'\mathcal{B}''} [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$. (4.7.9)
- If $\mathbf{T} \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ is invertible in the sense that $\mathbf{T}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$ for some $\mathbf{T}^{-1} \in \mathcal{L}(\mathcal{U}, \mathcal{U})$, then for every basis $\mathcal{B}$ of $\mathcal{U}$,
  - ▷ $[\mathbf{T}^{-1}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}}^{-1}$. (4.7.10)

Proof. The first three properties (4.7.7)–(4.7.9) follow directly from (4.7.6). For example, to prove (4.7.9), let $\mathbf{u}$ be any vector in $\mathcal{U}$, and write

$$[\mathbf{LT}]_{\mathcal{B}\mathcal{B}''} [\mathbf{u}]_{\mathcal{B}} = \big[\mathbf{LT}(\mathbf{u})\big]_{\mathcal{B}''} = \big[\mathbf{L}\big(\mathbf{T}(\mathbf{u})\big)\big]_{\mathcal{B}''} = [\mathbf{L}]_{\mathcal{B}'\mathcal{B}''} \big[\mathbf{T}(\mathbf{u})\big]_{\mathcal{B}'} = [\mathbf{L}]_{\mathcal{B}'\mathcal{B}''} [\mathbf{T}]_{\mathcal{B}\mathcal{B}'} [\mathbf{u}]_{\mathcal{B}}.$$

This is true for all $\mathbf{u} \in \mathcal{U}$, so $[\mathbf{LT}]_{\mathcal{B}\mathcal{B}''} = [\mathbf{L}]_{\mathcal{B}'\mathcal{B}''} [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$ (see Exercise 3.5.5). Proving (4.7.7) and (4.7.8) is similar—details are omitted. To prove (4.7.10), note that if $\dim \mathcal{U} = n$, then $[\mathbf{I}]_{\mathcal{B}} = \mathbf{I}_n$ for all bases $\mathcal{B}$, so property (4.7.9) implies $\mathbf{I}_n = [\mathbf{I}]_{\mathcal{B}} = [\mathbf{T}\mathbf{T}^{-1}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}} [\mathbf{T}^{-1}]_{\mathcal{B}}$, and thus $[\mathbf{T}^{-1}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}}^{-1}$. ■

Example 4.7.6

Problem: Form the composition $\mathbf{C} = \mathbf{LT}$ of the two linear transformations $\mathbf{T} : \Re^3 \to \Re^2$ and $\mathbf{L} : \Re^2 \to \Re^2$ defined by

$$\mathbf{T}(x, y, z) = (x + y, \ \ y - z) \quad \text{and} \quad \mathbf{L}(u, v) = (2u - v, \ \ u),$$

and then verify (4.7.9) and (4.7.10) using the standard bases $\mathcal{S}_2$ and $\mathcal{S}_3$ for $\Re^2$ and $\Re^3$, respectively.

Solution: The composition $\mathbf{C} : \Re^3 \to \Re^2$ is the linear transformation

$$\mathbf{C}(x, y, z) = \mathbf{L}\big(\mathbf{T}(x, y, z)\big) = \mathbf{L}(x + y, \ \ y - z) = (2x + y + z, \ \ x + y).$$

The coordinate matrix representations of $\mathbf{C}$, $\mathbf{L}$, and $\mathbf{T}$ are

$$[\mathbf{C}]_{\mathcal{S}_3\mathcal{S}_2} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad [\mathbf{L}]_{\mathcal{S}_2} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad [\mathbf{T}]_{\mathcal{S}_3\mathcal{S}_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Property (4.7.9) is verified because $[\mathbf{LT}]_{\mathcal{S}_3\mathcal{S}_2} = [\mathbf{C}]_{\mathcal{S}_3\mathcal{S}_2} = [\mathbf{L}]_{\mathcal{S}_2}[\mathbf{T}]_{\mathcal{S}_3\mathcal{S}_2}$. Find $\mathbf{L}^{-1}$ by looking for scalars β_{ij} in $\mathbf{L}^{-1}(u,v) = (\beta_{11}u + \beta_{12}v,\ \beta_{21}u + \beta_{22}v)$ such that $\mathbf{L}\mathbf{L}^{-1} = \mathbf{L}^{-1}\mathbf{L} = \mathbf{I}$ or, equivalently,

$$\mathbf{L}\big(\mathbf{L}^{-1}(u,v)\big) = \mathbf{L}^{-1}\big(\mathbf{L}(u,v)\big) = (u,v) \quad \text{for all } (u,v) \in \Re^2.$$

Computation reveals $\mathbf{L}^{-1}(u,v) = (v,\ 2v-u)$, and (4.7.10) is verified by noting

$$[\mathbf{L}^{-1}]_{\mathcal{S}_2} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = [\mathbf{L}]_{\mathcal{S}_2}^{-1}.$$

Exercises for section 4.7

4.7.1. Determine which of the following functions are linear operators on $\Re^2$.

(a) $\mathbf{T}(x,y) = (x, 1+y)$, (b) $\mathbf{T}(x,y) = (y,x)$,
(c) $\mathbf{T}(x,y) = (0, xy)$, (d) $\mathbf{T}(x,y) = (x^2, y^2)$,
(e) $\mathbf{T}(x,y) = (x, \sin y)$, (f) $\mathbf{T}(x,y) = (x+y, x-y)$.

4.7.2. For $\mathbf{A} \in \Re^{n\times n}$, determine which of the following functions are linear transformations.

(a) $\mathbf{T}(\mathbf{X}_{n\times n}) = \mathbf{AX} - \mathbf{XA}$, (b) $\mathbf{T}(\mathbf{x}_{n\times 1}) = \mathbf{Ax} + \mathbf{b}$ for $\mathbf{b} \neq \mathbf{0}$,
(c) $\mathbf{T}(\mathbf{A}) = \mathbf{A}^T$, (d) $\mathbf{T}(\mathbf{X}_{n\times n}) = (\mathbf{X} + \mathbf{X}^T)/2$.

4.7.3. Explain why $\mathbf{T}(\mathbf{0}) = \mathbf{0}$ for every linear transformation $\mathbf{T}$.

4.7.4. Determine which of the following mappings are linear operators on $\mathcal{P}_n$, the vector space of polynomials of degree n or less.

(a) $\mathbf{T} = \xi_k\mathbf{D}^k + \xi_{k-1}\mathbf{D}^{k-1} + \cdots + \xi_1\mathbf{D} + \xi_0\mathbf{I}$, where $\mathbf{D}^k$ is the k^{th}-order differentiation operator (i.e., $\mathbf{D}^k p(t) = d^k p/dt^k$).
(b) $\mathbf{T}\big(p(t)\big) = t^n p'(0) + t$.

4.7.5. Let $\mathbf{v}$ be a fixed vector in $\Re^{n\times 1}$ and let $\mathbf{T} : \Re^{n\times 1} \to \Re$ be the mapping defined by $\mathbf{T}(\mathbf{x}) = \mathbf{v}^{\mathbf{T}}\mathbf{x}$ (i.e., the standard inner product).

(a) Is $\mathbf{T}$ a linear operator?
(b) Is $\mathbf{T}$ a linear transformation?

4.7.6. For the operator $\mathbf{T} : \Re^2 \to \Re^2$ defined by $\mathbf{T}(x,y) = (x+y,\ -2x+4y)$, determine $[\mathbf{T}]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

4.7.7. Let $\mathbf{T} : \Re^2 \to \Re^3$ be the linear transformation defined by $\mathbf{T}(x, y) = (x + 3y,\ 0,\ 2x - 4y)$.

(a) Determine $[\mathbf{T}]_{\mathcal{S}\mathcal{S}'}$, where $\mathcal{S}$ and $\mathcal{S}'$ are the standard bases for $\Re^2$ and $\Re^3$, respectively.

(b) Determine $[\mathbf{T}]_{\mathcal{S}\mathcal{S}''}$, where $\mathcal{S}''$ is the basis for $\Re^3$ obtained by permuting the standard basis according to $\mathcal{S}'' = \{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1\}$.

4.7.8. Let $\mathbf{T}$ be the operator on $\Re^3$ defined by $\mathbf{T}(x, y, z) = (x-y,\ y-x,\ x-z)$ and consider the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and the basis} \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(a) Determine $[\mathbf{T}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

(b) Compute $[\mathbf{T}(\mathbf{v})]_{\mathcal{B}}$, and then verify that $[\mathbf{T}]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{T}(\mathbf{v})]_{\mathcal{B}}$.

4.7.9. For $\mathbf{A} \in \Re^{n\times n}$, let $\mathbf{T}$ be the linear operator on $\Re^{n\times 1}$ defined by $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. That is, $\mathbf{T}$ is the operator defined by matrix multiplication. With respect to the standard basis $\mathcal{S}$, show that $[\mathbf{T}]_{\mathcal{S}} = \mathbf{A}$.

4.7.10. If $\mathbf{T}$ is a linear operator on a space $\mathcal{V}$ with basis $\mathcal{B}$, explain why $[\mathbf{T}^k]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}}^k$ for all nonnegative integers k.

4.7.11. Let $\mathbf{P}$ be the projector that maps each point $\mathbf{v} \in \Re^2$ to its orthogonal projection on the line $y = x$ as depicted in Figure 4.7.4.

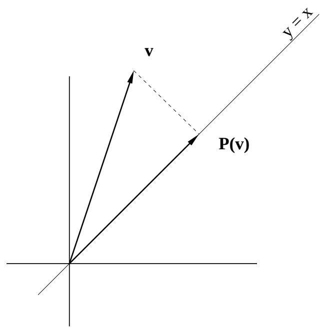

FIGURE 4.7.4

(a) Determine the coordinate matrix of $\mathbf{P}$ with respect to the standard basis.

(b) Determine the orthogonal projection of $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ onto the line $y = x$.

4.7.12. For the standard basis $\mathcal{S} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\Re^{2\times 2}$, determine the matrix representation $[\mathbf{T}]_{\mathcal{S}}$ for each of the following linear operators on $\Re^{2\times 2}$, and then verify $[\mathbf{T}(\mathbf{U})]_{\mathcal{S}} = [\mathbf{T}]_{\mathcal{S}}[\mathbf{U}]_{\mathcal{S}}$ for $\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) $\mathbf{T}(\mathbf{X}_{2\times 2}) = \dfrac{\mathbf{X} + \mathbf{X}^T}{2}$.

(b) $\mathbf{T}(\mathbf{X}_{2\times 2}) = \mathbf{AX} - \mathbf{XA}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

4.7.13. For $\mathcal{P}_2$ and $\mathcal{P}_3$ (the spaces of polynomials of degrees less than or equal to two and three, respectively), let $\mathbf{S} : \mathcal{P}_2 \to \mathcal{P}_3$ be the linear transformation defined by $\mathbf{S}(p) = \int_0^t p(x)dx$. Determine $[\mathbf{S}]_{\mathcal{B}\mathcal{B}'}$, where $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{B}' = \{1, t, t^2, t^3\}$.

4.7.14. Let $\mathbf{Q}$ be the linear operator on $\Re^2$ that rotates each point counterclockwise through an angle θ, and let $\mathbf{R}$ be the linear operator on $\Re^2$ that reflects each point about the x-axis.

(a) Determine the matrix of the composition $[\mathbf{RQ}]_{\mathcal{S}}$ relative to the standard basis $\mathcal{S}$.

(b) Relative to the standard basis, determine the matrix of the linear operator that rotates each point in $\Re^2$ counterclockwise through an angle 2θ.

4.7.15. Let $\mathbf{P} : \mathcal{U} \to \mathcal{V}$ and $\mathbf{Q} : \mathcal{U} \to \mathcal{V}$ be two linear transformations, and let $\mathcal{B}$ and $\mathcal{B}'$ be arbitrary bases for $\mathcal{U}$ and $\mathcal{V}$, respectively.

(a) Provide the details to explain why $[\mathbf{P}+\mathbf{Q}]_{\mathcal{B}\mathcal{B}'} = [\mathbf{P}]_{\mathcal{B}\mathcal{B}'} + [\mathbf{Q}]_{\mathcal{B}\mathcal{B}'}$.

(b) Provide the details to explain why $[\alpha\mathbf{P}]_{\mathcal{B}\mathcal{B}'} = \alpha[\mathbf{P}]_{\mathcal{B}\mathcal{B}'}$, where α is an arbitrary scalar.

4.7.16. Let $\mathbf{I}$ be the identity operator on an n-dimensional space $\mathcal{V}$.

(a) Explain why

$$[\mathbf{I}]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

regardless of the choice of basis $\mathcal{B}$.

(b) Let $\mathcal{B} = \{\mathbf{x}_i\}_{i=1}^n$ and $\mathcal{B}' = \{\mathbf{y}_i\}_{i=1}^n$ be two different bases for $\mathcal{V}$, and let $\mathbf{T}$ be the linear operator on $\mathcal{V}$ that maps vectors from $\mathcal{B}'$ to vectors in $\mathcal{B}$ according to the rule $\mathbf{T}(\mathbf{y}_i) = \mathbf{x}_i$ for $i = 1, 2, \ldots, n$. Explain why

$$[\mathbf{I}]_{\mathcal{B}\mathcal{B}'} = [\mathbf{T}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}'} = \Big([\mathbf{x}_1]_{\mathcal{B}'} \,\Big|\, [\mathbf{x}_2]_{\mathcal{B}'} \,\Big|\, \cdots \,\Big|\, [\mathbf{x}_n]_{\mathcal{B}'} \Big).$$

(c) When $\mathcal{V} = \Re^3$, determine $[\mathbf{I}]_{\mathcal{B}\mathcal{B}'}$ for

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

4.7.17. Let $\mathbf{T} : \Re^3 \to \Re^3$ be the linear operator defined by

$$\mathbf{T}(x, y, z) = (2x - y, \;\; -x + 2y - z, \;\; z - y).$$

(a) Determine $\mathbf{T}^{-1}(x, y, z)$.
(b) Determine $[\mathbf{T}^{-1}]_{\mathcal{S}}$, where $\mathcal{S}$ is the standard basis for $\Re^3$.

4.7.18. Let $\mathbf{T}$ be a linear operator on an n-dimensional space $\mathcal{V}$. Show that the following statements are equivalent.
(1) $\mathbf{T^{-1}}$ exists.
(2) $\mathbf{T}$ is a one-to-one mapping (i.e., $\mathbf{T}(\mathbf{x}) = \mathbf{T}(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$).
(3) $N(\mathbf{T}) = \{\mathbf{0}\}$.
(4) $\mathbf{T}$ is an onto mapping (i.e., for each $\mathbf{v} \in \mathcal{V}$, there is an $\mathbf{x} \in \mathcal{V}$ such that $\mathbf{T}(\mathbf{x}) = \mathbf{v}$).
Hint: Show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (2), and then show (2) and (4) $\implies$ (1).

4.7.19. Let $\mathcal{V}$ be an n-dimensional space with a basis $\mathcal{B} = \{\mathbf{u}_i\}_{i=1}^n$.
(a) Prove that a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\} \subseteq \mathcal{V}$ is linearly independent if and only if the set of coordinate vectors

$$\left\{ [\mathbf{x}_1]_{\mathcal{B}}, \;\; [\mathbf{x}_2]_{\mathcal{B}}, \;\; \ldots, \;\; [\mathbf{x}_r]_{\mathcal{B}} \right\} \subseteq \Re^{n \times 1}$$

is a linearly independent set.
(b) If $\mathbf{T}$ is a linear operator on $\mathcal{V}$, then the ***range*** of $\mathbf{T}$ is the set

$$R(\mathbf{T}) = \{\mathbf{T}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{V}\}.$$

Suppose that the basic columns of $[\mathbf{T}]_{\mathcal{B}}$ occur in positions $b_1, b_2, \ldots, b_r$. Explain why $\left\{ \mathbf{T}(\mathbf{u}_{b_1}), \mathbf{T}(\mathbf{u}_{b_2}), \ldots, \mathbf{T}(\mathbf{u}_{b_r}) \right\}$ is a basis for $R(\mathbf{T})$.

4.8 CHANGE OF BASIS AND SIMILARITY

By their nature, coordinate matrix representations are basis dependent. However, it's desirable to study linear transformations without reference to particular bases because some bases may force a coordinate matrix representation to exhibit special properties that are not present in the coordinate matrix relative to other bases. To divorce the study from the choice of bases it's necessary to somehow identify properties of coordinate matrices that are invariant among all bases—these are properties intrinsic to the transformation itself, and they are the ones on which to focus. The purpose of this section is to learn how to sort out these basis-independent properties.

The discussion is limited to a single finite-dimensional space $\mathcal{V}$ and to linear operators on $\mathcal{V}$. Begin by examining how the coordinates of $\mathbf{v} \in \mathcal{V}$ change as the basis for $\mathcal{V}$ changes. Consider two different bases

$$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \quad \text{and} \quad \mathcal{B}' = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}.$$

It's convenient to regard $\mathcal{B}$ as an *old basis* for $\mathcal{V}$ and $\mathcal{B}'$ as a *new basis* for $\mathcal{V}$. Throughout this section $\mathbf{T}$ will denote the linear operator such that

$$\mathbf{T}(\mathbf{y}_i) = \mathbf{x}_i \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.8.1}$$

$\mathbf{T}$ is called the ***change of basis operator*** because it maps the new basis vectors in $\mathcal{B}'$ to the old basis vectors in $\mathcal{B}$. Notice that $[\mathbf{T}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}'} = [\mathbf{I}]_{\mathcal{B}\mathcal{B}'}$. To see this, observe that

$$\mathbf{x}_i = \sum_{j=1}^{n} \alpha_j \mathbf{y}_j \quad \Longrightarrow \quad \mathbf{T}(\mathbf{x}_i) = \sum_{j=1}^{n} \alpha_j \mathbf{T}(\mathbf{y}_j) = \sum_{j=1}^{n} \alpha_j \mathbf{x}_j,$$

which means $[\mathbf{x}_i]_{\mathcal{B}'} = [\mathbf{T}(\mathbf{x}_i)]_{\mathcal{B}}$, so, according to (4.7.4),

$$[\mathbf{T}]_{\mathcal{B}} = \Big([\mathbf{T}(\mathbf{x}_1)]_{\mathcal{B}} \ [\mathbf{T}(\mathbf{x}_2)]_{\mathcal{B}} \ \cdots [\mathbf{T}(\mathbf{x}_n)]_{\mathcal{B}} \Big) = \Big([\mathbf{x}_1]_{\mathcal{B}'} \ [\mathbf{x}_2]_{\mathcal{B}'} \cdots [\mathbf{x}_n]_{\mathcal{B}'} \Big) = [\mathbf{T}]_{\mathcal{B}'}.$$

The fact that $[\mathbf{I}]_{\mathcal{B}\mathcal{B}'} = [\mathbf{T}]_{\mathcal{B}}$ follows because $[\mathbf{I}(\mathbf{x}_i)]_{\mathcal{B}'} = [\mathbf{x}_i]_{\mathcal{B}'}$. The matrix

$$\mathbf{P} = [\mathbf{I}]_{\mathcal{B}\mathcal{B}'} = [\mathbf{T}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}'} \tag{4.8.2}$$

will hereafter be referred to as a ***change of basis matrix.*** **Caution!** $[\mathbf{I}]_{\mathcal{B}\mathcal{B}'}$ is not necessarily the identity matrix—see Exercise 4.7.16—and $[\mathbf{I}]_{\mathcal{B}\mathcal{B}'} \neq [\mathbf{I}]_{\mathcal{B}'\mathcal{B}}$.

We are now in a position to see how the coordinates of a vector change as the basis for the underlying space changes.

Changing Vector Coordinates

Let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $\mathcal{B}' = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be bases for $\mathcal{V}$, and let $\mathbf{T}$ and $\mathbf{P}$ be the associated change of basis operator and change of basis matrix, respectively—i.e., $\mathbf{T}(\mathbf{y}_i) = \mathbf{x}_i$, for each i, and

$$\mathbf{P} = [\mathbf{T}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}'} = [\mathbf{I}]_{\mathcal{B}\mathcal{B}'} = \left([\mathbf{x}_1]_{\mathcal{B}'} \,\middle|\, [\mathbf{x}_2]_{\mathcal{B}'} \,\middle|\, \cdots \,\middle|\, [\mathbf{x}_n]_{\mathcal{B}'} \right). \tag{4.8.3}$$

- $[\mathbf{v}]_{\mathcal{B}'} = \mathbf{P}[\mathbf{v}]_{\mathcal{B}}$ for all $\mathbf{v} \in \mathcal{V}$. (4.8.4)
- $\mathbf{P}$ is nonsingular.
- No other matrix can be used in place of $\mathbf{P}$ in (4.8.4).

Proof. Use (4.7.6) to write $[\mathbf{v}]_{\mathcal{B}'} = [\mathbf{I}(\mathbf{v})]_{\mathcal{B}'} = [\mathbf{I}]_{\mathcal{B}\mathcal{B}'}[\mathbf{v}]_{\mathcal{B}} = \mathbf{P}[\mathbf{v}]_{\mathcal{B}}$, which is (4.8.4). $\mathbf{P}$ is nonsingular because $\mathbf{T}$ is invertible (in fact, $\mathbf{T}^{-1}(\mathbf{x}_i) = \mathbf{y}_i$), and because (4.7.10) insures $[\mathbf{T}^{-1}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}}^{-1} = \mathbf{P}^{-1}$. $\mathbf{P}$ is unique because if $\mathbf{W}$ is another matrix satisfying (4.8.4) for all $\mathbf{v} \in \mathcal{V}$, then $(\mathbf{P} - \mathbf{W})[\mathbf{v}]_{\mathcal{B}} = \mathbf{0}$ for all $\mathbf{v}$. Taking $\mathbf{v} = \mathbf{x}_i$ yields $(\mathbf{P} - \mathbf{W})\mathbf{e}_i = \mathbf{0}$ for each i, so $\mathbf{P} - \mathbf{W} = \mathbf{0}$. ■

If we think of $\mathcal{B}$ as the *old* basis and $\mathcal{B}'$ as the *new* basis, then the change of basis operator $\mathbf{T}$ acts as

$$\mathbf{T}(\text{new basis}) = \text{old basis},$$

while the change of basis matrix $\mathbf{P}$ acts as

$$\text{new coordinates} = \mathbf{P}(\text{old coordinates}).$$

For this reason, $\mathbf{T}$ should be referred to as the change of basis operator *from* $\mathcal{B}'$ *to* $\mathcal{B}$, while $\mathbf{P}$ is called the change of basis matrix *from* $\mathcal{B}$ *to* $\mathcal{B}'$.

Example 4.8.1

Problem: For the space $\mathcal{P}_2$ of polynomials of degree 2 or less, determine the change of basis matrix $\mathbf{P}$ from $\mathcal{B}$ to $\mathcal{B}'$, where

$$\mathcal{B} = \{1, t, t^2\} \quad \text{and} \quad \mathcal{B}' = \{1,\ 1+t,\ 1+t+t^2\},$$

and then find the coordinates of $q(t) = 3 + 2t + 4t^2$ relative to $\mathcal{B}'$.

Solution: According to (4.8.3), the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is

$$\mathbf{P} = \left([\mathbf{x}_1]_{\mathcal{B}'} \,\middle|\, [\mathbf{x}_2]_{\mathcal{B}'} \,\middle|\, [\mathbf{x}_3]_{\mathcal{B}'} \right).$$

In this case, $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = t$, and $\mathbf{x}_3 = t^2$, and $\mathbf{y}_1 = 1$, $\mathbf{y}_2 = 1 + t$, and $\mathbf{y}_3 = 1 + t + t^2$, so the coordinates $[\mathbf{x}_i]_{\mathcal{B}'}$ are computed as follows:

$$\begin{aligned} 1 &= 1(1) + 0(1+t) + 0(1+t+t^2) = 1\mathbf{y}_1 + 0\mathbf{y}_2 + 0\mathbf{y}_3, \\ t &= -1(1) + 1(1+t) + 0(1+t+t^2) = -1\mathbf{y}_1 + 1\mathbf{y}_2 + 0\mathbf{y}_3, \\ t^2 &= 0(1) - 1(1+t) + 1(1+t+t^2) = 0\mathbf{y}_1 - 1\mathbf{y}_2 + 1\mathbf{y}_3. \end{aligned}$$

Therefore,

$$\mathbf{P} = \left([\mathbf{x}_1]_{\mathcal{B}'} \,\Big|\, [\mathbf{x}_2]_{\mathcal{B}'} \,\Big|\, [\mathbf{x}_3]_{\mathcal{B}'} \right) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

and the coordinates of $\mathbf{q} = q(t) = 3 + 2t + 4t^2$ with respect to $\mathcal{B}'$ are

$$[\mathbf{q}]_{\mathcal{B}'} = \mathbf{P}[\mathbf{q}]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

To independently check that these coordinates are correct, simply verify that

$$q(t) = 1(1) - 2(1+t) + 4(1+t+t^2).$$

It's now rather easy to describe how the coordinate matrix of a linear operator changes as the underlying basis changes.

Changing Matrix Coordinates

Let $\mathbf{A}$ be a linear operator on $\mathcal{V}$, and let $\mathcal{B}$ and $\mathcal{B}'$ be two bases for $\mathcal{V}$. The coordinate matrices $[\mathbf{A}]_{\mathcal{B}}$ and $[\mathbf{A}]_{\mathcal{B}'}$ are related as follows.

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}[\mathbf{A}]_{\mathcal{B}'}\mathbf{P}, \quad \text{where} \quad \mathbf{P} = [\mathbf{I}]_{\mathcal{B}\mathcal{B}'} \tag{4.8.5}$$

is the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$. Equivalently,

$$[\mathbf{A}]_{\mathcal{B}'} = \mathbf{Q}^{-1}[\mathbf{A}]_{\mathcal{B}}\mathbf{Q}, \quad \text{where} \quad \mathbf{Q} = [\mathbf{I}]_{\mathcal{B}'\mathcal{B}} = \mathbf{P}^{-1} \tag{4.8.6}$$

is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$.

Proof. Let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $\mathcal{B}' = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$, and observe that for each j, (4.7.6) can be used to write

$$\Big[\mathbf{A}(\mathbf{x}_j)\Big]_{\mathcal{B}'} = [\mathbf{A}]_{\mathcal{B}'}\,[\mathbf{x}_j]_{\mathcal{B}'} = [\mathbf{A}]_{\mathcal{B}'}\mathbf{P}_{*j} = \Big[[\mathbf{A}]_{\mathcal{B}'}\mathbf{P}\Big]_{*j}.$$

Now use the change of coordinates rule (4.8.4) together with the fact that $\big[\mathbf{A}(\mathbf{x}_j)\big]_{\mathcal{B}} = \big[[\mathbf{A}]_{\mathcal{B}}\big]_{*j}$ (see (4.7.4)) to write

$$\Big[\mathbf{A}(\mathbf{x}_j)\Big]_{\mathcal{B}'} = \mathbf{P}\Big[\mathbf{A}(\mathbf{x}_j)\Big]_{\mathcal{B}} = \mathbf{P}\Big[[\mathbf{A}]_{\mathcal{B}}\Big]_{*j} = \Big[\mathbf{P}[\mathbf{A}]_{\mathcal{B}}\Big]_{*j}.$$

Consequently, $\big[[\mathbf{A}]_{\mathcal{B}'}\mathbf{P}\big]_{*j} = \big[\mathbf{P}[\mathbf{A}]_{\mathcal{B}}\big]_{*j}$ for each j, so $[\mathbf{A}]_{\mathcal{B}'}\mathbf{P} = \mathbf{P}[\mathbf{A}]_{\mathcal{B}}$. Since the change of basis matrix $\mathbf{P}$ is nonsingular, it follows that $[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}[\mathbf{A}]_{\mathcal{B}'}\mathbf{P}$, and (4.8.5) is proven. Setting $\mathbf{Q} = \mathbf{P}^{-1}$ in (4.8.5) yields $[\mathbf{A}]_{\mathcal{B}'} = \mathbf{Q}^{-1}[\mathbf{A}]_{\mathcal{B}}\mathbf{Q}$. The matrix $\mathbf{Q} = \mathbf{P}^{-1}$ is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$ because if $\mathbf{T}$ is the change of basis operator from $\mathcal{B}'$ to $\mathcal{B}$ (i.e., $\mathbf{T}(\mathbf{y}_i) = \mathbf{x}_i$), then $\mathbf{T}^{-1}$ is the change of basis operator from $\mathcal{B}$ to $\mathcal{B}'$ (i.e., $\mathbf{T}^{-1}(\mathbf{x}_i) = \mathbf{y}_i$), and according to (4.8.3), the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$ is

$$[\mathbf{I}]_{\mathcal{B}'\mathcal{B}} = \Big(\, [\mathbf{y}_1]_{\mathcal{B}} \,\Big|\, [\mathbf{y}_2]_{\mathcal{B}} \,\Big|\, \cdots \,\Big|\, [\mathbf{y}_n]_{\mathcal{B}} \Big) = [\mathbf{T}^{-1}]_{\mathcal{B}} = [\mathbf{T}]_{\mathcal{B}}^{-1} = \mathbf{P}^{-1} = \mathbf{Q}. \quad \blacksquare$$

Example 4.8.2

Problem: Consider the linear operator $\mathbf{A}(x, y) = (y,\ -2x + 3y)$ on $\Re^2$ along with the two bases

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{S}' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

First compute the coordinate matrix $[\mathbf{A}]_{\mathcal{S}}$ as well as the change of basis matrix $\mathbf{Q}$ from $\mathcal{S}'$ to $\mathcal{S}$, and then use these two matrices to determine $[\mathbf{A}]_{\mathcal{S}'}$.

Solution: The matrix of $\mathbf{A}$ relative to $\mathcal{S}$ is obtained by computing

$$\begin{aligned} \mathbf{A}(\mathbf{e}_1) &= \mathbf{A}(1, 0) = (0,\ -2) = (0)\mathbf{e}_1 + (-2)\mathbf{e}_2, \\ \mathbf{A}(\mathbf{e}_2) &= \mathbf{A}(0, 1) = (1, 3) = (1)\mathbf{e}_1 + (3)\mathbf{e}_2, \end{aligned}$$

so that $[\mathbf{A}]_{\mathcal{S}} = \big(\, [\mathbf{A}(\mathbf{e}_1)]_{\mathcal{S}} \,\big|\, [\mathbf{A}(\mathbf{e}_2)]_{\mathcal{S}} \,\big) = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. According to (4.8.6), the change of basis matrix from $\mathcal{S}'$ to $\mathcal{S}$ is

$$\mathbf{Q} = \Big(\, [\mathbf{y}_1]_{\mathcal{S}} \,\Big|\, [\mathbf{y}_2]_{\mathcal{S}} \,\Big) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and the matrix of $\mathbf{A}$ with respect to $\mathcal{S}'$ is

$$[\mathbf{A}]_{\mathcal{S}'} = \mathbf{Q}^{-1}[\mathbf{A}]_{\mathcal{S}}\mathbf{Q} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Notice that $[\mathbf{A}]_{\mathcal{S}'}$ is a diagonal matrix, whereas $[\mathbf{A}]_{\mathcal{S}}$ is not. This shows that the standard basis is not always the best choice for providing a simple matrix representation. Finding a basis so that the associated coordinate matrix is as simple as possible is one of the fundamental issues of matrix theory. Given an operator $\mathbf{A}$, the solution to the general problem of determining a basis $\mathcal{B}$ so that $[\mathbf{A}]_{\mathcal{B}}$ is diagonal is summarized on p. 520.

Example 4.8.3

Problem: Consider a matrix $\mathbf{M}_{n\times n}$ to be a linear operator on $\Re^n$ by defining $\mathbf{M}(\mathbf{v}) = \mathbf{M}\mathbf{v}$ (matrix–vector multiplication). If $\mathcal{S}$ is the standard basis for $\Re^n$, and if $\mathcal{S}' = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is any other basis, describe $[\mathbf{M}]_{\mathcal{S}}$ and $[\mathbf{M}]_{\mathcal{S}'}$.

Solution: The j^{th} column in $[\mathbf{M}]_{\mathcal{S}}$ is $[\mathbf{M}\mathbf{e}_j]_{\mathcal{S}} = [\mathbf{M}_{*j}]_{\mathcal{S}} = \mathbf{M}_{*j}$, and hence $[\mathbf{M}]_{\mathcal{S}} = \mathbf{M}$. That is, the coordinate matrix of $\mathbf{M}$ with respect to $\mathcal{S}$ is $\mathbf{M}$ itself. To find $[\mathbf{M}]_{\mathcal{S}'}$, use (4.8.6) to write $[\mathbf{M}]_{\mathcal{S}'} = \mathbf{Q}^{-1}[\mathbf{M}]_{\mathcal{S}}\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$, where

$$\mathbf{Q} = [\mathbf{I}]_{\mathcal{S}'\mathcal{S}} = \Big([\mathbf{q}_1]_{\mathcal{S}} \,\Big|\, [\mathbf{q}_2]_{\mathcal{S}} \,\Big|\, \cdots \,\Big|\, [\mathbf{q}_n]_{\mathcal{S}} \Big) = \Big(\mathbf{q}_1 \,\Big|\, \mathbf{q}_2 \,\Big|\, \cdots \,\Big|\, \mathbf{q}_n \Big).$$

Conclusion: The matrices $\mathbf{M}$ and $\mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$ represent the same linear operator (namely, $\mathbf{M}$), but with respect to two different bases (namely, $\mathcal{S}$ and $\mathcal{S}'$). So, when considering properties of $\mathbf{M}$ (as a linear operator), it's legitimate to replace $\mathbf{M}$ by $\mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$. Whenever the structure of $\mathbf{M}$ obscures its operator properties, look for a basis $\mathcal{S}' = \{\mathbf{Q}_{*1}, \mathbf{Q}_{*2}, \ldots, \mathbf{Q}_{*n}\}$ (or, equivalently, a nonsingular matrix $\mathbf{Q}$) such that $\mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$ has a simpler structure. This is an important theme throughout linear algebra and matrix theory.

For a linear operator $\mathbf{A}$, the special relationships between $[\mathbf{A}]_{\mathcal{B}}$ and $[\mathbf{A}]_{\mathcal{B}'}$ that are given in (4.8.5) and (4.8.6) motivate the following definitions.

Similarity

- Matrices $\mathbf{B}_{n\times n}$ and $\mathbf{C}_{n\times n}$ are said to be ***similar matrices*** whenever there exists a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{B} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}$. We write $\mathbf{B} \simeq \mathbf{C}$ to denote that $\mathbf{B}$ and $\mathbf{C}$ are similar.
- The linear operator $f : \Re^{n\times n} \to \Re^{n\times n}$ defined by $f(\mathbf{C}) = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}$ is called a ***similarity transformation.***

Equations (4.8.5) and (4.8.6) say that any two coordinate matrices of a given linear operator must be similar. But must any two similar matrices be coordinate matrices of the same linear operator? Yes, and here's why. Suppose $\mathbf{C} = \mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}$, and let $\mathbf{A}(\mathbf{v}) = \mathbf{B}\mathbf{v}$ be the linear operator defined by matrix–vector multiplication. If $\mathcal{S}$ is the standard basis, then it's straightforward to see that $[\mathbf{A}]_{\mathcal{S}} = \mathbf{B}$ (Exercise 4.7.9). If $\mathcal{B}' = \{\mathbf{Q}_{*1}, \mathbf{Q}_{*2}, \ldots, \mathbf{Q}_{*n}\}$ is the basis consisting of the columns of $\mathbf{Q}$, then (4.8.6) insures that $[\mathbf{A}]_{\mathcal{B}'} = [\mathbf{I}]_{\mathcal{B}'\mathcal{S}}^{-1}[\mathbf{A}]_{\mathcal{S}}[\mathbf{I}]_{\mathcal{B}'\mathcal{S}}$, where

$$[\mathbf{I}]_{\mathcal{B}'\mathcal{S}} = \Big([\mathbf{Q}_{*1}]_{\mathcal{S}} \,\Big|\, [\mathbf{Q}_{*2}]_{\mathcal{S}} \,\Big|\, \cdots \,\Big|\, [\mathbf{Q}_{*n}]_{\mathcal{S}} \Big) = \mathbf{Q}.$$

Therefore, $\mathbf{B} = [\mathbf{A}]_{\mathcal{S}}$ and $\mathbf{C} = \mathbf{Q}^{-1}\mathbf{B}\mathbf{Q} = \mathbf{Q}^{-1}[\mathbf{A}]_{\mathcal{S}}\mathbf{Q} = [\mathbf{A}]_{\mathcal{B}'}$, so $\mathbf{B}$ and $\mathbf{C}$ are both coordinate matrix representations of $\mathbf{A}$. In other words, *similar matrices represent the same linear operator.*

As stated at the beginning of this section, the goal is to isolate and study coordinate-independent properties of linear operators. They are the ones determined by sorting out those properties of coordinate matrices that are basis independent. But, as (4.8.5) and (4.8.6) show, all coordinate matrices for a given linear operator must be similar, so the coordinate-independent properties are exactly the ones that are ***similarity invariant*** (invariant under similarity transformations). Naturally, determining and studying similarity invariants is an important part of linear algebra and matrix theory.

Example 4.8.4

Problem: The trace of a square matrix $\mathbf{C}$ was defined in Example 3.3.1 to be the sum of the diagonal entries

$$\mathit{trace}\,(\mathbf{C}) = \sum_i c_{ii}.$$

Show that trace is a similarity invariant, and explain why it makes sense to talk about the *trace of a linear operator* without regard to any particular basis. Then determine the trace of the linear operator on $\Re^2$ that is defined by

$$\mathbf{A}(x, y) = (y,\ -2x + 3y). \tag{4.8.7}$$

Solution: As demonstrated in Example 3.6.5, $\mathit{trace}\,(\mathbf{BC}) = \mathit{trace}\,(\mathbf{CB})$, whenever the products are defined, so

$$\mathit{trace}\left(\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}\right) = \mathit{trace}\left(\mathbf{C}\mathbf{Q}\mathbf{Q}^{-1}\right) = \mathit{trace}\,(\mathbf{C}),$$

and thus trace is a similarity invariant. This allows us to talk about the trace of a linear operator $\mathbf{A}$ without regard to any particular basis because $\mathit{trace}\,([\mathbf{A}]_{\mathcal{B}})$ is the same number regardless of the choice of $\mathcal{B}$. For example, two coordinate matrices of the operator $\mathbf{A}$ in (4.8.7) were computed in Example 4.8.2 to be

$$[\mathbf{A}]_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad [\mathbf{A}]_{\mathcal{S}'} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

and it's clear that $\mathit{trace}\,([\mathbf{A}]_{\mathcal{S}}) = \mathit{trace}\,([\mathbf{A}]_{\mathcal{S}'}) = 3$. Since $\mathit{trace}\,([\mathbf{A}]_{\mathcal{B}}) = 3$ for all $\mathcal{B}$, we can legitimately define $\mathit{trace}\,(\mathbf{A}) = 3$.

Exercises for section 4.8

4.8.1. Explain why rank is a similarity invariant.

4.8.2. Explain why similarity is transitive in the sense that $\mathbf{A} \simeq \mathbf{B}$ and $\mathbf{B} \simeq \mathbf{C}$ implies $\mathbf{A} \simeq \mathbf{C}$.

4.8.3. $\mathbf{A}(x, y, z) = (x + 2y - z,\ -y,\ x + 7z)$ is a linear operator on $\Re^3$.
(a) Determine $[\mathbf{A}]_{\mathcal{S}}$, where $\mathcal{S}$ is the standard basis.
(b) Determine $[\mathbf{A}]_{\mathcal{S}'}$ as well as the nonsingular matrix $\mathbf{Q}$ such that $[\mathbf{A}]_{\mathcal{S}'} = \mathbf{Q}^{-1}[\mathbf{A}]_{\mathcal{S}}\mathbf{Q}$ for $\mathcal{S}' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

4.8.4. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$. Consider $\mathbf{A}$ as a linear operator on $\Re^{n \times 1}$ by means of matrix multiplication $\mathbf{A}(\mathbf{x}) = \mathbf{Ax}$, and determine $[\mathbf{A}]_{\mathcal{B}}$.

4.8.5. Show that $\mathbf{C} = \begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & -3 \\ 6 & 10 \end{pmatrix}$ are similar matrices, and find a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{C} = \mathbf{Q}^{-1}\mathbf{BQ}$. **Hint:** Consider $\mathbf{B}$ as a linear operator on $\Re^2$, and compute $[\mathbf{B}]_{\mathcal{S}}$ and $[\mathbf{B}]_{\mathcal{S}'}$, where $\mathcal{S}$ is the standard basis, and $\mathcal{S}' = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}$.

4.8.6. Let $\mathbf{T}$ be the linear operator $\mathbf{T}(x, y) = (-7x - 15y,\ 6x + 12y)$. Find a basis $\mathcal{B}$ such that $[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, and determine a matrix $\mathbf{Q}$ such that $[\mathbf{T}]_{\mathcal{B}} = \mathbf{Q}^{-1}[\mathbf{T}]_{\mathcal{S}}\mathbf{Q}$, where $\mathcal{S}$ is the standard basis.

4.8.7. By considering the rotator $\mathbf{P}(x, y) = (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta)$ described in Example 4.7.1 and Figure 4.7.1, show that the matrices

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

are similar over the complex field. **Hint:** In case you have forgotten (or didn't know), $e^{i\theta} = \cos\theta + i \sin\theta$.

4.8.8. Let λ be a scalar such that $(\mathbf{C}-\lambda\mathbf{I})_{n\times n}$ is singular.

(a) If $\mathbf{B}\simeq\mathbf{C}$, prove that $(\mathbf{B}-\lambda\mathbf{I})$ is also singular.

(b) Prove that $(\mathbf{B}-\lambda_i\mathbf{I})$ is singular whenever $\mathbf{B}_{n\times n}$ is similar to

$$\mathbf{D}=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{pmatrix}.$$

4.8.9. If $\mathbf{A}\simeq\mathbf{B}$, show that $\mathbf{A}^k\simeq\mathbf{B}^k$ for all nonnegative integers k.

4.8.10. Suppose $\mathcal{B}=\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\}$ and $\mathcal{B}'=\{\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n\}$ are bases for an n-dimensional subspace $\mathcal{V}\subseteq\Re^{m\times 1}$, and let $\mathbf{X}_{m\times n}$ and $\mathbf{Y}_{m\times n}$ be the matrices whose columns are the vectors from $\mathcal{B}$ and $\mathcal{B}'$, respectively.

(a) Explain why $\mathbf{Y}^T\mathbf{Y}$ is nonsingular, and prove that the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is $\mathbf{P}=\left(\mathbf{Y}^T\mathbf{Y}\right)^{-1}\mathbf{Y}^T\mathbf{X}$.

(b) Describe $\mathbf{P}$ when $m=n$.

4.8.11. (a) $\mathbf{N}$ is *nilpotent of index k* when $\mathbf{N}^k=\mathbf{0}$ but $\mathbf{N}^{k-1}\neq\mathbf{0}$. If $\mathbf{N}$ is a nilpotent operator of index n on $\Re^n$, and if $\mathbf{N}^{n-1}(\mathbf{y})\neq\mathbf{0}$, show $\mathcal{B}=\left\{\mathbf{y},\mathbf{N}(\mathbf{y}),\mathbf{N}^2(\mathbf{y}),\ldots,\mathbf{N}^{n-1}(\mathbf{y})\right\}$ is a basis for $\Re^n$, and then demonstrate that

$$[\mathbf{N}]_{\mathcal{B}}=\mathbf{J}=\begin{pmatrix}0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}.$$

(b) If $\mathbf{A}$ and $\mathbf{B}$ are any two $n\times n$ nilpotent matrices of index n, explain why $\mathbf{A}\simeq\mathbf{B}$.

(c) Explain why all $n\times n$ nilpotent matrices of index n must have a zero trace and be of rank $n-1$.

4.8.12. $\mathbf{E}$ is *idempotent* when $\mathbf{E}^2=\mathbf{E}$. For an idempotent operator $\mathbf{E}$ on $\Re^n$, let $\mathcal{X}=\{\mathbf{x}_i\}_{i=1}^{r}$ and $\mathcal{Y}=\{\mathbf{y}_i\}_{i=1}^{n-r}$ be bases for $R(\mathbf{E})$ and $N(\mathbf{E})$, respectively.

(a) Prove that $\mathcal{B}=\mathcal{X}\cup\mathcal{Y}$ is a basis for $\Re^n$. **Hint:** Show $\mathbf{E}\mathbf{x}_i=\mathbf{x}_i$ and use this to deduce that $\mathcal{B}$ is linearly independent.

(b) Show that $[\mathbf{E}]_{\mathcal{B}}=\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$.

(c) Explain why two $n\times n$ idempotent matrices of the same rank must be similar.

(d) If $\mathbf{F}$ is an idempotent matrix, prove that $rank(\mathbf{F})=trace(\mathbf{F})$.

4.9 INVARIANT SUBSPACES

For a linear operator $\mathbf{T}$ on a vector space $\mathcal{V}$, and for $\mathcal{X} \subseteq \mathcal{V}$,

$$\mathbf{T}(\mathcal{X}) = \{\mathbf{T}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}\}$$

is the set of all possible images of vectors from $\mathcal{X}$ under the transformation $\mathbf{T}$. Notice that $\mathbf{T}(\mathcal{V}) = R(\mathbf{T})$. When $\mathcal{X}$ is a subspace of $\mathcal{V}$, it follows that $\mathbf{T}(\mathcal{X})$ is also a subspace of $\mathcal{V}$, but $\mathbf{T}(\mathcal{X})$ is usually not related to $\mathcal{X}$. However, in some special cases it can happen that $\mathbf{T}(\mathcal{X}) \subseteq \mathcal{X}$, and such subspaces are the focus of this section.

Invariant Subspaces

- For a linear operator $\mathbf{T}$ on $\mathcal{V}$, a subspace $\mathcal{X} \subseteq \mathcal{V}$ is said to be an ***invariant subspace*** under $\mathbf{T}$ whenever $\mathbf{T}(\mathcal{X}) \subseteq \mathcal{X}$.
- In such a situation, $\mathbf{T}$ can be considered as a linear operator on $\mathcal{X}$ by forgetting about everything else in $\mathcal{V}$ and restricting $\mathbf{T}$ to act only on vectors from $\mathcal{X}$. Hereafter, this ***restricted operator*** will be denoted by $\mathbf{T}_{/\mathcal{X}}$.

Example 4.9.1

Problem: For

$$\mathbf{A} = \begin{pmatrix} 4 & 4 & 4 \\ -2 & -2 & -5 \\ 1 & 2 & 5 \end{pmatrix}, \qquad \mathbf{x}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix},$$

show that the subspace $\mathcal{X}$ spanned by $\mathcal{B} = \{\mathbf{x}_1, \ \mathbf{x}_2\}$ is an invariant subspace under $\mathbf{A}$. Then describe the restriction $\mathbf{A}_{/\mathcal{X}}$ and determine the coordinate matrix of $\mathbf{A}_{/\mathcal{X}}$ relative to $\mathcal{B}$.

Solution: Observe that $\mathbf{A}\mathbf{x}_1 = 2\mathbf{x}_1 \in \mathcal{X}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{x}_1 + 2\mathbf{x}_2 \in \mathcal{X}$, so the image of any $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \in \mathcal{X}$ is back in $\mathcal{X}$ because

$$\mathbf{A}\mathbf{x} = \mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha\mathbf{A}\mathbf{x}_1 + \beta\mathbf{A}\mathbf{x}_2 = 2\alpha\mathbf{x}_1 + \beta(\mathbf{x}_1 + 2\mathbf{x}_2) = (2\alpha + \beta)\mathbf{x}_1 + 2\beta\mathbf{x}_2.$$

This equation completely describes the action of $\mathbf{A}$ restricted to $\mathcal{X}$, so

$$\mathbf{A}_{/\mathcal{X}}(\mathbf{x}) = (2\alpha + \beta)\mathbf{x}_1 + 2\beta\mathbf{x}_2 \quad \text{for each} \quad \mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \in \mathcal{X}.$$

Since $\mathbf{A}_{/\mathcal{X}}(\mathbf{x}_1) = 2\mathbf{x}_1$ and $\mathbf{A}_{/\mathcal{X}}(\mathbf{x}_2) = \mathbf{x}_1 + 2\mathbf{x}_2$, we have

$$\left[\mathbf{A}_{/\mathcal{X}}\right]_{\mathcal{B}} = \left(\left[\mathbf{A}_{/\mathcal{X}}(\mathbf{x}_1)\right]_{\mathcal{B}} \,\middle|\, \left[\mathbf{A}_{/\mathcal{X}}(\mathbf{x}_2)\right]_{\mathcal{B}} \right) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The invariant subspaces for a linear operator $\mathbf{T}$ are important because they produce simplified coordinate matrix representations of $\mathbf{T}$. To understand how this occurs, suppose $\mathcal{X}$ is an invariant subspace under $\mathbf{T}$, and let

$$\mathcal{B}_{\mathcal{X}} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$$

be a basis for $\mathcal{X}$ that is part of a basis

$$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_q\}$$

for the entire space $\mathcal{V}$. To compute $[\mathbf{T}]_{\mathcal{B}}$, recall from the definition of coordinate matrices that

$$[\mathbf{T}]_{\mathcal{B}} = \Big([\mathbf{T}(\mathbf{x}_1)]_{\mathcal{B}} \Big| \cdots \Big| [\mathbf{T}(\mathbf{x}_r)]_{\mathcal{B}} \Big| [\mathbf{T}(\mathbf{y}_1)]_{\mathcal{B}} \Big| \cdots \Big| [\mathbf{T}(\mathbf{y}_q)]_{\mathcal{B}} \Big). \tag{4.9.1}$$

Because each $\mathbf{T}(\mathbf{x}_j)$ is contained in $\mathcal{X}$, only the first r vectors from $\mathcal{B}$ are needed to represent each $\mathbf{T}(\mathbf{x}_j)$, so, for $j = 1, 2, \ldots, r$,

$$\mathbf{T}(\mathbf{x}_j) = \sum_{i=1}^{r} \alpha_{ij}\mathbf{x}_i \quad \text{and} \quad [\mathbf{T}(\mathbf{x}_j)]_{\mathcal{B}} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{rj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.9.2}$$

The space

$$\mathcal{Y} = span\,\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_q\} \tag{4.9.3}$$

may not be an invariant subspace for $\mathbf{T}$, so all the basis vectors in $\mathcal{B}$ may be needed to represent the $\mathbf{T}(\mathbf{y}_j)$'s. Consequently, for $j = 1, 2, \ldots, q$,

$$\mathbf{T}(\mathbf{y}_j) = \sum_{i=1}^{r} \beta_{ij}\mathbf{x}_i + \sum_{i=1}^{q} \gamma_{ij}\mathbf{y}_i \quad \text{and} \quad [\mathbf{T}(\mathbf{y}_j)]_{\mathcal{B}} = \begin{pmatrix} \beta_{1j} \\ \vdots \\ \beta_{rj} \\ \gamma_{1j} \\ \vdots \\ \gamma_{qj} \end{pmatrix}. \tag{4.9.4}$$

Using (4.9.2) and (4.9.4) in (4.9.1) produces the block-triangular matrix

$$[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1r} & \beta_{11} & \cdots & \beta_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} & \beta_{r1} & \cdots & \beta_{rq} \\ 0 & \cdots & 0 & \gamma_{11} & \cdots & \gamma_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma_{q1} & \cdots & \gamma_{qq} \end{pmatrix}. \tag{4.9.5}$$

The equations $\mathbf{T}(\mathbf{x}_j) = \sum_{i=1}^{r} \alpha_{ij}\mathbf{x}_i$ in (4.9.2) mean that

$$\left[\mathbf{T}_{/\mathcal{X}}(\mathbf{x}_j)\right]_{\mathcal{B}_\mathcal{X}} = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{rj} \end{pmatrix}, \quad \text{so} \quad \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_\mathcal{X}} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \end{pmatrix},$$

and thus the matrix in (4.9.5) can be written as

$$[\mathbf{T}]_\mathcal{B} = \begin{pmatrix} \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_\mathcal{X}} & \mathbf{B}_{r\times q} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix}. \tag{4.9.6}$$

In other words, (4.9.6) says that the matrix representation for $\mathbf{T}$ can be made to be block triangular whenever a basis for an invariant subspace is available.

The more invariant subspaces we can find, the more tools we have to construct simplified matrix representations. For example, if the space $\mathcal{Y}$ in (4.9.3) is also an invariant subspace for $\mathbf{T}$, then $\mathbf{T}(\mathbf{y}_j) \in \mathcal{Y}$ for each $j = 1, 2, \ldots, q$, and only the $\mathbf{y}_i$'s are needed to represent $\mathbf{T}(\mathbf{y}_j)$ in (4.9.4). Consequently, the β_{ij} 's are all zero, and $[\mathbf{T}]_\mathcal{B}$ has the block-diagonal form

$$[\mathbf{T}]_\mathcal{B} = \begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix} = \begin{pmatrix} \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_x} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{T}_{/\mathcal{Y}}\right]_{\mathcal{B}_y} \end{pmatrix}.$$

This notion easily generalizes in the sense that if $\mathcal{B} = \mathcal{B}_\mathcal{X} \cup \mathcal{B}_\mathcal{Y} \cup \cdots \cup \mathcal{B}_\mathcal{Z}$ is a basis for $\mathcal{V}$, where $\mathcal{B}_\mathcal{X}, \mathcal{B}_\mathcal{Y}, \ldots, \mathcal{B}_\mathcal{Z}$ are bases for invariant subspaces under $\mathbf{T}$ that have dimensions $r_1, r_2, \ldots, r_k$, respectively, then $[\mathbf{T}]_\mathcal{B}$ has the block-diagonal form

$$[\mathbf{T}]_\mathcal{B} = \begin{pmatrix} \mathbf{A}_{r_1\times r_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{r_2\times r_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_{r_k\times r_k} \end{pmatrix},$$

where

$$\mathbf{A} = \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_x}, \quad \mathbf{B} = \left[\mathbf{T}_{/\mathcal{Y}}\right]_{\mathcal{B}_y}, \quad \ldots, \quad \mathbf{C} = \left[\mathbf{T}_{/\mathcal{Z}}\right]_{\mathcal{B}_z}.$$

The situations discussed above are also reversible in the sense that if the matrix representation of $\mathbf{T}$ has a block-triangular form

$$[\mathbf{T}]_\mathcal{B} = \begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{B}_{r\times q} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix}$$

relative to some basis

$$\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\},$$

then the r-dimensional subspace $\mathcal{U} = span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ spanned by the first r vectors in $\mathcal{B}$ must be an invariant subspace under $\mathbf{T}$. Furthermore, if the matrix representation of $\mathbf{T}$ has a block-diagonal form

$$[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix}$$

relative to $\mathcal{B}$, then both

$$\mathcal{U} = span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\} \quad \text{and} \quad \mathcal{W} = span\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$$

must be invariant subspaces for $\mathbf{T}$. The details are left as exercises.

The general statement concerning invariant subspaces and coordinate matrix representations is given below.

Invariant Subspaces and Matrix Representations

Let $\mathbf{T}$ be a linear operator on an n-dimensional space $\mathcal{V}$, and let $\mathcal{X}, \mathcal{Y}, \ldots, \mathcal{Z}$ be subspaces of $\mathcal{V}$ with respective dimensions $r_1, r_2, \ldots, r_k$ and bases $\mathcal{B}_\mathcal{X}, \mathcal{B}_\mathcal{Y}, \ldots, \mathcal{B}_\mathcal{Z}$. Furthermore, suppose that $\sum_i r_i = n$ and $\mathcal{B} = \mathcal{B}_\mathcal{X} \cup \mathcal{B}_\mathcal{Y} \cup \cdots \cup \mathcal{B}_\mathcal{Z}$ is a basis for $\mathcal{V}$.

- The subspace $\mathcal{X}$ is an invariant subspace under $\mathbf{T}$ if and only if $[\mathbf{T}]_\mathcal{B}$ has the block-triangular form

$$[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} \mathbf{A}_{r_1\times r_1} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}, \quad \text{in which case} \quad \mathbf{A} = \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_\mathcal{X}}. \tag{4.9.7}$$

- The subspaces $\mathcal{X}, \mathcal{Y}, \ldots, \mathcal{Z}$ are all invariant under $\mathbf{T}$ if and only if $[\mathbf{T}]_\mathcal{B}$ has the block-diagonal form

$$[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} \mathbf{A}_{r_1\times r_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{r_2\times r_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_{r_k\times r_k} \end{pmatrix}, \tag{4.9.8}$$

  in which case

$$\mathbf{A} = \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_x}, \quad \mathbf{B} = \left[\mathbf{T}_{/\mathcal{Y}}\right]_{\mathcal{B}_y}, \quad \ldots, \quad \mathbf{C} = \left[\mathbf{T}_{/\mathcal{Z}}\right]_{\mathcal{B}_z}.$$

An important corollary concerns the special case in which the linear operator $\mathbf{T}$ is in fact an $n \times n$ matrix and $\mathbf{T}(\mathbf{v}) = \mathbf{T}\mathbf{v}$ is a matrix–vector multiplication.

Triangular and Diagonal Block Forms

When $\mathbf{T}$ is an $n \times n$ matrix, the following two statements are true.

- $\mathbf{Q}$ is a nonsingular matrix such that

$$\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q} = \begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{B}_{r\times q} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix} \tag{4.9.9}$$

  if and only if the first r columns in $\mathbf{Q}$ span an invariant subspace under $\mathbf{T}$.

- $\mathbf{Q}$ is a nonsingular matrix such that

$$\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q} = \begin{pmatrix} \mathbf{A}_{r_1\times r_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{r_2\times r_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_{r_k\times r_k} \end{pmatrix} \tag{4.9.10}$$

  if and only if $\mathbf{Q} = \left(\mathbf{Q}_1 \,|\, \mathbf{Q}_2 \,|\, \cdots \,|\, \mathbf{Q}_k\right)$ in which $\mathbf{Q}_i$ is $n \times r_i$, and the columns of each $\mathbf{Q}_i$ span an invariant subspace under $\mathbf{T}$.

Proof. We know from Example 4.8.3 that if $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ is a basis for $\Re^n$, and if $\mathbf{Q} = \left(\mathbf{q}_1 \,|\, \mathbf{q}_2 \,|\, \cdots \,|\, \mathbf{q}_n\right)$ is the matrix containing the vectors from $\mathcal{B}$ as its columns, then $[\mathbf{T}]_{\mathcal{B}} = \mathbf{Q}^{-1}\mathbf{T}\mathbf{Q}$. Statements (4.9.9) and (4.9.10) are now direct consequences of statements (4.9.7) and (4.9.8), respectively. ■

Example 4.9.2

Problem: For

$$\mathbf{T} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -5 & -16 & -22 \\ 0 & 3 & 10 & 14 \\ 4 & 8 & 12 & 14 \end{pmatrix}, \quad \mathbf{q}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{q}_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix},$$

verify that $\mathcal{X} = span\,\{\mathbf{q}_1, \mathbf{q}_2\}$ is an invariant subspace under $\mathbf{T}$, and then find a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q}$ has the block-triangular form

$$\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q} = \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right).$$

Solution: $\mathcal{X}$ is invariant because $\mathbf{T}\mathbf{q}_1 = \mathbf{q}_1 + 3\mathbf{q}_2$ and $\mathbf{T}\mathbf{q}_2 = 2\mathbf{q}_1 + 4\mathbf{q}_2$ insure that for all α and β, the images

$$\mathbf{T}(\alpha\mathbf{q}_1 + \beta\mathbf{q}_2) = (\alpha + 2\beta)\mathbf{q}_1 + (3\alpha + 4\beta)\mathbf{q}_2$$

lie in $\mathcal{X}$. The desired matrix $\mathbf{Q}$ is constructed by extending $\{\mathbf{q}_1, \mathbf{q}_2\}$ to a basis $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ for $\Re^4$. If the extension technique described in Solution 2 of Example 4.4.5 is used, then

$$\mathbf{q}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{q}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and

$$\mathbf{Q} = \left(\mathbf{q}_1 \mid \mathbf{q}_2 \mid \mathbf{q}_3 \mid \mathbf{q}_4\right) = \begin{pmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the first two columns of $\mathbf{Q}$ span a space that is invariant under $\mathbf{T}$, it follows from (4.9.9) that $\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q}$ must be in block-triangular form. This is easy to verify by computing

$$\mathbf{Q}^{-1} = \begin{pmatrix} 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q}^{-1}\mathbf{T}\mathbf{Q} = \left(\begin{array}{cc|cc} 1 & 2 & 0 & -6 \\ 3 & 4 & 0 & -14 \\ \hline 0 & 0 & -1 & -3 \\ 0 & 0 & 4 & 14 \end{array}\right).$$

In passing, notice that the upper-left-hand block is

$$\left[\mathbf{T}_{/\mathcal{X}}\right]_{\{\mathbf{q}_1, \mathbf{q}_2\}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Example 4.9.3

Consider again the matrices of Example 4.9.2:

$$\mathbf{T} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -5 & -16 & -22 \\ 0 & 3 & 10 & 14 \\ 4 & 8 & 12 & 14 \end{pmatrix}, \quad \mathbf{q}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{q}_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix}.$$

There are infinitely many extensions of $\{\mathbf{q}_1, \mathbf{q}_2\}$ to a basis $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ for $\Re^4$—the extension used in Example 4.9.2 is only one possibility. Another extension is

$$\mathbf{q}_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{q}_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

This extension might be preferred over that of Example 4.9.2 because the spaces $\mathcal{X} = span\{\mathbf{q}_1, \mathbf{q}_2\}$ and $\mathcal{Y} = span\{\mathbf{q}_3, \mathbf{q}_4\}$ are both invariant under $\mathbf{T}$, and therefore it follows from (4.9.10) that $\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q}$ is block diagonal. Indeed, it is not difficult to verify that

$$\mathbf{Q}^{-1}\mathbf{T}\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -5 & -16 & -22 \\ 0 & 3 & 10 & 14 \\ 4 & 8 & 12 & 14 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{array}\right).$$

Notice that the diagonal blocks must be the matrices of the restrictions in the sense that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \left[\mathbf{T}_{/\mathcal{X}}\right]_{\{\mathbf{q}_1,\mathbf{q}_2\}} \quad \text{and} \quad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \left[\mathbf{T}_{/\mathcal{Y}}\right]_{\{\mathbf{q}_3,\mathbf{q}_4\}}.$$

Example 4.9.4

Problem: Find all subspaces of $\Re^2$ that are invariant under

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

Solution: The trivial subspace $\{\mathbf{0}\}$ is the only zero-dimensional invariant subspace, and the entire space $\Re^2$ is the only two-dimensional invariant subspace. The real problem is to find all one-dimensional invariant subspaces. If $\mathcal{M}$ is a one-dimensional subspace spanned by $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}(\mathcal{M}) \subseteq \mathcal{M}$, then

$$\mathbf{A}\mathbf{x} \in \mathcal{M} \implies \text{there is a scalar } \lambda \text{ such that } \mathbf{A}\mathbf{x} = \lambda\mathbf{x} \implies (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}.$$

In other words, $\mathcal{M} \subseteq N(\mathbf{A} - \lambda\mathbf{I})$. Since $\dim \mathcal{M} = 1$, it must be the case that $N(\mathbf{A} - \lambda\mathbf{I}) \neq \{\mathbf{0}\}$, and consequently λ must be a scalar such that $(\mathbf{A} - \lambda\mathbf{I})$ is a singular matrix. Row operations produce

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 3-\lambda \\ -\lambda & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 3-\lambda \\ 0 & 1+(\lambda^2-3\lambda)/2 \end{pmatrix},$$

and it is clear that $(\mathbf{A} - \lambda\mathbf{I})$ is singular if and only if $1+(\lambda^2-3\lambda)/2 = 0$—i.e., if and only if λ is a root of

$$\lambda^2 - 3\lambda + 2 = 0.$$

Thus $\lambda = 1$ and $\lambda = 2$, and straightforward computation yields the two one-dimensional invariant subspaces

$$\mathcal{M}_1 = N(\mathbf{A} - \mathbf{I}) = span\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \quad \text{and} \quad \mathcal{M}_2 = N(\mathbf{A} - 2\mathbf{I}) = span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}.$$

In passing, notice that $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ is a basis for $\Re^2$, and

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \text{where} \quad \mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

In general, scalars λ for which $(\mathbf{A} - \lambda\mathbf{I})$ is singular are called the ***eigenvalues*** of $\mathbf{A}$, and the nonzero vectors in $N(\mathbf{A} - \lambda\mathbf{I})$ are known as the associated ***eigenvectors*** for $\mathbf{A}$. As this example indicates, eigenvalues and eigenvectors are of fundamental importance in identifying invariant subspaces and reducing matrices by means of similarity transformations. Eigenvalues and eigenvectors are discussed at length in Chapter 7.

Exercises for section 4.9

4.9.1. Let $\mathbf{T}$ be an arbitrary linear operator on a vector space $\mathcal{V}$.

(a) Is the trivial subspace $\{\mathbf{0}\}$ invariant under $\mathbf{T}$?

(b) Is the entire space $\mathcal{V}$ invariant under $\mathbf{T}$?

4.9.2. Describe all of the subspaces that are invariant under the identity operator $\mathbf{I}$ on a space $\mathcal{V}$.

4.9.3. Let $\mathbf{T}$ be the linear operator on $\Re^4$ defined by

$$\mathbf{T}(x_1, x_2, x_3, x_4) = (x_1 + x_2 + 2x_3 - x_4, \quad x_2 + x_4, \quad 2x_3 - x_4, \quad x_3 + x_4),$$

and let $\mathcal{X} = span\{\mathbf{e}_1, \mathbf{e}_2\}$ be the subspace that is spanned by the first two unit vectors in $\Re^4$.

(a) Explain why $\mathcal{X}$ is invariant under $\mathbf{T}$.

(b) Determine $\left[\mathbf{T}_{/\mathcal{X}}\right]_{\{\mathbf{e}_1, \mathbf{e}_2\}}$.

(c) Describe the structure of $[\mathbf{T}]_{\mathcal{B}}$, where $\mathcal{B}$ is any basis obtained from an extension of $\{\mathbf{e}_1, \mathbf{e}_2\}$.

4.9.4. Let $\mathbf{T}$ and $\mathbf{Q}$ be the matrices

$$\mathbf{T} = \begin{pmatrix} -2 & -1 & -5 & -2 \\ -9 & 0 & -8 & -2 \\ 2 & 3 & 11 & 5 \\ 3 & -5 & -13 & -7 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 3 & -4 \\ -2 & 0 & 1 & 0 \\ 3 & -1 & -4 & 3 \end{pmatrix}.$$

(a) Explain why the columns of $\mathbf{Q}$ are a basis for $\Re^4$.
(b) Verify that $\mathcal{X} = span\{\mathbf{Q}_{*1}, \mathbf{Q}_{*2}\}$ and $\mathcal{Y} = span\{\mathbf{Q}_{*3}, \mathbf{Q}_{*4}\}$ are each invariant subspaces under $\mathbf{T}$.
(c) Describe the structure of $\mathbf{Q}^{-1}\mathbf{TQ}$ without doing any computation.
(d) Now compute the product $\mathbf{Q}^{-1}\mathbf{TQ}$ to determine

$$\left[\mathbf{T}_{/\mathcal{X}}\right]_{\{\mathbf{Q}_{*1},\mathbf{Q}_{*2}\}} \quad \text{and} \quad \left[\mathbf{T}_{/\mathcal{Y}}\right]_{\{\mathbf{Q}_{*3},\mathbf{Q}_{*4}\}}.$$

4.9.5. Let $\mathbf{T}$ be a linear operator on a space $\mathcal{V}$, and suppose that

$$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r, \ \mathbf{w}_1, \ldots, \mathbf{w}_q\}$$

is a basis for $\mathcal{V}$ such that $[\mathbf{T}]_\mathcal{B}$ has the block-diagonal form

$$[\mathbf{T}]_\mathcal{B} = \begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix}.$$

Explain why $\mathcal{U} = span\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ and $\mathcal{W} = span\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ must each be invariant subspaces under $\mathbf{T}$.

4.9.6. If $\mathbf{T}_{n\times n}$ and $\mathbf{P}_{n\times n}$ are matrices such that

$$\mathbf{P}^{-1}\mathbf{TP} = \begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{q\times q} \end{pmatrix},$$

explain why

$$\mathcal{U} = span\{\mathbf{P}_{*1}, \ldots, \mathbf{P}_{*r}\} \quad \text{and} \quad \mathcal{W} = span\{\mathbf{P}_{*r+1}, \ldots, \mathbf{P}_{*n}\}$$

are each invariant subspaces under $\mathbf{T}$.

4.9.7. If $\mathbf{A}$ is an $n \times n$ matrix and λ is a scalar such that $(\mathbf{A} - \lambda\mathbf{I})$ is singular (i.e., λ is an eigenvalue), explain why the associated space of eigenvectors $N(\mathbf{A} - \lambda\mathbf{I})$ is an invariant subspace under $\mathbf{A}$.

4.9.8. Consider the matrix $\mathbf{A} = \begin{pmatrix} -9 & 4 \\ -24 & 11 \end{pmatrix}$.
(a) Determine the eigenvalues of $\mathbf{A}$.
(b) Identify all subspaces of $\Re^2$ that are invariant under $\mathbf{A}$.
(c) Find a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{AQ}$ is a diagonal matrix.

We share a philosophy about linear algebra: we think basis-free,
but when the chips are down we close the office door
and compute with matrices like fury.
— *Irving Kaplansky* (1917–) *speaking about Paul Halmos* (1916–)