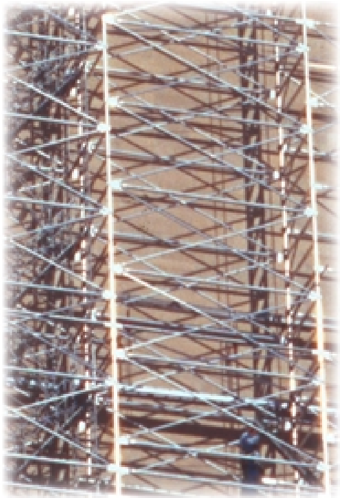

CHAPTER 2

Rectangular Systems and Echelon Forms

2.1 ROW ECHELON FORM AND RANK

We are now ready to analyze more general linear systems consisting of m linear equations involving n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

where m may be different from n. If we do not know for sure that m and n are the same, then the system is said to be ***rectangular.*** The case $m = n$ is still allowed in the discussion—statements concerning rectangular systems also are valid for the special case of square systems.

The first goal is to extend the Gaussian elimination technique from square systems to completely general rectangular systems. Recall that for a square system with a unique solution, the pivotal positions are always located along the ***main diagonal***—the diagonal line from the upper-left-hand corner to the lower-right-hand corner—in the coefficient matrix $\mathbf{A}$ so that Gaussian elimination results in a reduction of $\mathbf{A}$ to a ***triangular matrix***, such as that illustrated below for the case $n = 4$:

$$\mathbf{T} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Remember that a pivot must always be a nonzero number. For square systems possessing a unique solution, it is a fact (proven later) that one can always bring a nonzero number into each pivotal position along the main diagonal.[8] However, in the case of a general rectangular system, it is not always possible to have the pivotal positions lying on a straight diagonal line in the coefficient matrix. This means that the final result of Gaussian elimination will *not* be triangular in form. For example, consider the following system:

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 &= 5,\\
2x_1 + 4x_2 + 4x_4 + 4x_5 &= 6,\\
x_1 + 2x_2 + 3x_3 + 5x_4 + 5x_5 &= 9,\\
2x_1 + 4x_2 + 4x_4 + 7x_5 &= 9.
\end{aligned}$$

Focus your attention on the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}, \tag{2.1.1}$$

and ignore the right-hand side for a moment. Applying Gaussian elimination to $\mathbf{A}$ yields the following result:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}.$$

In the basic elimination process, the strategy is to move down and to the right to the next pivotal position. If a zero occurs in this position, an interchange with a row below the pivotal row is executed so as to bring a nonzero number into the pivotal position. However, in this example, it is clearly impossible to bring a nonzero number into the $(2,2)$-position by interchanging the second row with a lower row.

In order to handle this situation, the elimination process is modified as follows.

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[8] This discussion is for exact arithmetic. If floating-point arithmetic is used, this may no longer be true. Part (a) of Exercise 1.6.1 is one such example.

Modified Gaussian Elimination

Suppose that $\mathbf{U}$ is the augmented matrix associated with the system after $i-1$ elimination steps have been completed. To execute the i^{th} step, proceed as follows:

- Moving from left to right in $\mathbf{U}$, locate the first column that contains a nonzero entry on or below the i^{th} position—say it is $\mathbf{U}_{*j}$.
- The pivotal position for the i^{th} step is the (i,j)-position.
- If necessary, interchange the i^{th} row with a lower row to bring a nonzero number into the (i,j)-position, and then annihilate all entries below this pivot.
- If row $\mathbf{U}_{i*}$ as well as all rows in $\mathbf{U}$ below $\mathbf{U}_{i*}$ consist entirely of zeros, then the elimination process is completed.

Illustrated below is the result of applying this modified version of Gaussian elimination to the matrix given in (2.1.1).

Example 2.1.1

Problem: Apply modified Gaussian elimination to the following matrix and circle the pivot positions:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the final result of applying Gaussian elimination in the above example is not a purely triangular form but rather a jagged or "stair-step" type of triangular form. Hereafter, a matrix that exhibits this stair-step structure will be said to be in ***row echelon form.***

Row Echelon Form

An $m \times n$ matrix $\mathbf{E}$ with rows $\mathbf{E}_{i*}$ and columns $\mathbf{E}_{*j}$ is said to be in ***row echelon form*** provided the following two conditions hold.

- If $\mathbf{E}_{i*}$ consists entirely of zeros, then all rows below $\mathbf{E}_{i*}$ are also entirely zero; i.e., all zero rows are at the bottom.

- If the first nonzero entry in $\mathbf{E}_{i*}$ lies in the j^{th} position, then all entries below the i^{th} position in columns $\mathbf{E}_{*1}, \mathbf{E}_{*2}, \ldots, \mathbf{E}_{*j}$ are zero.

These two conditions say that the nonzero entries in an echelon form must lie on or above a stair-step line that emanates from the upper-left-hand corner and slopes down and to the right. The pivots are the first nonzero entries in each row. A typical structure for a matrix in row echelon form is illustrated below with the pivots circled.

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Because of the flexibility in choosing row operations to reduce a matrix $\mathbf{A}$ to a row echelon form $\mathbf{E}$, the entries in $\mathbf{E}$ are not uniquely determined by $\mathbf{A}$. Nevertheless, it can be proven that the "form" of $\mathbf{E}$ is unique in the sense that *the positions of the pivots in* $\mathbf{E}$ $($*and* $\mathbf{A})$ *are uniquely determined by the entries in* $\mathbf{A}$.[9] Because the pivotal positions are unique, it follows that the number of pivots, which is the same as the number of nonzero rows in $\mathbf{E}$, is also uniquely determined by the entries in $\mathbf{A}$. This number is called the ***rank***[10] of $\mathbf{A}$, and it

[9] The fact that the pivotal positions are unique should be intuitively evident. If it isn't, take the matrix given in (2.1.1) and try to force some different pivotal positions by a different sequence of row operations.

[10] The word "rank" was introduced in 1879 by the German mathematician Ferdinand Georg Frobenius (p. 662), who thought of it as the size of the largest nonzero minor determinant in $\mathbf{A}$. But the concept had been used as early as 1851 by the English mathematician James J. Sylvester (1814–1897).

is extremely important in the development of our subject.

Rank of a Matrix

Suppose $\mathbf{A}_{m\times n}$ is reduced by row operations to an echelon form $\mathbf{E}$. The ***rank*** of $\mathbf{A}$ is defined to be the number

$$
\begin{aligned}
rank\,(\mathbf{A}) &= \text{number of pivots} \\
&= \text{number of nonzero rows in } \mathbf{E} \\
&= \text{number of basic columns in } \mathbf{A},
\end{aligned}
$$

where the ***basic columns*** of $\mathbf{A}$ are defined to be those columns in $\mathbf{A}$ that contain the pivotal positions.

Example 2.1.2

Problem: Determine the rank, and identify the basic columns in

$$
\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix}.
$$

Solution: Reduce $\mathbf{A}$ to row echelon form as shown below:

$$
\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E}.
$$

Consequently, $rank\,(\mathbf{A}) = 2$. The pivotal positions lie in the first and fourth columns so that the basic columns of $\mathbf{A}$ are $\mathbf{A}_{*1}$ and $\mathbf{A}_{*4}$. That is,

$$
\text{Basic Columns} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}.
$$

Pay particular attention to the fact that the basic columns are extracted from $\mathbf{A}$ and not from the row echelon form $\mathbf{E}$.

Exercises for section 2.1

2.1.1. Reduce each of the following matrices to row echelon form, determine the rank, and identify the basic columns.

$$\text{(a)}\ \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{pmatrix} \quad \text{(b)}\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{pmatrix} \quad \text{(c)}\ \begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 4 & 2 & 4 & 4 & 1 & 5 & 5 \\ 2 & 1 & 3 & 1 & 0 & 4 & 3 \\ 6 & 3 & 4 & 8 & 1 & 9 & 5 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 8 & 4 & 2 & 14 & 1 & 13 & 3 \end{pmatrix}$$

2.1.2. Determine which of the following matrices are in row echelon form:

$$\text{(a)}\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}. \qquad \text{(b)}\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\text{(c)}\ \begin{pmatrix} 2 & 2 & 3 & -4 \\ 0 & 0 & 7 & -8 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \text{(d)}\ \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.1.3. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Give a short explanation of why each of the following statements is true.

(a) $rank(\mathbf{A}) \leq \min\{m, n\}$.
(b) $rank(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is entirely zero.
(c) $rank(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is a multiple of another row.
(d) $rank(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is a combination of other rows.
(e) $rank(\mathbf{A}) < n$ if one column in $\mathbf{A}$ is entirely zero.

2.1.4. Let $\mathbf{A} = \begin{pmatrix} .1 & .2 & .3 \\ .4 & .5 & .6 \\ .7 & .8 & .901 \end{pmatrix}$.

(a) Use exact arithmetic to determine $rank(\mathbf{A})$.
(b) Now use 3-digit floating-point arithmetic (without partial pivoting or scaling) to determine $rank(\mathbf{A})$. This number might be called the "3-digit numerical rank."
(c) What happens if partial pivoting is incorporated?

2.1.5. How many different "forms" are possible for a 3×4 matrix that is in row echelon form?

2.1.6. Suppose that $[\mathbf{A}|\mathbf{b}]$ is reduced to a matrix $[\mathbf{E}|\mathbf{c}]$.

(a) Is $[\mathbf{E}|\mathbf{c}]$ in row echelon form if $\mathbf{E}$ is?
(b) If $[\mathbf{E}|\mathbf{c}]$ is in row echelon form, must $\mathbf{E}$ be in row echelon form?

2.2 REDUCED ROW ECHELON FORM

At each step of the Gauss–Jordan method, the pivot is forced to be a 1, and then all entries above and below the pivotal 1 are annihilated. If $\mathbf{A}$ is the coefficient matrix for a square system with a unique solution, then the end result of applying the Gauss–Jordan method to $\mathbf{A}$ is a matrix with 1's on the main diagonal and 0's everywhere else. That is,

$$\mathbf{A} \xrightarrow{\text{Gauss–Jordan}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

But if the Gauss–Jordan technique is applied to a more general $m \times n$ matrix, then the final result is not necessarily the same as described above. The following example illustrates what typically happens in the rectangular case.

Example 2.2.1

Problem: Apply Gauss–Jordan elimination to the following 4×5 matrix and circle the pivot positions. This is the same matrix used in Example 2.1.1:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compare the results of this example with the results of Example 2.1.1, and notice that the "form" of the final matrix is the same in both examples, which indeed must be the case because of the uniqueness of "form" mentioned in the previous section. The only difference is in the numerical value of some of the entries. By the nature of Gauss–Jordan elimination, each pivot is 1 and all entries above and below each pivot are 0. Consequently, the row echelon form produced by the Gauss–Jordan method contains a reduced number of nonzero entries, so it seems only natural to refer to this as a ***reduced row echelon form.***[11]

Reduced Row Echelon Form

A matrix $\mathbf{E}_{m\times n}$ is said to be in ***reduced row echelon form*** provided that the following three conditions hold.

- $\mathbf{E}$ is in row echelon form.
- The first nonzero entry in each row (i.e., each pivot) is 1.
- All entries above each pivot are 0.

A typical structure for a matrix in reduced row echelon form is illustrated below, where entries marked * can be either zero or nonzero numbers:

$$\begin{pmatrix} 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As previously stated, if matrix $\mathbf{A}$ is transformed to a row echelon form by row operations, then the "form" is uniquely determined by $\mathbf{A}$, but the individual entries in the form are not unique. However, if $\mathbf{A}$ is transformed by row operations to a *reduced* row echelon form $\mathbf{E_A}$, then it can be shown[12] that both the "form" as well as the individual entries in $\mathbf{E_A}$ are uniquely determined by $\mathbf{A}$. In other words, the reduced row echelon form $\mathbf{E_A}$ produced from $\mathbf{A}$ is independent of whatever elimination scheme is used. Producing an unreduced form is computationally more efficient, but the uniqueness of $\mathbf{E_A}$ makes it more useful for theoretical purposes.

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11 In some of the older books this is called the *Hermite normal form* in honor of the French mathematician Charles Hermite (1822–1901), who, around 1851, investigated reducing matrices by row operations.

12 A formal uniqueness proof must wait until Example 3.9.2, but you can make this intuitively clear right now with some experiments. Try to produce two different reduced row echelon forms from the same matrix.

$\mathbf{E_A}$ Notation

For a matrix $\mathbf{A}$, the symbol $\mathbf{E_A}$ will hereafter denote the unique reduced row echelon form derived from $\mathbf{A}$ by means of row operations.

Example 2.2.2

Problem: Determine $\mathbf{E_A}$, deduce $rank\,(\mathbf{A})$, and identify the basic columns of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, $rank\,(\mathbf{A}) = 3$, and $\{\mathbf{A}_{*1}, \mathbf{A}_{*3}, \mathbf{A}_{*5}\}$ are the three basic columns.

The above example illustrates another important feature of $\mathbf{E_A}$ and explains why the basic columns are indeed "basic." Each nonbasic column is expressible as a combination of basic columns. In Example 2.2.2,

$$\mathbf{A}_{*2} = 2\mathbf{A}_{*1} \quad \text{and} \quad \mathbf{A}_{*4} = \mathbf{A}_{*1} + \mathbf{A}_{*3}. \tag{2.2.1}$$

Notice that exactly the same set of relationships hold in $\mathbf{E_A}$. That is,

$$\mathbf{E}_{*2} = 2\mathbf{E}_{*1} \quad \text{and} \quad \mathbf{E}_{*4} = \mathbf{E}_{*1} + \mathbf{E}_{*3}. \tag{2.2.2}$$

This is no coincidence—it's characteristic of what happens in general. There's more to observe. The relationships between the nonbasic and basic columns in a general matrix $\mathbf{A}$ are usually obscure, but the relationships among the columns in $\mathbf{E_A}$ are absolutely transparent. For example, notice that the multipliers used in the relationships (2.2.1) and (2.2.2) appear explicitly in the two nonbasic columns in $\mathbf{E_A}$—they are just the nonzero entries in these nonbasic columns. This is important because it means that $\mathbf{E_A}$ can be used as a "map" or "key" to discover or unlock the hidden relationships among the columns of $\mathbf{A}$.

Finally, observe from Example 2.2.2 that only the basic columns *to the left* of a given nonbasic column are needed in order to express the nonbasic column as a combination of basic columns—e.g., representing $\mathbf{A}_{*2}$ requires only $\mathbf{A}_{*1}$ and not $\mathbf{A}_{*3}$ or $\mathbf{A}_{*5}$, while representing $\mathbf{A}_{*4}$ requires only $\mathbf{A}_{*1}$ and $\mathbf{A}_{*3}$. This too is typical. For the time being, we accept the following statements to be true. A rigorous proof is given later on p. 136.

Column Relationships in A and $\mathbf{E_A}$

- Each nonbasic column $\mathbf{E}_{*k}$ in $\mathbf{E_A}$ is a combination (a sum of multiples) of the basic columns in $\mathbf{E_A}$ to the left of $\mathbf{E}_{*k}$. That is,

$$\begin{aligned}\mathbf{E}_{*k} &= \mu_1\mathbf{E}_{*b_1} + \mu_2\mathbf{E}_{*b_2} + \cdots + \mu_j\mathbf{E}_{*b_j}\\ &= \mu_1\begin{pmatrix}1\\0\\\vdots\\0\\\vdots\\0\end{pmatrix} + \mu_2\begin{pmatrix}0\\1\\\vdots\\0\\\vdots\\0\end{pmatrix} + \cdots + \mu_j\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix} = \begin{pmatrix}\mu_1\\\mu_2\\\vdots\\\mu_j\\\vdots\\0\end{pmatrix},\end{aligned}$$

  where the $\mathbf{E}_{*b_i}$'s are the basic columns to the left of $\mathbf{E}_{*k}$ and where the multipliers μ_i are the first j entries in $\mathbf{E}_{*k}$.

- The relationships that exist among the columns of $\mathbf{A}$ are exactly the same as the relationships that exist among the columns of $\mathbf{E_A}$. In particular, if $\mathbf{A}_{*k}$ is a nonbasic column in $\mathbf{A}$, then

$$\mathbf{A}_{*k} = \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j}, \tag{2.2.3}$$

  where the $\mathbf{A}_{*b_i}$'s are the basic columns to the left of $\mathbf{A}_{*k}$, and where the multipliers μ_i are as described above—the first j entries in $\mathbf{E}_{*k}$.

Example 2.2.3

Problem: Write each nonbasic column as a combination of basic columns in

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix}.$$

Solution: Transform $\mathbf{A}$ to $\mathbf{E_A}$ as shown below.

$$\begin{pmatrix} 2 & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & 3 & \frac{3}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & 3 & \frac{3}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 0 & 4 & 12 & -9 & \frac{7}{2} \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 2 & 7 & \frac{15}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 0 & 0 & 0 & -17 & -\frac{17}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 7 & \frac{15}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

The third and fifth columns are nonbasic. Looking at the columns in $\mathbf{E_A}$ reveals

$$\mathbf{E}_{*3} = 2\mathbf{E}_{*1} + 3\mathbf{E}_{*2} \quad \text{and} \quad \mathbf{E}_{*5} = 4\mathbf{E}_{*1} + 2\mathbf{E}_{*2} + \frac{1}{2}\mathbf{E}_{*4}.$$

The relationships that exist among the columns of $\mathbf{A}$ must be exactly the same as those in $\mathbf{E_A}$, so

$$\mathbf{A}_{*3} = 2\mathbf{A}_{*1} + 3\mathbf{A}_{*2} \quad \text{and} \quad \mathbf{A}_{*5} = 4\mathbf{A}_{*1} + 2\mathbf{A}_{*2} + \frac{1}{2}\mathbf{A}_{*4}.$$

You can easily check the validity of these equations by direct calculation.

In summary, the utility of $\mathbf{E_A}$ lies in its ability to reveal dependencies in data stored as columns in an array $\mathbf{A}$. The nonbasic columns in $\mathbf{A}$ represent redundant information in the sense that this information can always be expressed in terms of the data contained in the basic columns.

Although data compression is not the primary reason for introducing $\mathbf{E_A}$, the application to these problems is clear. For a large array of data, it may be more efficient to store only "independent data" (i.e., the basic columns of $\mathbf{A}$) along with the nonzero multipliers μ_i obtained from the nonbasic columns in $\mathbf{E_A}$. Then the redundant data contained in the nonbasic columns of $\mathbf{A}$ can always be reconstructed if and when it is called for.

Exercises for section 2.2

2.2.1. Determine the reduced row echelon form for each of the following matrices and then express each nonbasic column in terms of the basic columns:

$$\text{(a)} \quad \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{pmatrix}, \qquad \text{(b)} \quad \begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 4 & 2 & 4 & 4 & 1 & 5 & 5 \\ 2 & 1 & 3 & 1 & 0 & 4 & 3 \\ 6 & 3 & 4 & 8 & 1 & 9 & 5 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 8 & 4 & 2 & 14 & 1 & 13 & 3 \end{pmatrix}.$$

2.2.2. Construct a matrix $\mathbf{A}$ whose reduced row echelon form is

$$\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Is $\mathbf{A}$ unique?

2.2.3. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Give a short explanation of why $rank(\mathbf{A}) < n$ whenever one column in $\mathbf{A}$ is a combination of other columns in $\mathbf{A}$.

2.2.4. Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} .1 & .2 & .3 \\ .4 & .5 & .6 \\ .7 & .8 & .901 \end{pmatrix}.$$

(a) Use exact arithmetic to determine $\mathbf{E_A}$.
(b) Now use 3-digit floating-point arithmetic (without partial pivoting or scaling) to determine $\mathbf{E_A}$ and formulate a statement concerning "near relationships" between the columns of $\mathbf{A}$.

2.2.5. Consider the matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

You already know that $\mathbf{E}_{*3}$ can be expressed in terms of $\mathbf{E}_{*1}$ and $\mathbf{E}_{*2}$. However, this is not the only way to represent the column dependencies in $\mathbf{E}$. Show how to write $\mathbf{E}_{*1}$ in terms of $\mathbf{E}_{*2}$ and $\mathbf{E}_{*3}$ and then express $\mathbf{E}_{*2}$ as a combination of $\mathbf{E}_{*1}$ and $\mathbf{E}_{*3}$. **Note:** This exercise illustrates that the set of pivotal columns is not the only set that can play the role of "basic columns." Taking the basic columns to be the ones containing the pivots is a matter of convenience because everything becomes automatic that way.

2.3 CONSISTENCY OF LINEAR SYSTEMS

A system of m linear equations in n unknowns is said to be a ***consistent*** system if it possesses at least one solution. If there are no solutions, then the system is called ***inconsistent.*** The purpose of this section is to determine conditions under which a given system will be consistent.

Stating conditions for consistency of systems involving only two or three unknowns is easy. A linear equation in two unknowns represents a line in 2-space, and a linear equation in three unknowns is a plane in 3-space. Consequently, a linear system of m equations in two unknowns is consistent if and only if the m lines defined by the m equations have at least one common point of intersection. Similarly, a system of m equations in three unknowns is consistent if and only if the associated m planes have at least one common point of intersection. However, when m is large, these geometric conditions may not be easy to verify visually, and when $n > 3$, the generalizations of intersecting lines or planes are impossible to visualize with the eye.

Rather than depending on geometry to establish consistency, we use Gaussian elimination. If the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ is reduced by row operations to a matrix $[\mathbf{E}|\mathbf{c}]$ that is in row echelon form, then consistency—or lack of it—becomes evident. Suppose that somewhere in the process of reducing $[\mathbf{A}|\mathbf{b}]$ to $[\mathbf{E}|\mathbf{c}]$ a situation arises in which the only nonzero entry in a row appears on the right-hand side, as illustrated below:

$$\text{Row } i \longrightarrow \left(\begin{array}{cccccc|c} * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right) \longleftarrow \alpha \neq 0.$$

If this occurs in the i^{th} row, then the i^{th} equation of the associated system is

$$0x_1 + 0x_2 + \cdots + 0x_n = \alpha.$$

For $\alpha \neq 0$, this equation has no solution, and hence the original system must also be inconsistent (because row operations don't alter the solution set). The converse also holds. That is, if a system is inconsistent, then somewhere in the elimination process a row of the form

$$(0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha), \quad \alpha \neq 0 \tag{2.3.1}$$

must appear. Otherwise, the back substitution process can be completed and a solution is produced. There is *no* inconsistency indicated when a row of the form $(0\ 0\ \cdots\ 0\,|\,0)$ is encountered. This simply says that $0 = 0$, and although this is no help in determining the value of any unknown, it is nevertheless a true statement, so it doesn't indicate inconsistency in the system.

There are some other ways to characterize the consistency (or inconsistency) of a system. One of these is to observe that if the last column $\mathbf{b}$ in the augmented matrix $[\mathbf{A}|\mathbf{b}]$ is a nonbasic column, then no pivot can exist in the last column, and hence the system is consistent because the situation (2.3.1) cannot occur. Conversely, if the system is consistent, then the situation (2.3.1) never occurs during Gaussian elimination and consequently the last column cannot be basic. In other words, $[\mathbf{A}|\mathbf{b}]$ is consistent if and only if $\mathbf{b}$ is a nonbasic column.

Saying that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$ is equivalent to saying that all basic columns in $[\mathbf{A}|\mathbf{b}]$ lie in the coefficient matrix $\mathbf{A}$. Since the number of basic columns in a matrix is the rank, consistency may also be characterized by stating that a system is consistent if and only if $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$.

Recall from the previous section the fact that each nonbasic column in $[\mathbf{A}|\mathbf{b}]$ must be expressible in terms of the basic columns. Because a consistent system is characterized by the fact that the right-hand side $\mathbf{b}$ is a nonbasic column, it follows that a system is consistent if and only if the right-hand side $\mathbf{b}$ is a combination of columns from the coefficient matrix $\mathbf{A}$.

Each of the equivalent[13] ways of saying that a system is consistent is summarized below.

Consistency

Each of the following is equivalent to saying that $[\mathbf{A}|\mathbf{b}]$ is consistent.

- In row reducing $[\mathbf{A}|\mathbf{b}]$, a row of the following form never appears:

$$(\,0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha\,), \quad \text{where} \quad \alpha \neq 0. \tag{2.3.2}$$

- $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$. (2.3.3)
- $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. (2.3.4)
- $\mathbf{b}$ is a combination of the basic columns in $\mathbf{A}$. (2.3.5)

Example 2.3.1

Problem: Determine if the following system is consistent:

$$\begin{aligned}
x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\
3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3.
\end{aligned}$$

[13] Statements P and Q are said to be equivalent when (P implies Q) as well as its converse (Q implies P) are true statements. This is also the meaning of the phrase "P if and only if Q."

Solution: Apply Gaussian elimination to the augmented matrix $[\mathbf{A}|\mathbf{b}]$ as shown:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 4 & 3 & 1 \\ 2 & 2 & 4 & 4 & 2 & 2 \\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Because a row of the form $(0 \;\; 0 \;\; \cdots \;\; 0 \;\; | \;\; \alpha)$ with $\alpha \neq 0$ never emerges, the system is consistent. We might also observe that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$ so that $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. Finally, by completely reducing $\mathbf{A}$ to $\mathbf{E_A}$, it is possible to verify that $\mathbf{b}$ is indeed a combination of the basic columns $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*5}\}$.

Exercises for section 2.3

2.3.1. Determine which of the following systems are consistent.

(a) $$\begin{aligned} x + 2y + z &= 2, \\ 2x + 4y \qquad &= 2, \\ 3x + 6y + z &= 4. \end{aligned}$$

(b) $$\begin{aligned} 2x + 2y + 4z &= 0, \\ 3x + 2y + 5z &= 0, \\ 4x + 2y + 6z &= 0. \end{aligned}$$

(c) $$\begin{aligned} x - y + z &= 1, \\ x - y - z &= 2, \\ x + y - z &= 3, \\ x + y + z &= 4. \end{aligned}$$

(d) $$\begin{aligned} x - y + z &= 1, \\ x - y - z &= 2, \\ x + y - z &= 3, \\ x + y + z &= 2. \end{aligned}$$

(e) $$\begin{aligned} 2w + x + 3y + 5z &= 1, \\ 4w + \quad\;\; 4y + 8z &= 0, \\ w + x + 2y + 3z &= 0, \\ x + \;\; y + \;\; z &= 0. \end{aligned}$$

(f) $$\begin{aligned} 2w + x + 3y + 5z &= 7, \\ 4w + \quad\;\; 4y + 8z &= 8, \\ w + x + 2y + 3z &= 5, \\ x + \;\; y + \;\; z &= 3. \end{aligned}$$

2.3.2. Construct a 3×4 matrix $\mathbf{A}$ and 3×1 columns $\mathbf{b}$ and $\mathbf{c}$ such that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix for an inconsistent system, but $[\mathbf{A}|\mathbf{c}]$ is the augmented matrix for a consistent system.

2.3.3. If $\mathbf{A}$ is an $m \times n$ matrix with $rank\,(\mathbf{A}) = m$, explain why the system $[\mathbf{A}|\mathbf{b}]$ must be consistent for every right-hand side $\mathbf{b}$.

2.3.4. Consider two consistent systems whose augmented matrices are of the form $[\mathbf{A}|\mathbf{b}]$ and $[\mathbf{A}|\mathbf{c}]$. That is, they differ only on the right-hand side. Is the system associated with $[\mathbf{A} \mid \mathbf{b}+\mathbf{c}]$ also consistent? Explain why.

2.3.5. Is it possible for a parabola whose equation has the form $y = \alpha+\beta x+\gamma x^2$ to pass through the four points $(0,1)$, $(1,3)$, $(2,15)$, and $(3,37)$? Why?

2.3.6. Consider using floating-point arithmetic (without scaling) to solve the following system:

$$\begin{aligned} .835x + .667y &= .168, \\ .333x + .266y &= .067. \end{aligned}$$

(a) Is the system consistent when 5-digit arithmetic is used?
(b) What happens when 6-digit arithmetic is used?

2.3.7. In order to grow a certain crop, it is recommended that each square foot of ground be treated with 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose that there are three brands of fertilizer on the market— say brand $\mathcal{X}$, brand $\mathcal{Y}$, and brand $\mathcal{Z}$. One pound of brand $\mathcal{X}$ contains 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen. One pound of brand $\mathcal{Y}$ contains 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen. One pound of brand $\mathcal{Z}$ contains only 1 unit of phosphorous and 1 unit of nitrogen. Determine whether or not it is possible to meet exactly the recommendation by applying some combination of the three brands of fertilizer.

2.3.8. Suppose that an augmented matrix $[\mathbf{A}|\mathbf{b}]$ is reduced by means of Gaussian elimination to a row echelon form $[\mathbf{E}|\mathbf{c}]$. If a row of the form

$$(\,0 \quad 0 \quad \cdots \quad 0 \mid \alpha\,), \qquad \alpha \neq 0$$

does not appear in $[\mathbf{E}|\mathbf{c}]$, is it possible that rows of this form could have appeared at earlier stages in the reduction process? Why?

2.4 HOMOGENEOUS SYSTEMS

A system of m linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0, \end{aligned}$$

in which the right-hand side consists entirely of 0's is said to be a ***homogeneous system.*** If there is at least one nonzero number on the right-hand side, then the system is called ***nonhomogeneous.*** The purpose of this section is to examine some of the elementary aspects concerning homogeneous systems.

Consistency is never an issue when dealing with homogeneous systems because the zero solution $x_1 = x_2 = \cdots = x_n = 0$ is always one solution regardless of the values of the coefficients. Hereafter, the solution consisting of all zeros is referred to as the ***trivial solution.*** The only question is, "Are there solutions other than the trivial solution, and if so, how can we best describe them?" As before, Gaussian elimination provides the answer.

While reducing the augmented matrix $[\mathbf{A}|\mathbf{0}]$ of a homogeneous system to a row echelon form using Gaussian elimination, the zero column on the right-hand side can never be altered by any of the three elementary row operations. That is, any row echelon form derived from $[\mathbf{A}|\mathbf{0}]$ by means of row operations must also have the form $[\mathbf{E}|\mathbf{0}]$. This means that the last column of 0's is just excess baggage that is not necessary to carry along at each step. Just reduce the coefficient matrix $\mathbf{A}$ to a row echelon form $\mathbf{E}$, and remember that the right-hand side is entirely zero when you execute back substitution. The process is best understood by considering a typical example.

In order to examine the solutions of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0, \end{aligned} \tag{2.4.1}$$

reduce the coefficient matrix to a row echelon form.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E}.$$

Therefore, the original homogeneous system is equivalent to the following reduced homogeneous system:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0, \\ - 3x_3 - 3x_4 &= 0. \end{aligned} \tag{2.4.2}$$

Since there are four unknowns but only two equations in this reduced system, it is impossible to extract a unique solution for each unknown. The best we can do is to pick two "basic" unknowns—which will be called the ***basic variables*** and solve for these in terms of the other two unknowns—whose values must remain arbitrary or "free," and consequently they will be referred to as the ***free variables.*** Although there are several possibilities for selecting a set of basic variables, the convention is to *always solve for the unknowns corresponding to the pivotal positions*—or, equivalently, the unknowns corresponding to the basic columns. In this example, the pivots (as well as the basic columns) lie in the first and third positions, so the strategy is to apply back substitution to solve the reduced system (2.4.2) for the basic variables x_1 and x_3 in terms of the free variables x_2 and x_4. The second equation in (2.4.2) yields

$$x_3 = -x_4$$

and substitution back into the first equation produces

$$\begin{aligned} x_1 &= -2x_2 - 2x_3 - 3x_4, \\ &= -2x_2 - 2(-x_4) - 3x_4, \\ &= -2x_2 - x_4. \end{aligned}$$

Therefore, all solutions of the original homogeneous system can be described by saying

$$\begin{aligned} x_1 &= -2x_2 - x_4, \\ x_2 &\ \text{is "free,"} \\ x_3 &= -x_4, \\ x_4 &\ \text{is "free."} \end{aligned} \tag{2.4.3}$$

As the free variables x_2 and x_4 range over all possible values, the above expressions describe all possible solutions. For example, when x_2 and x_4 assume the values $x_2 = 1$ and $x_4 = -2$, then the particular solution

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = -2$$

is produced. When $x_2 = \pi$ and $x_4 = \sqrt{2}$, then another particular solution

$$x_1 = -2\pi - \sqrt{2}, \quad x_2 = \pi, \quad x_3 = -\sqrt{2}, \quad x_4 = \sqrt{2}$$

is generated.

Rather than describing the solution set as illustrated in (2.4.3), future developments will make it more convenient to express the solution set by writing

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \tag{2.4.4}$$

with the understanding that x_2 and x_4 are free variables that can range over all possible numbers. This representation will be called the ***general solution*** of the homogeneous system. This expression for the general solution emphasizes that every solution is some combination of the two particular solutions

$$\mathbf{h}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

The fact that $\mathbf{h}_1$ and $\mathbf{h}_2$ are each solutions is clear because $\mathbf{h}_1$ is produced when the free variables assume the values $x_2 = 1$ and $x_4 = 0$, whereas the solution $\mathbf{h}_2$ is generated when $x_2 = 0$ and $x_4 = 1$.

Now consider a general homogeneous system $[\mathbf{A}|\mathbf{0}]$ of m linear equations in n unknowns. If the coefficient matrix is such that $rank(\mathbf{A}) = r$, then it should be apparent from the preceding discussion that there will be exactly r basic variables—corresponding to the positions of the basic columns in $\mathbf{A}$—and exactly $n - r$ free variables—corresponding to the positions of the nonbasic columns in $\mathbf{A}$. Reducing $\mathbf{A}$ to a row echelon form using Gaussian elimination and then using back substitution to solve for the basic variables in terms of the free variables produces the ***general solution,*** which has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}, \tag{2.4.5}$$

where $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and where $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n \times 1$ columns that represent particular solutions of the system. As the free variables x_{f_i} range over all possible values, the general solution generates all possible solutions.

The general solution does not depend on which row echelon form is used in the sense that using back substitution to solve for the basic variables in terms of the nonbasic variables generates a unique set of particular solutions $\{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}\}$, regardless of which row echelon form is used. Without going into great detail, one can argue that this is true because using back substitution in any row echelon form to solve for the basic variables must produce exactly the same result as that obtained by completely reducing $\mathbf{A}$ to $\mathbf{E_A}$ and then solving the reduced homogeneous system for the basic variables. Uniqueness of $\mathbf{E_A}$ guarantees the uniqueness of the $\mathbf{h}_i$'s.

For example, if the coefficient matrix $\mathbf{A}$ associated with the system (2.4.1) is completely reduced by the Gauss–Jordan procedure to $\mathbf{E_A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E_A},$$

then we obtain the following reduced system:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 0, \\ x_3 + x_4 &= 0. \end{aligned}$$

Solving for the basic variables x_1 and x_3 in terms of x_2 and x_4 produces exactly the same result as given in (2.4.3) and hence generates exactly the same general solution as shown in (2.4.4).

Because it avoids the back substitution process, you may find it more convenient to use the Gauss–Jordan procedure to reduce $\mathbf{A}$ completely to $\mathbf{E_A}$ and then construct the general solution directly from the entries in $\mathbf{E_A}$. This approach usually will be adopted in the examples and exercises.

As was previously observed, all homogeneous systems are consistent because the trivial solution consisting of all zeros is always one solution. The natural question is, "When is the trivial solution the *only* solution?" In other words, we wish to know when a homogeneous system possesses a unique solution. The form of the general solution (2.4.5) makes the answer transparent. As long as there is at least one free variable, then it is clear from (2.4.5) that there will be an infinite number of solutions. Consequently, the trivial solution is the only solution if and only if there are no free variables. Because the number of free variables is given by $n - r$, where $r = rank(\mathbf{A})$, the previous statement can be reformulated to say that *a homogeneous system possesses a unique solution—the trivial solution—if and only if* $rank(\mathbf{A}) = n$.

Example 2.4.1

The homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 5x_2 + 7x_3 &= 0, \\ 3x_1 + 6x_2 + 8x_3 &= 0, \end{aligned}$$

has only the trivial solution because

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{E}$$

shows that $rank(\mathbf{A}) = n = 3$. Indeed, it is also obvious from $\mathbf{E}$ that applying back substitution in the system $[\mathbf{E}|\mathbf{0}]$ yields only the trivial solution.

Example 2.4.2

Problem: Explain why the following homogeneous system has infinitely many solutions, and exhibit the general solution:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 5x_2 + 7x_3 &= 0, \\ 3x_1 + 6x_2 + 6x_3 &= 0. \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{E}$$

shows that $rank(\mathbf{A}) = 2 < n = 3$. Since the basic columns lie in positions one and two, x_1 and x_2 are the basic variables while x_3 is free. Using back substitution on $[\mathbf{E}|\mathbf{0}]$ to solve for the basic variables in terms of the free variable produces $x_2 = -3x_3$ and $x_1 = -2x_2 - 2x_3 = 4x_3$, so the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \quad \text{where} \quad x_3 \text{ is free.}$$

That is, every solution is a multiple of the one particular solution $\mathbf{h}_1 = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$.

Summary

Let $\mathbf{A}_{m\times n}$ be the coefficient matrix for a homogeneous system of m linear equations in n unknowns, and suppose $rank(\mathbf{A}) = r$.

- The unknowns that correspond to the positions of the basic columns (i.e., the pivotal positions) are called the ***basic variables***, and the unknowns corresponding to the positions of the nonbasic columns are called the ***free variables.***
- There are exactly r basic variables and $n - r$ free variables.
- To describe all solutions, reduce $\mathbf{A}$ to a row echelon form using Gaussian elimination, and then use back substitution to solve for the basic variables in terms of the free variables. This produces the ***general solution*** that has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

  where the terms $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and where $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n \times 1$ columns that represent particular solutions of the homogeneous system. The $\mathbf{h}_i$'s are independent of which row echelon form is used in the back substitution process. As the free variables x_{f_i} range over all possible values, the general solution generates all possible solutions.
- A homogeneous system possesses a unique solution (the trivial solution) if and only if $rank(\mathbf{A}) = n$—i.e., if and only if there are no free variables.

Exercises for section 2.4

2.4.1. Determine the general solution for each of the following homogeneous systems.

$$\text{(a)}\quad \begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 0,\\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0,\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0. \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 2x + y + z &= 0,\\ 4x + 2y + z &= 0,\\ 6x + 3y + z &= 0,\\ 8x + 4y + z &= 0. \end{aligned}$$

$$\text{(c)}\quad \begin{aligned} x_1 + x_2 + 2x_3 \phantom{{}+3x_4} &= 0,\\ 3x_1 \phantom{{}+x_2} + 3x_3 + 3x_4 &= 0,\\ 2x_1 + x_2 + 3x_3 + x_4 &= 0,\\ x_1 + 2x_2 + 3x_3 - x_4 &= 0. \end{aligned} \qquad \text{(d)}\quad \begin{aligned} 2x + y + z &= 0,\\ 4x + 2y + z &= 0,\\ 6x + 3y + z &= 0,\\ 8x + 5y + z &= 0. \end{aligned}$$

2.4.2. Among all solutions that satisfy the homogeneous system

$$\begin{aligned} x + 2y + z &= 0,\\ 2x + 4y + z &= 0,\\ x + 2y - z &= 0, \end{aligned}$$

determine those that also satisfy the nonlinear constraint $y - xy = 2z$.

2.4.3. Consider a homogeneous system whose coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 1\\ 2 & 4 & -1 & 3 & 8\\ 1 & 2 & 3 & 5 & 7\\ 2 & 4 & 2 & 6 & 2\\ 3 & 6 & 1 & 7 & -3 \end{pmatrix}.$$

First transform $\mathbf{A}$ to an unreduced row echelon form to determine the general solution of the associated homogeneous system. Then reduce $\mathbf{A}$ to $\mathbf{E_A}$, and show that the same general solution is produced.

2.4.4. If $\mathbf{A}$ is the coefficient matrix for a homogeneous system consisting of four equations in eight unknowns and if there are five free variables, what is $rank\,(\mathbf{A})$?

2.4.5. Suppose that $\mathbf{A}$ is the coefficient matrix for a homogeneous system of four equations in six unknowns and suppose that $\mathbf{A}$ has at least one nonzero row.

(a) Determine the fewest number of free variables that are possible.

(b) Determine the maximum number of free variables that are possible.

2.4.6. Explain why a homogeneous system of m equations in n unknowns where $m < n$ must always possess an infinite number of solutions.

2.4.7. Construct a homogeneous system of three equations in four unknowns that has

$$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

as its general solution.

2.4.8. If $\mathbf{c}_1$ and $\mathbf{c}_2$ are columns that represent two particular solutions of the same homogeneous system, explain why the sum $\mathbf{c}_1 + \mathbf{c}_2$ must also represent a solution of this system.

2.5 NONHOMOGENEOUS SYSTEMS

Recall that a system of m linear equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

is said to be ***nonhomogeneous*** whenever $b_i \neq 0$ for at least one i. Unlike homogeneous systems, a nonhomogeneous system may be inconsistent and the techniques of §2.3 must be applied in order to determine if solutions do indeed exist. Unless otherwise stated, it is assumed that all systems in this section are consistent.

To describe the set of all possible solutions of a consistent nonhomogeneous system, construct a general solution by exactly the same method used for homogeneous systems as follows.

- Use Gaussian elimination to reduce the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ to a row echelon form $[\mathbf{E}|\mathbf{c}]$.
- Identify the basic variables and the free variables in the same manner described in §2.4.
- Apply back substitution to $[\mathbf{E}|\mathbf{c}]$ and solve for the basic variables in terms of the free variables.
- Write the result in the form

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}, \tag{2.5.1}$$

  where $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and $\mathbf{p}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n \times 1$ columns. This is the ***general solution*** of the nonhomogeneous system.

As the free variables x_{f_i} range over all possible values, the general solution (2.5.1) generates all possible solutions of the system $[\mathbf{A}|\mathbf{b}]$. Just as in the homogeneous case, the columns $\mathbf{h_i}$ and $\mathbf{p}$ are independent of which row echelon form $[\mathbf{E}|\mathbf{c}]$ is used. Therefore, $[\mathbf{A}|\mathbf{b}]$ may be completely reduced to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ by using the Gauss–Jordan method thereby avoiding the need to perform back substitution. We will use this approach whenever it is convenient.

The difference between the general solution of a nonhomogeneous system and the general solution of a homogeneous system is the column $\mathbf{p}$ that appears in (2.5.1). To understand why $\mathbf{p}$ appears and where it comes from, consider the nonhomogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 4, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 5, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 7, \end{aligned} \tag{2.5.2}$$

in which the coefficient matrix is the same as the coefficient matrix for the homogeneous system (2.4.1) used in the previous section. If $[\mathbf{A}|\mathbf{b}]$ is completely reduced by the Gauss–Jordan procedure to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$

$$[\mathbf{A}|\mathbf{b}] = \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 3 & 6 & 1 & 4 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]},$$

then the following reduced system is obtained:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 2, \\ x_3 + x_4 &= 1. \end{aligned}$$

Solving for the basic variables, x_1 and x_3, in terms of the free variables, x_2 and x_4, produces

$$\begin{aligned} &x_1 = 2 - 2x_2 - x_4, \\ &x_2 \text{ is "free,"} \\ &x_3 = 1 - x_4, \\ &x_4 \text{ is "free."} \end{aligned}$$

The general solution is obtained by writing these statements in the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 2x_2 - x_4 \\ x_2 \\ 1 - x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}. \tag{2.5.3}$$

As the free variables x_2 and x_4 range over all possible numbers, this generates all possible solutions of the nonhomogeneous system (2.5.2). Notice that the column $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ in (2.5.3) is a *particular solution* of the nonhomogeneous system (2.5.2)—it is the solution produced when the free variables assume the values $x_2 = 0$ and $x_4 = 0$.

Furthermore, recall from (2.4.4) that the general solution of the associated homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0, \end{aligned} \tag{2.5.4}$$

is given by

$$\begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

That is, the general solution of the associated homogeneous system (2.5.4) is a part of the general solution of the original nonhomogeneous system (2.5.2).

These two observations can be combined by saying that *the general solution of the nonhomogeneous system is given by a particular solution plus the general solution of the associated homogeneous system.*[14]

To see that the previous statement is always true, suppose $[\mathbf{A}|\mathbf{b}]$ represents a general $m \times n$ consistent system where $rank\,(\mathbf{A}) = r$. Consistency guarantees that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$, and hence the basic columns in $[\mathbf{A}|\mathbf{b}]$ are in the same positions as the basic columns in $[\mathbf{A}|\mathbf{0}]$ so that the nonhomogeneous system and the associated homogeneous system have exactly the same set of basic variables as well as free variables. Furthermore, it is not difficult to see that

$$\mathbf{E}_{[\mathbf{A}|\mathbf{0}]} = [\mathbf{E_A}|\mathbf{0}] \quad \text{and} \quad \mathbf{E}_{[\mathbf{A}|\mathbf{b}]} = [\mathbf{E_A}|\mathbf{c}],$$

where $\mathbf{c}$ is some column of the form $\mathbf{c} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. This means that if you solve the i^{th} equation in the reduced homogeneous system for the i^{th} basic variable x_{b_i} in terms of the free variables $x_{f_i}, x_{f_{i+1}}, \ldots, x_{f_{n-r}}$ to produce

$$x_{b_i} = \alpha_i x_{f_i} + \alpha_{i+1} x_{f_{i+1}} + \cdots + \alpha_{n-r} x_{f_{n-r}}, \tag{2.5.5}$$

then the solution for the i^{th} basic variable in the reduced nonhomogeneous system must have the form

$$x_{b_i} = \xi_i + \alpha_i x_{f_i} + \alpha_{i+1} x_{f_{i+1}} + \cdots + \alpha_{n-r} x_{f_{n-r}}. \tag{2.5.6}$$

---

[14] For those students who have studied differential equations, this statement should have a familiar ring. Exactly the same situation holds for the general solution to a linear differential equation. This is no accident—it is due to the inherent linearity in both problems. More will be said about this issue later in the text.

That is, the two solutions differ only in the fact that the latter contains the constant ξ_i. Consider organizing the expressions (2.5.5) and (2.5.6) so as to construct the respective general solutions. If the general solution of the homogeneous system has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

then it is apparent that the general solution of the nonhomogeneous system must have a similar form

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r} \tag{2.5.7}$$

in which the column $\mathbf{p}$ contains the constants ξ_i along with some 0's—the ξ_i 's occupy positions in $\mathbf{p}$ that correspond to the positions of the basic columns, and 0's occupy all other positions. The column $\mathbf{p}$ represents one particular solution to the nonhomogeneous system because it is the solution produced when the free variables assume the values $x_{f_1} = x_{f_2} = \cdots = x_{f_{n-r}} = 0$.

Example 2.5.1

Problem: Determine the general solution of the following nonhomogeneous system and compare it with the general solution of the associated homogeneous system:

$$\begin{aligned}
x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\
3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3.
\end{aligned}$$

Solution: Reducing the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ yields

$$\mathbf{A} = \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 2 & 2 & 4 & 4 & 3 & 1\\ 2 & 2 & 4 & 4 & 2 & 2\\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 2 & 2 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]}.$$

Observe that the system is indeed consistent because the last column is nonbasic. Solve the reduced system for the basic variables x_1, x_2, and x_5 in terms of the free variables x_3 and x_4 to obtain

$$\begin{aligned}
&x_1 = 1 - x_3 - 2x_4,\\
&x_2 = 1 - x_3,\\
&x_3 \text{ is "free,"}\\
&x_4 \text{ is "free,"}\\
&x_5 = -1.
\end{aligned}$$

The general solution to the nonhomogeneous system is

$$\mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - x_3 - 2x_4\\ 1 - x_3\\ x_3\\ x_4\\ -1 \end{pmatrix} = \begin{pmatrix} 1\\ 1\\ 0\\ 0\\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1\\ -1\\ 1\\ 0\\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2\\ 0\\ 0\\ 1\\ 0 \end{pmatrix}.$$

The general solution of the associated homogeneous system is

$$\mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4\\ -x_3\\ x_3\\ x_4\\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1\\ -1\\ 1\\ 0\\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2\\ 0\\ 0\\ 1\\ 0 \end{pmatrix}.$$

You should verify for yourself that

$$\mathbf{p} = \begin{pmatrix} 1\\ 1\\ 0\\ 0\\ -1 \end{pmatrix}$$

is indeed a particular solution to the nonhomogeneous system and that

$$\mathbf{h}_3 = \begin{pmatrix} -1\\ -1\\ 1\\ 0\\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_4 = \begin{pmatrix} -2\\ 0\\ 0\\ 1\\ 0 \end{pmatrix}$$

are particular solutions to the associated homogeneous system.

Now turn to the question, "When does a consistent system have a unique solution?" It is known from (2.5.7) that the general solution of a consistent $m \times n$ nonhomogeneous system $[\mathbf{A}|\mathbf{b}]$ with $rank(\mathbf{A}) = r$ is given by

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

where

$$x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}$$

is the general solution of the associated homogeneous system. Consequently, it is evident that the nonhomogeneous system $[\mathbf{A}|\mathbf{b}]$ will have a unique solution (namely, $\mathbf{p}$) if and only if there are no free variables—i.e., if and only if $r = n$ (= number of unknowns)—this is equivalent to saying that the associated homogeneous system $[\mathbf{A}|\mathbf{0}]$ has only the trivial solution.

Example 2.5.2

Consider the following nonhomogeneous system:

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2, \\ x_1 + 2x_2 + 3x_3 &= 1, \\ x_1 \qquad\quad + \ x_3 &= -3, \\ 2x_1 + 4x_2 \qquad\quad &= 8. \end{aligned}$$

Reducing $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ yields

$$[\mathbf{A}|\mathbf{b}] = \left(\begin{array}{ccc|c} 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & -3 \\ 2 & 4 & 0 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]}.$$

The system is consistent because the last column is nonbasic. There are several ways to see that the system has a unique solution. Notice that

$$rank(\mathbf{A}) = 3 = \text{ number of unknowns},$$

which is the same as observing that there are no free variables. Furthermore, the associated homogeneous system clearly has only the trivial solution. Finally, because we completely reduced $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$, it is obvious that there is only one solution possible and that it is given by $\mathbf{p} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$.

Summary

Let $[\mathbf{A}|\mathbf{b}]$ be the augmented matrix for a consistent $m \times n$ nonhomogeneous system in which $rank\,(\mathbf{A}) = r$.

- Reducing $[\mathbf{A}|\mathbf{b}]$ to a row echelon form using Gaussian elimination and then solving for the basic variables in terms of the free variables leads to the ***general solution***

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}.$$

  As the free variables x_{f_i} range over all possible values, this general solution generates all possible solutions of the system.

- Column $\mathbf{p}$ is a particular solution of the nonhomogeneous system.
- The expression $x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}$ is the general solution of the associated homogeneous system.
- Column $\mathbf{p}$ as well as the columns $\mathbf{h}_i$ are independent of the row echelon form to which $[\mathbf{A}|\mathbf{b}]$ is reduced.
- The system possesses a unique solution if and only if any of the following is true.
  - ▷ $rank\,(\mathbf{A}) = n =$ number of unknowns.
  - ▷ There are no free variables.
  - ▷ The associated homogeneous system possesses only the trivial solution.

Exercises for section 2.5

2.5.1. Determine the general solution for each of the following nonhomogeneous systems.

(a)
$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 3,\\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 4,\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 5. \end{aligned}$$

(b)
$$\begin{aligned} 2x + y + z &= 4,\\ 4x + 2y + z &= 6,\\ 6x + 3y + z &= 8,\\ 8x + 4y + z &= 10. \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 + 2x_3 \phantom{{}+3x_4} &= 1,\\ 3x_1 \phantom{{}+x_2} + 3x_3 + 3x_4 &= 6,\\ 2x_1 + x_2 + 3x_3 + x_4 &= 3,\\ x_1 + 2x_2 + 3x_3 - x_4 &= 0. \end{aligned}$$

(d)
$$\begin{aligned} 2x + y + z &= 2,\\ 4x + 2y + z &= 5,\\ 6x + 3y + z &= 8,\\ 8x + 5y + z &= 8. \end{aligned}$$

2.5.2. Among the solutions that satisfy the set of linear equations

$$\begin{aligned} x_1 + \;\; x_2 + 2x_3 + 2x_4 + \;\; x_5 &= 1, \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1, \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2, \\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3, \end{aligned}$$

find all those that also satisfy the following two constraints:

$$\begin{aligned} (x_1 - x_2)^2 - 4x_5^2 &= 0, \\ x_3^2 - x_5^2 &= 0. \end{aligned}$$

2.5.3. In order to grow a certain crop, it is recommended that each square foot of ground be treated with 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose that there are three brands of fertilizer on the market—say brand $\mathcal{X}$, brand $\mathcal{Y}$, and brand $\mathcal{Z}$. One pound of brand $\mathcal{X}$ contains 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen. One pound of brand $\mathcal{Y}$ contains 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen. One pound of brand $\mathcal{Z}$ contains only 1 unit of phosphorous and 1 unit of nitrogen.

(a) Take into account the obvious fact that a negative number of pounds of any brand can never be applied, and suppose that because of the way fertilizer is sold only an integral number of pounds of each brand will be applied. Under these constraints, determine all possible combinations of the three brands that can be applied to satisfy the recommendations exactly.

(b) Suppose that brand $\mathcal{X}$ costs \$1 per pound, brand $\mathcal{Y}$ costs \$6 per pound, and brand $\mathcal{Z}$ costs \$3 per pound. Determine the least expensive solution that will satisfy the recommendations exactly as well as the constraints of part (a).

2.5.4. Consider the following system:

$$\begin{aligned} 2x + 2y + \;\; 3z &= 0, \\ 4x + 8y + 12z &= -4, \\ 6x + 2y + \;\; \alpha z &= 4. \end{aligned}$$

(a) Determine all values of α for which the system is consistent.

(b) Determine all values of α for which there is a unique solution, and compute the solution for these cases.

(c) Determine all values of α for which there are infinitely many different solutions, and give the general solution for these cases.

2.5.5. If columns $\mathbf{s}_1$ and $\mathbf{s}_2$ are particular solutions of the same nonhomogeneous system, must it be the case that the sum $\mathbf{s}_1 + \mathbf{s}_2$ is also a solution?

2.5.6. Suppose that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix for a consistent system of m equations in n unknowns where $m \geq n$. What must $\mathbf{E_A}$ look like when the system possesses a unique solution?

2.5.7. Construct a nonhomogeneous system of three equations in four unknowns that has

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

as its general solution.

2.5.8. Consider using floating-point arithmetic (without partial pivoting or scaling) to solve the system represented by the following augmented matrix:

$$\left(\begin{array}{ccc|c} .835 & .667 & .5 & .168 \\ .333 & .266 & .1994 & .067 \\ 1.67 & 1.334 & 1.1 & .436 \end{array}\right).$$

(a) Determine the 4-digit general solution.
(b) Determine the 5-digit general solution.
(c) Determine the 6-digit general solution.

2.6 ELECTRICAL CIRCUITS

The theory of electrical circuits is an important application that naturally gives rise to rectangular systems of linear equations. Because the underlying mathematics depends on several of the concepts discussed in the preceding sections, you may find it interesting and worthwhile to make a small excursion into the elementary mathematical analysis of electrical circuits. However, the continuity of the text is not compromised by omitting this section.

In a direct current circuit containing resistances and sources of electromotive force (abbreviated EMF) such as batteries, a point at which three or more conductors are joined is called a ***node*** or ***branch point*** of the circuit, and a closed conduction path is called a ***loop.*** Any part of a circuit between two adjoining nodes is called a ***branch*** of the circuit. The circuit shown in Figure 2.6.1 is a typical example that contains four nodes, seven loops, and six branches.

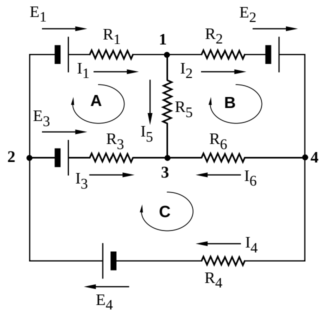


FIGURE 2.6.1

The problem is to relate the currents I_k in each branch to the resistances R_k and the EMFs E_k.[15] This is accomplished by using ***Ohm's law*** in conjunction with ***Kirchhoff's rules*** to produce a system of linear equations.

Ohm's Law

Ohm's law states that for a current of I amps, the voltage drop (in volts) across a resistance of R ohms is given by $V = IR$.

Kirchhoff's rules—formally stated below—are the two fundamental laws that govern the study of electrical circuits.

15 For an EMF source of magnitude E and a current I, there is always a small internal resistance in the source, and the voltage drop across it is $V = E - I \times (\textit{internal resistance})$. But internal source resistance is usually negligible, so the voltage drop across the source can be taken as $V = E$. When internal resistance cannot be ignored, its effects may be incorporated into existing external resistances, or it can be treated as a separate external resistance.

Kirchhoff's Rules

NODE RULE: *The algebraic sum of currents toward each node is zero.* That is, the total incoming current must equal the total outgoing current. This is simply a statement of conservation of charge.

LOOP RULE: *The algebraic sum of the EMFs around each loop must equal the algebraic sum of the IR products in the same loop.* That is, assuming internal source resistances have been accounted for, the algebraic sum of the voltage drops over the sources equals the algebraic sum of the voltage drops over the resistances in each loop. This is a statement of conservation of energy.

Kirchhoff's rules may be used without knowing the directions of the currents and EMFs in advance. You may arbitrarily assign directions. If negative values emerge in the final solution, then the actual direction is opposite to that assumed. To apply the node rule, consider a current to be *positive* if its direction is toward the node—otherwise, consider the current to be *negative.* It should be clear that the node rule will always generate a homogeneous system. For example, applying the node rule to the circuit in Figure 2.6.1 yields four homogeneous equations in six unknowns—the unknowns are the I_k 's:

$$\begin{aligned}
\text{Node 1:} &\quad I_1 - I_2 - I_5 = 0,\\
\text{Node 2:} &\quad -I_1 - I_3 + I_4 = 0,\\
\text{Node 3:} &\quad I_3 + I_5 + I_6 = 0,\\
\text{Node 4:} &\quad I_2 - I_4 - I_6 = 0.
\end{aligned}$$

To apply the loop rule, some direction (clockwise or counterclockwise) must be chosen as the positive direction, and all EMFs and currents in that direction are considered *positive* and those in the opposite direction are *negative.* It is possible for a current to be considered positive for the node rule but considered negative when it is used in the loop rule. If the positive direction is considered to be clockwise in each case, then applying the loop rule to the three indicated loops A, B, and C in the circuit shown in Figure 2.6.1 produces the three nonhomogeneous equations in six unknowns—the I_k 's are treated as the unknowns, while the R_k's and E_k's are assumed to be known.

$$\begin{aligned}
\text{Loop A:} &\quad I_1R_1 - I_3R_3 + I_5R_5 = E_1 - E_3,\\
\text{Loop B:} &\quad I_2R_2 - I_5R_5 + I_6R_6 = E_2,\\
\text{Loop C:} &\quad I_3R_3 + I_4R_4 - I_6R_6 = E_3 + E_4.
\end{aligned}$$

There are 4 additional loops that also produce loop equations thereby making a total of 11 equations (4 nodal equations and 7 loop equations) in 6 unknowns. Although this appears to be a rather general 11×6 system of equations, it really is not. If the circuit is in a state of equilibrium, then the physics of the situation dictates that for each set of EMFs E_k, the corresponding currents I_k must be uniquely determined. In other words, physics guarantees that the 11×6 system produced by applying the two Kirchhoff rules must be *consistent* and possess a *unique* solution.

Suppose that $[\mathbf{A}|\mathbf{b}]$ represents the augmented matrix for the 11×6 system generated by Kirchhoff's rules. From the results in §2.5, we know that the system has a unique solution if and only if

$$rank\,(\mathbf{A}) = \text{ number of unknowns } = 6.$$

Furthermore, it was demonstrated in §2.3 that the system is consistent if and only if

$$rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A}).$$

Combining these two facts allows us to conclude that

$$rank[\mathbf{A}|\mathbf{b}] = 6$$

so that when $[\mathbf{A}|\mathbf{b}]$ is reduced to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$, there will be exactly 6 nonzero rows and 5 zero rows. Therefore, 5 of the original 11 equations are redundant in the sense that they can be "zeroed out" by forming combinations of some particular set of 6 "independent" equations. It is desirable to know beforehand which of the 11 equations will be redundant and which can act as the "independent" set.

Notice that in using the node rule, the equation corresponding to node 4 is simply the negative sum of the equations for nodes 1, 2, and 3, and that the first three equations are independent in the sense that no one of the three can be written as a combination of any other two. This situation is typical. For a general circuit with n nodes, it can be demonstrated that the equations for the first $n-1$ nodes are independent, and the equation for the last node is redundant.

The loop rule also can generate redundant equations. Only ***simple loops***—loops not containing smaller loops—give rise to independent equations. For example, consider the loop consisting of the three exterior branches in the circuit shown in Figure 2.6.1. Applying the loop rule to this large loop will produce no new information because the large loop can be constructed by "adding" the three simple loops A, B, and C contained within. The equation associated with the large outside loop is

$$I_1R_1 + I_2R_2 + I_4R_4 = E_1 + E_2 + E_4,$$

which is precisely the sum of the equations that correspond to the three component loops A, B, and C. This phenomenon will hold in general so that only the simple loops need to be considered when using the loop rule.

The point of this discussion is to conclude that the more general 11×6 rectangular system can be replaced by an equivalent 6×6 square system that has a unique solution by dropping the last nodal equation and using only the simple loop equations. This is characteristic of practical work in general. The physics of a problem together with natural constraints can usually be employed to replace a general rectangular system with one that is square and possesses a unique solution.

One of the goals in our study is to understand more clearly the notion of "independence" that emerged in this application. So far, independence has been an intuitive idea, but this example helps make it clear that independence is a fundamentally important concept that deserves to be nailed down more firmly. This is done in §4.3, and the general theory for obtaining independent equations from electrical circuits is developed in Examples 4.4.6 and 4.4.7.

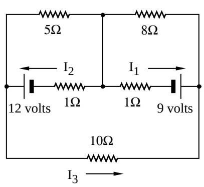
Exercises for section 2.6

2.6.1. Suppose that $R_i = i$ ohms and $E_i = i$ volts in the circuit shown in Figure 2.6.1.

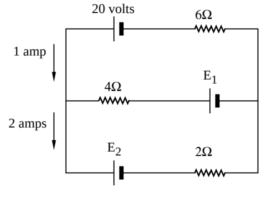
(a) Determine the six indicated currents.

(b) Select node number 1 to use as a reference point and fix its potential to be 0 volts. With respect to this reference, calculate the potentials at the other three nodes. Check your answer by verifying the loop rule for each loop in the circuit.

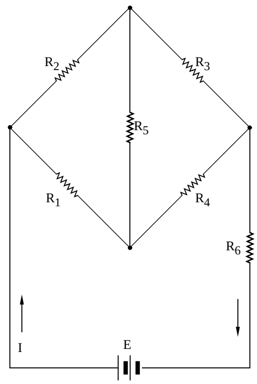
2.6.2. Determine the three currents indicated in the following circuit.



2.6.3. Determine the two unknown EMFs in the following circuit.



2.6.4. Consider the circuit shown below and answer the following questions.



(a) How many nodes does the circuit contain?
(b) How many branches does the circuit contain?
(c) Determine the total number of loops and then determine the number of simple loops.
(d) Demonstrate that the simple loop equations form an "independent" system of equations in the sense that there are no redundant equations.
(e) Verify that any three of the nodal equations constitute an "independent" system of equations.
(f) Verify that the loop equation associated with the loop containing R_1, R_2, R_3, and R_4 can be expressed as the sum of the two equations associated with the two simple loops contained in the larger loop.
(g) Determine the indicated current I if $R_1 = R_2 = R_3 = R_4 = 1$ ohm, $R_5 = R_6 = 5$ ohms, and $E = 5$ volts.

Life is good for only two things, discovering mathematics and teaching mathematics.
— *Siméon D. Poisson* (1781–1840)