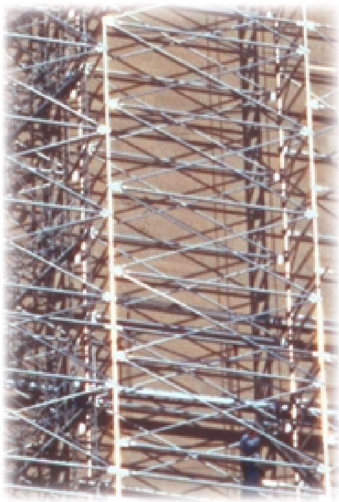

CHAPTER 3

Matrix Algebra

3.1 FROM ANCIENT CHINA TO ARTHUR CAYLEY

The ancient Chinese appreciated the advantages of array manipulation in dealing with systems of linear equations, and they possessed the seed that might have germinated into a genuine theory of matrices. Unfortunately, in the year 213 B.C., emperor Shih Hoang-ti ordered that "all books be burned and all scholars be buried." It is presumed that the emperor wanted all knowledge and written records to begin with him and his regime. The edict was carried out, and it will never be known how much knowledge was lost. The book *Chiu-chang Suan-shu* (*Nine Chapters on Arithmetic*), mentioned in the introduction to Chapter 1, was compiled on the basis of remnants that survived.

More than a millennium passed before further progress was documented. The Chinese counting board with its colored rods and its applications involving array manipulation to solve linear systems eventually found its way to Japan. Seki Kowa (1642–1708), whom many Japanese consider to be one of the greatest mathematicians that their country has produced, carried forward the Chinese principles involving "rule of thumb" elimination methods on arrays of numbers. His understanding of the elementary operations used in the Chinese elimination process led him to formulate the concept of what we now call the determinant. While formulating his ideas concerning the solution of linear systems, Seki Kowa anticipated the fundamental concepts of array operations that today form the basis for matrix algebra. However, there is no evidence that he developed his array operations to actually construct an algebra for matrices.

From the middle 1600s to the middle 1800s, while Europe was flowering in mathematical development, the study of array manipulation was exclusively dedicated to the theory of determinants. Curiously, matrix algebra did not evolve along with the study of determinants.

It was not until the work of the British mathematician Arthur Cayley (1821–1895) that the matrix was singled out as a separate entity, distinct from the notion of a determinant, and algebraic operations between matrices were defined. In an 1855 paper, Cayley first introduced his basic ideas that were presented mainly to simplify notation. Finally, in 1857, Cayley expanded on his original ideas and wrote *A Memoir on the Theory of Matrices.* This laid the foundations for the modern theory and is generally credited for being the birth of the subjects of matrix analysis and linear algebra.

Arthur Cayley began his career by studying literature at Trinity College, Cambridge (1838–1842), but developed a side interest in mathematics, which he studied in his spare time. This "hobby" resulted in his first mathematical paper in 1841 when he was only 20 years old. To make a living, he entered the legal profession and practiced law for 14 years. However, his main interest was still mathematics. During the legal years alone, Cayley published almost 300 papers in mathematics.

In 1850 Cayley crossed paths with James J. Sylvester, and between the two of them matrix theory was born and nurtured. The two have been referred to as the "invariant twins." Although Cayley and Sylvester shared many mathematical interests, they were quite different people, especially in their approach to mathematics. Cayley had an insatiable hunger for the subject, and he read everything that he could lay his hands on. Sylvester, on the other hand, could not stand the sight of papers written by others. Cayley never forgot anything he had read or seen—he became a living encyclopedia. Sylvester, so it is said, would frequently fail to remember even his own theorems.

In 1863, Cayley was given a chair in mathematics at Cambridge University, and thereafter his mathematical output was enormous. Only Cauchy and Euler were as prolific. Cayley often said, "I really love my subject," and all indications substantiate that this was indeed the way he felt. He remained a working mathematician until his death at age 74.

Because the idea of the determinant preceded concepts of matrix algebra by at least two centuries, Morris Kline says in his book *Mathematical Thought from Ancient to Modern Times* that "the subject of matrix theory was well developed before it was created." This must have indeed been the case because immediately after the publication of Cayley's memoir, the subjects of matrix theory and linear algebra virtually exploded and quickly evolved into a discipline that now occupies a central position in applied mathematics.

3.2 ADDITION AND TRANSPOSITION

In the previous chapters, matrix language and notation were used simply to formulate some of the elementary concepts surrounding linear systems. The purpose now is to turn this language into a mathematical theory.[16]

Unless otherwise stated, a ***scalar*** is a complex number. Real numbers are a subset of the complex numbers, and hence real numbers are also scalar quantities. In the early stages, there is little harm in thinking only in terms of real scalars. Later on, however, the necessity for dealing with complex numbers will be unavoidable. Throughout the text, $\Re$ will denote the set of real numbers, and $\mathcal{C}$ will denote the complex numbers. The set of all n-tuples of real numbers will be denoted by $\Re^n$, and the set of all complex n-tuples will be denoted by $\mathcal{C}^n$. For example, $\Re^2$ is the set of all ordered pairs of real numbers (i.e., the standard cartesian plane), and $\Re^3$ is ordinary 3-space. Analogously, $\Re^{m\times n}$ and $\mathcal{C}^{m\times n}$ denote the $m \times n$ matrices containing real numbers and complex numbers, respectively.

Matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are defined to be ***equal matrices*** when $\mathbf{A}$ and $\mathbf{B}$ have the same shape and corresponding entries are equal. That is, $a_{ij} = b_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In particular, this definition applies to arrays such as $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Even though $\mathbf{u}$ and $\mathbf{v}$ describe exactly the same point in 3-space, we cannot consider them to be equal matrices because they have different shapes. An array (or matrix) consisting of a single column, such as $\mathbf{u}$, is called a ***column vector***, while an array consisting of a single row, such as $\mathbf{v}$, is called a ***row vector***.

Addition of Matrices

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, the ***sum*** of $\mathbf{A}$ and $\mathbf{B}$ is defined to be the $m \times n$ matrix $\mathbf{A}+\mathbf{B}$ obtained by adding corresponding entries. That is,

$$[\mathbf{A}+\mathbf{B}]_{ij} = [\mathbf{A}]_{ij} + [\mathbf{B}]_{ij} \quad \text{for each } i \text{ and } j.$$

For example,

$$\begin{pmatrix} -2 & x & 3 \\ z+3 & 4 & -y \end{pmatrix} + \begin{pmatrix} 2 & 1-x & -2 \\ -3 & 4+x & 4+y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ z & 8+x & 4 \end{pmatrix}.$$

[16] The great French mathematician Pierre-Simon Laplace (1749–1827) said that, “Such is the advantage of a well-constructed language that its simplified notation often becomes the source of profound theories.” The theory of matrices is a testament to the validity of Laplace’s statement.

The symbol "+" is used two different ways—it denotes addition between scalars in some places and addition between matrices at other places. Although these are two distinct algebraic operations, no ambiguities will arise if the context in which "+" appears is observed. Also note that the requirement that $\mathbf{A}$ and $\mathbf{B}$ have the same shape prevents adding a row to a column, even though the two may contain the same number of entries.

The matrix $(-\mathbf{A})$, called the ***additive inverse*** of $\mathbf{A}$, is defined to be the matrix obtained by negating each entry of $\mathbf{A}$. That is, if $\mathbf{A} = [a_{ij}]$, then $-\mathbf{A} = [-a_{ij}]$. This allows matrix subtraction to be defined in the natural way. For two matrices of the same shape, the ***difference*** $\mathbf{A} - \mathbf{B}$ is defined to be the matrix $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ so that

$$[\mathbf{A} - \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} - [\mathbf{B}]_{ij} \quad \text{for each } i \text{ and } j.$$

Since matrix addition is defined in terms of scalar addition, the familiar algebraic properties of scalar addition are inherited by matrix addition as detailed below.

Properties of Matrix Addition

For $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, the following properties hold.

Closure property: $\mathbf{A} + \mathbf{B}$ is again an $m \times n$ matrix.

Associative property: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Commutative property: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Additive identity: The $m \times n$ matrix $\mathbf{0}$ consisting of all zeros has the property that $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

Additive inverse: The $m \times n$ matrix $(-\mathbf{A})$ has the property that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

Another simple operation that is derived from scalar arithmetic is as follows.

Scalar Multiplication

The product of a scalar α times a matrix $\mathbf{A}$, denoted by $\alpha\mathbf{A}$, is defined to be the matrix obtained by multiplying each entry of $\mathbf{A}$ by α. That is, $[\alpha\mathbf{A}]_{ij} = \alpha[\mathbf{A}]_{ij}$ for each i and j.

For example,

$$2\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \\ 2 & 8 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 2 \end{pmatrix}.$$

The rules for combining addition and scalar multiplication are what you might suspect they should be. Some of the important ones are listed below.

Properties of Scalar Multiplication

For $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ and for scalars α and β, the following properties hold.

Closure property:	$\alpha\mathbf{A}$ is again an $m \times n$ matrix.
Associative property:	$(\alpha\beta)\mathbf{A} = \alpha(\beta\mathbf{A})$.
Distributive property:	$\alpha(\mathbf{A}+\mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$. Scalar multiplication is distributed over matrix addition.
Distributive property:	$(\alpha+\beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$. Scalar multiplication is distributed over scalar addition.
Identity property:	$1\mathbf{A} = \mathbf{A}$. The number 1 is an identity element under scalar multiplication.

Other properties such as $\alpha\mathbf{A} = \mathbf{A}\alpha$ could have been listed, but the properties singled out pave the way for the definition of a vector space on p. 160.

A matrix operation that's not derived from scalar arithmetic is *transposition* as defined below.

Transpose

The ***transpose*** of $\mathbf{A}_{m\times n}$ is defined to be the $n \times m$ matrix $\mathbf{A}^T$ obtained by interchanging rows and columns in $\mathbf{A}$. More precisely, if $\mathbf{A} = [a_{ij}]$, then $[\mathbf{A}^T]_{ij} = a_{ji}$. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

It should be evident that for all matrices, $\left(\mathbf{A}^T\right)^T = \mathbf{A}$.

Whenever a matrix contains complex entries, the operation of complex conjugation almost always accompanies the transpose operation. (Recall that the complex conjugate of $z = a + \mathrm{i}b$ is defined to be $\overline{z} = a - \mathrm{i}b$.)

Conjugate Transpose

For $\mathbf{A} = [a_{ij}]$, the ***conjugate matrix*** is defined to be $\overline{\mathbf{A}} = [\overline{a}_{ij}]$, and the ***conjugate transpose*** of $\mathbf{A}$ is defined to be $\bar{\mathbf{A}}^T = \overline{\mathbf{A}^T}$. From now on, $\bar{\mathbf{A}}^T$ will be denoted by $\mathbf{A}^*$, so $[\mathbf{A}^*]_{ij} = \overline{a}_{ji}$. For example,

$$\begin{pmatrix} 1-4\mathrm{i} & \mathrm{i} & 2 \\ 3 & 2+\mathrm{i} & 0 \end{pmatrix}^* = \begin{pmatrix} 1+4\mathrm{i} & 3 \\ -\mathrm{i} & 2-\mathrm{i} \\ 2 & 0 \end{pmatrix}.$$

$(\mathbf{A}^*)^* = \mathbf{A}$ for all matrices, and $\mathbf{A}^* = \mathbf{A}^T$ whenever $\mathbf{A}$ contains only real entries. Sometimes the matrix $\mathbf{A}^*$ is called the ***adjoint*** of $\mathbf{A}$.

The transpose (and conjugate transpose) operation is easily combined with matrix addition and scalar multiplication. The basic rules are given below.

Properties of the Transpose

If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same shape, and if α is a scalar, then each of the following statements is true.

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \quad \text{and} \quad (\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*. \tag{3.2.1}$$

$$(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T \quad \text{and} \quad (\alpha\mathbf{A})^* = \overline{\alpha}\mathbf{A}^*. \tag{3.2.2}$$

Proof.[17] We will prove that (3.2.1) and (3.2.2) hold for the transpose operation. The proofs of the statements involving conjugate transposes are similar and are left as exercises. For each i and j, it is true that

$$[(\mathbf{A}+\mathbf{B})^T]_{ij} = [\mathbf{A}+\mathbf{B}]_{ji} = [\mathbf{A}]_{ji} + [\mathbf{B}]_{ji} = [\mathbf{A}^T]_{ij} + [\mathbf{B}^T]_{ij} = [\mathbf{A}^T + \mathbf{B}^T]_{ij}.$$

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[17] Computers can outperform people in many respects in that they do arithmetic much faster and more accurately than we can, and they are now rather adept at symbolic computation and mechanical manipulation of formulas. But computers can't do mathematics—people still hold the monopoly. Mathematics emanates from the uniquely human capacity to reason abstractly in a creative and logical manner, and learning mathematics goes hand-in-hand with learning how to reason abstractly and create logical arguments. This is true regardless of whether your orientation is applied or theoretical. For this reason, formal proofs will appear more frequently as the text evolves, and it is expected that your level of comprehension as well as your ability to create proofs will grow as you proceed.

This proves that corresponding entries in $(\mathbf{A}+\mathbf{B})^T$ and $\mathbf{A}^T+\mathbf{B}^T$ are equal, so it must be the case that $(\mathbf{A}+\mathbf{B})^T=\mathbf{A}^T+\mathbf{B}^T$. Similarly, for each i and j,

$$[(\alpha\mathbf{A})^T]_{ij}=[\alpha\mathbf{A}]_{ji}=\alpha[\mathbf{A}]_{ji}=\alpha[\mathbf{A}^T]_{ij} \implies (\alpha\mathbf{A})^T=\alpha\mathbf{A}^T. \quad \blacksquare$$

Sometimes transposition doesn't change anything. For example, if

$$\mathbf{A}=\begin{pmatrix}1&2&3\\2&4&5\\3&5&6\end{pmatrix}, \quad \text{then} \quad \mathbf{A}^T=\mathbf{A}.$$

This is because the entries in $\mathbf{A}$ are symmetrically located about the ***main diagonal***—the line from the upper-left-hand corner to the lower-right-hand corner.

Matrices of the form $\mathbf{D}=\begin{pmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\lambda_n\end{pmatrix}$ are called ***diagonal matrices***, and they are clearly symmetric in the sense that $\mathbf{D}=\mathbf{D}^T$. This is one of several kinds of symmetries described below.

Symmetries

Let $\mathbf{A}=[a_{ij}]$ be a square matrix.

- $\mathbf{A}$ is said to be a ***symmetric matrix*** whenever $\mathbf{A}=\mathbf{A}^T$, i.e., whenever $a_{ij}=a_{ji}$.
- $\mathbf{A}$ is said to be a ***skew-symmetric matrix*** whenever $\mathbf{A}=-\mathbf{A}^T$, i.e., whenever $a_{ij}=-a_{ji}$.
- $\mathbf{A}$ is said to be a ***hermitian matrix*** whenever $\mathbf{A}=\mathbf{A}^*$, i.e., whenever $a_{ij}=\overline{a}_{ji}$. This is the complex analog of symmetry.
- $\mathbf{A}$ is said to be a ***skew-hermitian matrix*** when $\mathbf{A}=-\mathbf{A}^*$, i.e., whenever $a_{ij}=-\overline{a}_{ji}$. This is the complex analog of skew symmetry.

For example, consider

$$\mathbf{A}=\begin{pmatrix}1&2+4\mathrm{i}&1-3\mathrm{i}\\2-4\mathrm{i}&3&8+6\mathrm{i}\\1+3\mathrm{i}&8-6\mathrm{i}&5\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}1&2+4\mathrm{i}&1-3\mathrm{i}\\2+4\mathrm{i}&3&8+6\mathrm{i}\\1-3\mathrm{i}&8+6\mathrm{i}&5\end{pmatrix}.$$

Can you see that $\mathbf{A}$ is hermitian but not symmetric, while $\mathbf{B}$ is symmetric but not hermitian?

Nature abounds with symmetry, and very often physical symmetry manifests itself as a symmetric matrix in a mathematical model. The following example is an illustration of this principle.

Example 3.2.1

Consider two springs that are connected as shown in Figure 3.2.1.

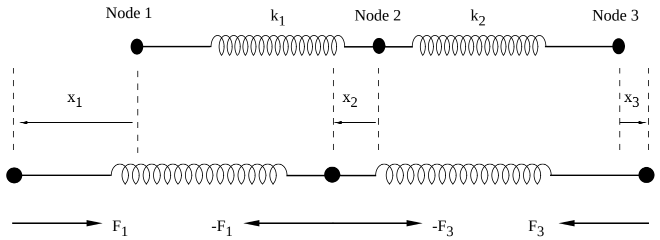


FIGURE 3.2.1

The springs at the top represent the "no tension" position in which no force is being exerted on any of the nodes. Suppose that the springs are stretched or compressed so that the nodes are displaced as indicated in the lower portion of Figure 3.2.1. Stretching or compressing the springs creates a force on each node according to Hooke's law[18] that says that the force exerted by a spring is $F = kx$, where x is the distance the spring is stretched or compressed and where k is a *stiffness constant* inherent to the spring. Suppose our springs have stiffness constants k_1 and k_2, and let F_i be the force on node i when the springs are stretched or compressed. Let's agree that a displacement to the left is positive, while a displacement to the right is negative, and consider a force directed to the right to be positive while one directed to the left is negative. If node 1 is displaced x_1 units, and if node 2 is displaced x_2 units, then the left-hand spring is stretched (or compressed) by a total amount of $x_1 - x_2$ units, so the force on node 1 is

$$F_1 = k_1(x_1 - x_2).$$

Similarly, if node 2 is displaced x_2 units, and if node 3 is displaced x_3 units, then the right-hand spring is stretched by a total amount of $x_2 - x_3$ units, so the force on node 3 is

$$F_3 = -k_2(x_2 - x_3).$$

The minus sign indicates the force is directed to the left. The force on the left-hand side of node 2 is the opposite of the force on node 1, while the force on the right-hand side of node 2 must be the opposite of the force on node 3. That is,

$$F_2 = -F_1 - F_3.$$

[18] Hooke's law is named for Robert Hooke (1635–1703), an English physicist, but it was generally known to several people (including Newton) before Hooke's 1678 claim to it was made. Hooke was a creative person who is credited with several inventions, including the wheel barometer, but he was reputed to be a man of "terrible character." This characteristic virtually destroyed his scientific career as well as his personal life. It is said that he lacked mathematical sophistication and that he left much of his work in incomplete form, but he bitterly resented people who built on his ideas by expressing them in terms of elegant mathematical formulations.

Organize the above three equations as a linear system:

$$\begin{aligned} k_1x_1 \qquad - k_1x_2 \qquad\qquad &= F_1, \\ -k_1x_1 + (k_1+k_2)x_2 - k_2x_3 &= F_2, \\ -k_2x_2 + k_2x_3 &= F_3, \end{aligned}$$

and observe that the coefficient matrix, called the ***stiffness matrix,***

$$\mathbf{K} = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix},$$

is a *symmetric* matrix. The point of this example is that symmetry in the physical problem translates to symmetry in the mathematics by way of the symmetric matrix $\mathbf{K}$. When the two springs are identical (i.e., when $k_1 = k_2 = k$), even more symmetry is present, and in this case

$$\mathbf{K} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Exercises for section 3.2

3.2.1. Determine the unknown quantities in the following expressions.

(a) $3\mathbf{X} = \begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$. (b) $2\begin{pmatrix} x+2 & y+3 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ y & z \end{pmatrix}^T$.

3.2.2. Identify each of the following as symmetric, skew symmetric, or neither.

(a) $\begin{pmatrix} 1 & -3 & 3 \\ -3 & 4 & -3 \\ 3 & 3 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 0 & -3 & -3 \\ 3 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}$.

(c) $\begin{pmatrix} 0 & -3 & -3 \\ -3 & 0 & 3 \\ -3 & 3 & 1 \end{pmatrix}$. (d) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$.

3.2.3. Construct an example of a 3×3 matrix $\mathbf{A}$ that satisfies the following conditions.

(a) $\mathbf{A}$ is both symmetric and skew symmetric.
(b) $\mathbf{A}$ is both hermitian and symmetric.
(c) $\mathbf{A}$ is skew hermitian.

3.2.4. Explain why the set of all $n \times n$ symmetric matrices is closed under matrix addition. That is, explain why the sum of two $n \times n$ symmetric matrices is again an $n \times n$ symmetric matrix. Is the set of all $n \times n$ skew-symmetric matrices closed under matrix addition?

3.2.5. Prove that each of the following statements is true.

(a) If $\mathbf{A} = [a_{ij}]$ is skew symmetric, then $a_{jj} = 0$ for each j.

(b) If $\mathbf{A} = [a_{ij}]$ is skew hermitian, then each a_{jj} is a pure imaginary number—i.e., a multiple of the imaginary unit i.

(c) If $\mathbf{A}$ is real and symmetric, then $\mathbf{B} = \mathrm{i}\mathbf{A}$ is skew hermitian.

3.2.6. Let $\mathbf{A}$ be any square matrix.

(a) Show that $\mathbf{A}+\mathbf{A}^T$ is symmetric and $\mathbf{A}-\mathbf{A}^T$ is skew symmetric.

(b) Prove that there is one and only one way to write $\mathbf{A}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

3.2.7. If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same shape, prove that each of the following statements is true.

(a) $(\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$.

(b) $(\alpha\mathbf{A})^* = \overline{\alpha}\mathbf{A}^*$.

3.2.8. Using the conventions given in Example 3.2.1, determine the stiffness matrix for a system of n identical springs, with stiffness constant k, connected in a line similar to that shown in Figure 3.2.1.

3.3 LINEARITY

The concept of linearity is the underlying theme of our subject. In elementary mathematics the term "linear function" refers to straight lines, but in higher mathematics linearity means something much more general. Recall that a function f is simply a rule for associating points in one set $\mathcal{D}$—called the ***domain*** of f—to points in another set $\mathcal{R}$—the ***range*** of f. A *linear* function is a particular type of function that is characterized by the following two properties.

Linear Functions

Suppose that $\mathcal{D}$ and $\mathcal{R}$ are sets that possess an addition operation as well as a scalar multiplication operation—i.e., a multiplication between scalars and set members. A function f that maps points in $\mathcal{D}$ to points in $\mathcal{R}$ is said to be a ***linear function*** whenever f satisfies the conditions that

$$f(x+y) = f(x) + f(y) \tag{3.3.1}$$

and

$$f(\alpha x) = \alpha f(x) \tag{3.3.2}$$

for every x and y in $\mathcal{D}$ and for all scalars α. These two conditions may be combined by saying that f is a linear function whenever

$$f(\alpha x + y) = \alpha f(x) + f(y) \tag{3.3.3}$$

for all scalars α and for all $x, y \in \mathcal{D}$.

One of the simplest linear functions is $f(x) = \alpha x$, whose graph in $\Re^2$ is a straight line through the origin. You should convince yourself that f is indeed a linear function according to the above definition. However, $f(x) = \alpha x + \beta$ does not qualify for the title "linear function"—it is a linear function that has been translated by a constant β. Translations of linear functions are referred to as ***affine functions.*** Virtually all information concerning affine functions can be derived from an understanding of linear functions, and consequently we will focus only on issues of linearity.

In $\Re^3$, the surface described by a function of the form

$$f(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$$

is a plane through the origin, and it is easy to verify that f is a linear function. For $\beta \neq 0$, the graph of $f(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2 + \beta$ is a plane *not* passing through the origin, and f is no longer a linear function—it is an affine function.

In $\Re^2$ and $\Re^3$, the graphs of linear functions are lines and planes through the origin, and there seems to be a pattern forming. Although we cannot visualize higher dimensions with our eyes, it seems reasonable to suggest that a general linear function of the form

$$f(x_1, x_2, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

somehow represents a "linear" or "flat" surface passing through the origin $\mathbf{0} = (0, 0, \ldots, 0)$ in $\Re^{n+1}$. One of the goals of the next chapter is to learn how to better interpret and understand this statement.

Linearity is encountered at every turn. For example, the familiar operations of differentiation and integration may be viewed as linear functions. Since

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} \quad \text{and} \quad \frac{d(\alpha f)}{dx} = \alpha \frac{df}{dx},$$

the differentiation operator $D_x(f) = df/dx$ is linear. Similarly,

$$\int (f+g)dx = \int f dx + \int g dx \quad \text{and} \quad \int \alpha f dx = \alpha \int f dx$$

means that the integration operator $I(f) = \int f dx$ is linear.

There are several important matrix functions that are linear. For example, the transposition function $f(\mathbf{X}_{m\times n}) = \mathbf{X}^T$ is linear because

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \quad \text{and} \quad (\alpha\mathbf{A})^T = \alpha\mathbf{A}^T$$

(recall (3.2.1) and (3.2.2)). Another matrix function that is linear is the *trace* function presented below.

Example 3.3.1

The ***trace*** of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is defined to be the sum of the entries lying on the main diagonal of $\mathbf{A}$. That is,

$$\mathit{trace}\,(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

Problem: Show that $f(\mathbf{X}_{n\times n}) = \mathit{trace}\,(\mathbf{X})$ is a linear function.

Solution: Let's be efficient by showing that (3.3.3) holds. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, and write

$$\begin{aligned} f(\alpha\mathbf{A}+\mathbf{B}) &= \mathit{trace}\,(\alpha\mathbf{A}+\mathbf{B}) = \sum_{i=1}^{n}[\alpha\mathbf{A}+\mathbf{B}]_{ii} = \sum_{i=1}^{n}(\alpha a_{ii} + b_{ii}) \\ &= \sum_{i=1}^{n}\alpha a_{ii} + \sum_{i=1}^{n} b_{ii} = \alpha\sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \alpha\,\mathit{trace}\,(\mathbf{A}) + \mathit{trace}\,(\mathbf{B}) \\ &= \alpha f(\mathbf{A}) + f(\mathbf{B}). \end{aligned}$$

Example 3.3.2

Consider a linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & u_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & u_2, \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & u_m, \end{array}$$

to be a function $\mathbf{u} = f(\mathbf{x})$ that maps $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \Re^n$ to $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \in \Re^m$.

Problem: Show that $\mathbf{u} = f(\mathbf{x})$ is linear.

Solution: Let $\mathbf{A} = [a_{ij}]$ be the matrix of coefficients, and write

$$f(\alpha\mathbf{x} + \mathbf{y}) = f\begin{pmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \vdots \\ \alpha x_n + y_n \end{pmatrix} = \sum_{j=1}^{n} (\alpha x_j + y_j)\mathbf{A}_{*j} = \sum_{j=1}^{n} (\alpha x_j \mathbf{A}_{*j} + y_j \mathbf{A}_{*j})$$

$$= \sum_{j=1}^{n} \alpha x_j \mathbf{A}_{*j} + \sum_{j=1}^{n} y_j \mathbf{A}_{*j} = \alpha \sum_{j=1}^{n} x_j \mathbf{A}_{*j} + \sum_{j=1}^{n} y_j \mathbf{A}_{*j}$$

$$= \alpha f(\mathbf{x}) + f(\mathbf{y}).$$

According to (3.3.3), the function f is linear.

The following terminology will be used from now on.

Linear Combinations

For scalars α_j and matrices $\mathbf{X}_j$, the expression

$$\alpha_1\mathbf{X}_1 + \alpha_2\mathbf{X}_2 + \cdots + \alpha_n\mathbf{X}_n = \sum_{j=1}^{n} \alpha_j \mathbf{X}_j$$

is called a ***linear combination*** of the $\mathbf{X}_j$'s.

Exercises for section 3.3

3.3.1. Each of the following is a function from $\Re^2$ into $\Re^2$. Determine which are linear functions.

(a) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 1+y \end{pmatrix}$. (b) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$.

(c) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ xy \end{pmatrix}$. (d) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$.

(e) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \sin y \end{pmatrix}$. (f) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$.

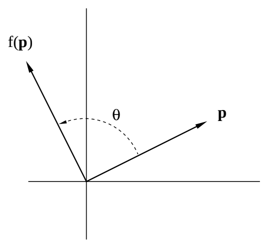
3.3.2. For $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and for constants ξ_i, verify that

$$f(\mathbf{x}) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$$

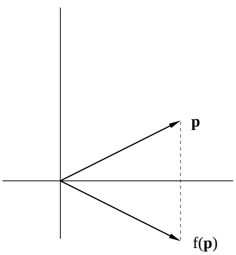
is a linear function.

3.3.3. Give examples of at least two different physical principles or laws that can be characterized as being linear phenomena.

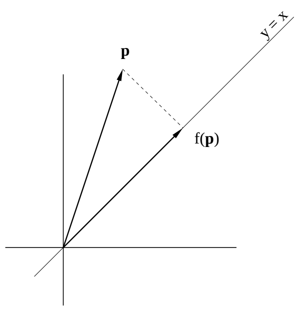
3.3.4. Determine which of the following three transformations in $\Re^2$ are linear.



Rotate counterclockwise through an angle θ.



Reflect about the x-axis.



Project onto the line $y = x$.

3.4 WHY DO IT THIS WAY

If you were given the task of formulating a definition for composing two matrices $\mathbf{A}$ and $\mathbf{B}$ in some sort of "natural" multiplicative fashion, your first attempt would probably be to compose $\mathbf{A}$ and $\mathbf{B}$ by multiplying corresponding entries—much the same way matrix addition is defined. Asked then to defend the usefulness of such a definition, you might be hard pressed to provide a truly satisfying response. Unless a person is in the right frame of mind, the issue of deciding how to best define matrix multiplication is not at all transparent, especially if it is insisted that the definition be both "natural" and "useful." The world had to wait for Arthur Cayley to come to this proper frame of mind.

As mentioned in §3.1, matrix algebra appeared late in the game. Manipulation on arrays and the theory of determinants existed long before Cayley and his theory of matrices. Perhaps this can be attributed to the fact that the "correct" way to multiply two matrices eluded discovery for such a long time.

Around 1855, Cayley became interested in composing linear functions.[19] In particular, he was investigating linear functions of the type discussed in Example 3.3.2. Typical examples of two such functions are

$$f(\mathbf{x}) = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \quad \text{and} \quad g(\mathbf{x}) = g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}.$$

Consider, as Cayley did, composing f and g to create another linear function

$$h(\mathbf{x}) = f\big(g(\mathbf{x})\big) = f\begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix} = \begin{pmatrix} (aA + bC)x_1 + (aB + bD)x_2 \\ (cA + dC)x_1 + (cB + dD)x_2 \end{pmatrix}.$$

It was Cayley's idea to use matrices of coefficients to represent these linear functions. That is, f, g, and h are represented by

$$\mathbf{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix}.$$

After making this association, it was only natural for Cayley to call $\mathbf{H}$ the *composition* (or *product*) of $\mathbf{F}$ and $\mathbf{G}$, and to write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix}. \tag{3.4.1}$$

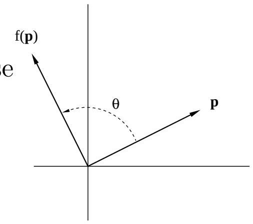
In other words, the product of two matrices represents the composition of the two associated linear functions. By means of this observation, Cayley brought to life the subjects of matrix analysis and linear algebra.

[19] Cayley was not the first to compose linear functions. In fact, Gauss used these compositions as early as 1801, but not in the form of an array of coefficients. Cayley was the first to make the connection between composition of linear functions and the composition of the associated matrices. Cayley's work from 1855 to 1857 is regarded as being the birth of our subject.

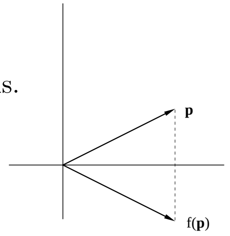
Exercises for section 3.4

Each problem in this section concerns the following three linear transformations in $\Re^2$.

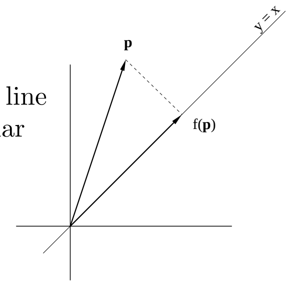
Rotation: Rotate points counterclockwise through an angle θ.



Reflection: Reflect points about the x-axis.



Projection: Project points onto the line $y = x$ in a perpendicular manner.



3.4.1. Determine the matrix associated with each of these linear functions. That is, determine the a_{ij}'s such that

$$f(\mathbf{p}) = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

3.4.2. By using matrix multiplication, determine the linear function obtained by performing a rotation followed by a reflection.

3.4.3. By using matrix multiplication, determine the linear function obtained by first performing a reflection, then a rotation, and finally a projection.

3.5 MATRIX MULTIPLICATION

The purpose of this section is to further develop the concept of matrix multiplication as intorduced in the previous section. In order to do this, it is helpful to begin by composing a single row with a single column. If

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

the ***standard inner product*** of $\mathbf{R}$ with $\mathbf{C}$ is defined to be the scalar

$$\mathbf{RC} = r_1c_1 + r_2c_2 + \cdots + r_nc_n = \sum_{i=1}^{n} r_ic_i.$$

For example,

$$\begin{pmatrix} 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2)(1) + (4)(2) + (-2)(3) = 4.$$

Recall from (3.4.1) that the product of two 2×2 matrices

$$\mathbf{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

was defined naturally by writing

$$\mathbf{FG} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix} = \mathbf{H}.$$

Notice that the (i, j)-entry in the product $\mathbf{H}$ can be described as the inner product of the i^{th} row of $\mathbf{F}$ with the j^{th} column in $\mathbf{G}$. That is,

$$h_{11} = \mathbf{F}_{1*}\mathbf{G}_{*1} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}, \qquad h_{12} = \mathbf{F}_{1*}\mathbf{G}_{*2} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix},$$

$$h_{21} = \mathbf{F}_{2*}\mathbf{G}_{*1} = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}, \qquad h_{22} = \mathbf{F}_{2*}\mathbf{G}_{*2} = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}.$$

This is exactly the way that the general definition of matrix multiplication is formulated.

Matrix Multiplication

- Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be ***conformable*** for multiplication in the order $\mathbf{AB}$ whenever $\mathbf{A}$ has exactly as many columns as $\mathbf{B}$ has rows—i.e., $\mathbf{A}$ is $m \times p$ and $\mathbf{B}$ is $p \times n$.

- For conformable matrices $\mathbf{A}_{m\times p} = [a_{ij}]$ and $\mathbf{B}_{p\times n} = [b_{ij}]$, the ***matrix product*** $\mathbf{AB}$ is defined to be the $m \times n$ matrix whose (i,j)-entry is the inner product of the i^{th} row of $\mathbf{A}$ with the j^{th} column in $\mathbf{B}$. That is,

$$[\mathbf{AB}]_{ij} = \mathbf{A}_{i*}\mathbf{B}_{*j} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

- In case $\mathbf{A}$ and $\mathbf{B}$ fail to be conformable—i.e., $\mathbf{A}$ is $m \times p$ and $\mathbf{B}$ is $q \times n$ with $p \neq q$—then no product $\mathbf{AB}$ is defined.

For example, if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix}_{3\times 4}$$

inside ones match

shape of the product

then the product $\mathbf{AB}$ exists and has shape 2×4. Consider a typical entry of this product, say, the (2,3)-entry. The definition says $[\mathbf{AB}]_{23}$ is obtained by forming the inner product of the second row of $\mathbf{A}$ with the third column of $\mathbf{B}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix},$$

so

$$[\mathbf{AB}]_{23} = \mathbf{A}_{2*}\mathbf{B}_{*3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = \sum_{k=1}^{3} a_{2k}b_{k3}.$$

For example,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ -3 & 0 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 3 & -3 & 2 \\ 2 & 5 & -1 & 8 \\ -1 & 2 & 0 & 2 \end{pmatrix} \Longrightarrow \mathbf{AB} = \begin{pmatrix} 8 & 3 & -7 & 4 \\ -8 & 1 & 9 & 4 \end{pmatrix}.$$

Notice that in spite of the fact that the product $\mathbf{AB}$ exists, the product $\mathbf{BA}$ is not defined—matrix $\mathbf{B}$ is 3×4 and $\mathbf{A}$ is 2×3, and the inside dimensions don't match in this order. Even when the products $\mathbf{AB}$ and $\mathbf{BA}$ each exist and have the same shape, they need not be equal. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Longrightarrow \mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{BA} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}. \tag{3.5.1}$$

This disturbing feature is a primary difference between scalar and matrix algebra.

Matrix Multiplication Is Not Commutative

Matrix multiplication is a noncommutative operation—i.e., it is possible for $\mathbf{AB} \neq \mathbf{BA}$, even when both products exist and have the same shape.

There are other major differences between multiplication of matrices and multiplication of scalars. For scalars,

$$\alpha\beta = 0 \quad \text{implies} \quad \alpha = 0 \quad \text{or} \quad \beta = 0. \tag{3.5.2}$$

However, the analogous statement for matrices does not hold—the matrices given in (3.5.1) show that it is possible for $\mathbf{AB} = \mathbf{0}$ with $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$. Related to this issue is a rule sometimes known as the ***cancellation law.*** For scalars, this law says that

$$\alpha\beta = \alpha\gamma \quad \text{and} \quad \alpha \neq 0 \quad \text{implies} \quad \beta = \gamma. \tag{3.5.3}$$

This is true because we invoke (3.5.2) to deduce that $\alpha(\beta - \gamma) = 0$ implies $\beta - \gamma = 0$. Since (3.5.2) does not hold for matrices, we cannot expect (3.5.3) to hold for matrices.

Example 3.5.1

The cancellation law (3.5.3) fails for matrix multiplication. If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

then

$$\mathbf{AB} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \mathbf{AC} \quad \text{but} \quad \mathbf{B} \neq \mathbf{C}$$

in spite of the fact that $\mathbf{A} \neq \mathbf{0}$.

There are various ways to express the individual rows and columns of a matrix product. For example, the i^{th} row of $\mathbf{AB}$ is

$$\begin{aligned}[\mathbf{AB}]_{i*} &= \big[\mathbf{A}_{i*}\mathbf{B}_{*1} \,|\, \mathbf{A}_{i*}\mathbf{B}_{*2} \,|\, \cdots \,|\, \mathbf{A}_{i*}\mathbf{B}_{*n}\big] = \mathbf{A}_{i*}\mathbf{B} \\ &= \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{1*} \\ \mathbf{B}_{2*} \\ \vdots \\ \mathbf{B}_{p*} \end{pmatrix} = a_{i1}\mathbf{B}_{1*} + a_{i2}\mathbf{B}_{2*} + \cdots + a_{ip}\mathbf{B}_{p*}.\end{aligned}$$

As shown below, there are similar representations for the individual columns.

Rows and Columns of a Product

Suppose that $\mathbf{A} = [a_{ij}]$ is $m \times p$ and $\mathbf{B} = [b_{ij}]$ is $p \times n$.

- $[\mathbf{AB}]_{i*} = \mathbf{A}_{i*}\mathbf{B}$ $\big[(\, i^{th}$ row of $\mathbf{AB}\,)=(\, i^{th}$ row of $\mathbf{A}\,) \times \mathbf{B}\big]$. (3.5.4)
- $[\mathbf{AB}]_{*j} = \mathbf{AB}_{*j}$ $\big[(\, j^{th}$ col of $\mathbf{AB}\,)= \mathbf{A} \times (\, j^{th}$ col of $\mathbf{B}\,)\big]$. (3.5.5)
- $[\mathbf{AB}]_{i*} = a_{i1}\mathbf{B}_{1*} + a_{i2}\mathbf{B}_{2*} + \cdots + a_{ip}\mathbf{B}_{p*} = \sum_{k=1}^{p} a_{ik}\mathbf{B}_{k*}$. (3.5.6)
- $[\mathbf{AB}]_{*j} = \mathbf{A}_{*1}b_{1j} + \mathbf{A}_{*2}b_{2j} + \cdots + \mathbf{A}_{*p}b_{pj} = \sum_{k=1}^{p} \mathbf{A}_{*k}b_{kj}$. (3.5.7)

These last two equations show that rows of $\mathbf{AB}$ are combinations of rows of $\mathbf{B}$, while columns of $\mathbf{AB}$ are combinations of columns of $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -5 & 1 \\ 2 & -7 & 2 \\ 1 & -2 & 0 \end{pmatrix}$, then the second row of $\mathbf{AB}$ is

$$[\mathbf{AB}]_{2*} = \mathbf{A}_{2*}\mathbf{B} = \begin{pmatrix} 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} 3 & -5 & 1 \\ 2 & -7 & 2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 3 & -5 \end{pmatrix},$$

and the second column of $\mathbf{AB}$ is

$$[\mathbf{AB}]_{*2} = \mathbf{AB}_{*2} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ -7 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$

This example makes the point that it is wasted effort to compute the entire product if only one row or column is called for. Although it's not necessary to compute the complete product, you may wish to verify that

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} 3 & -5 & 1 \\ 2 & -7 & 2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 9 & -3 \\ 6 & 3 & -5 \end{pmatrix}.$$

Matrix multiplication provides a convenient representation for a linear system of equations. For example, the 3×4 system

$$
\begin{aligned}
2x_1 + 3x_2 + 4x_3 + 8x_4 &= 7,\\
3x_1 + 5x_2 + 6x_3 + 2x_4 &= 6,\\
4x_1 + 2x_2 + 4x_3 + 9x_4 &= 4,
\end{aligned}
$$

can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$
\mathbf{A}_{3\times 4} = \begin{pmatrix} 2 & 3 & 4 & 8\\ 3 & 5 & 6 & 2\\ 4 & 2 & 4 & 9 \end{pmatrix}, \quad \mathbf{x}_{4\times 1} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_{3\times 1} = \begin{pmatrix} 7\\ 6\\ 4 \end{pmatrix}.
$$

And this example generalizes to become the following statement.

Linear Systems

Every linear system of m equations in n unknowns

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1,\\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2,\\
 & & & & \vdots & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m,
\end{array}
$$

can be written as a single matrix equation $\mathbf{Ax} = \mathbf{b}$ in which

$$
\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_m \end{pmatrix}.
$$

Conversely, every matrix equation of the form $\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ represents a system of m linear equations in n unknowns.

The numerical solution of a linear system was presented earlier in the text without the aid of matrix multiplication because the operation of matrix multiplication is not an integral part of the arithmetical process used to extract a solution by means of Gaussian elimination. Viewing a linear system as a single matrix equation $\mathbf{Ax} = \mathbf{b}$ is more of a notational convenience that can be used to uncover theoretical properties and to prove general theorems concerning linear systems.

For example, a very concise proof of the fact (2.3.5) stating that a system of equations $\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns in $\mathbf{A}$ is obtained by noting that the system is consistent if and only if there exists a column $\mathbf{s}$ that satisfies

$$\mathbf{b} = \mathbf{As} = \begin{pmatrix} \mathbf{A}_{*1} & \mathbf{A}_{*2} & \cdots & \mathbf{A}_{*n} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \mathbf{A}_{*1}s_1 + \mathbf{A}_{*2}s_2 + \cdots + \mathbf{A}_{*n}s_n.$$

The following example illustrates a common situation in which matrix multiplication arises naturally.

Example 3.5.2

An airline serves five cities, say, A, B, C, D, and H, in which H is the "hub city." The various routes between the cities are indicated in Figure 3.5.1.

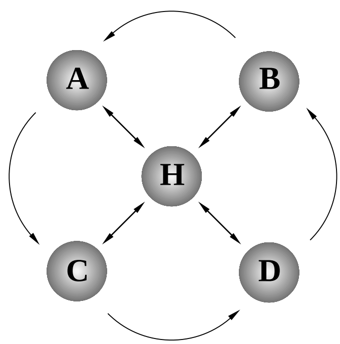


FIGURE 3.5.1

Suppose you wish to travel from city A to city B so that at least two connecting flights are required to make the trip. Flights $(A \rightarrow H)$ and $(H \rightarrow B)$ provide the minimal number of connections. However, if space on either of these two flights is not available, you will have to make at least three flights. Several questions arise. How many routes from city A to city B require *exactly* three connecting flights? How many routes require *no more than* four flights—and so forth? Since this particular network is small, these questions can be answered by "eyeballing" the diagram, but the "eyeball method" won't get you very far with the large networks that occur in more practical situations. Let's see how matrix algebra can be applied. Begin by creating a ***connectivity matrix*** $\mathbf{C} = [c_{ij}]$ (also known as an ***adjacency matrix***) in which

$$c_{ij} = \begin{cases} 1 & \text{if there is a flight from city } i \text{ to city } j, \\ 0 & \text{otherwise.} \end{cases}$$

For the network depicted in Figure 3.5.1,

$$\mathbf{C} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ H \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} A & B & C & D & H \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

The matrix $\mathbf{C}$ together with its powers $\mathbf{C}^2, \mathbf{C}^3, \mathbf{C}^4, \ldots$ will provide all of the information needed to analyze the network. To see how, notice that since c_{ik} is the number of direct routes from city i to city k, and since c_{kj} is the number of direct routes from city k to city j, it follows that $c_{ik}c_{kj}$ must be the number of 2-flight routes from city i to city j that have a connection at city k. Consequently, the (i,j)-entry in the product $\mathbf{C}^2 = \mathbf{CC}$ is

$$[\mathbf{C}^2]_{ij} = \sum_{k=1}^{5} c_{ik}c_{kj} = \text{ the total number of 2-flight routes from city } i \text{ to city } j.$$

Similarly, the (i,j)-entry in the product $\mathbf{C}^3 = \mathbf{CCC}$ is

$$[\mathbf{C}^3]_{ij} = \sum_{k_1,k_2=1}^{5} c_{ik_1}c_{k_1k_2}c_{k_2j} = \text{ number of 3-flight routes from city } i \text{ to city } j,$$

and, in general,

$$[\mathbf{C}^n]_{ij} = \sum_{k_1,k_2,\cdots,k_{n-1}=1}^{5} c_{ik_1}c_{k_1k_2}\cdots c_{k_{n-2}k_{n-1}}c_{k_{n-1}j}$$

is the total number of n-flight routes from city i to city j. Therefore, the total number of routes from city i to city j that require *no more* than n flights must be given by

$$[\mathbf{C}]_{ij} + [\mathbf{C}^2]_{ij} + [\mathbf{C}^3]_{ij} + \cdots + [\mathbf{C}^n]_{ij} = [\mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots + \mathbf{C}^n]_{ij}.$$

For our particular network,

$$\mathbf{C}^2 = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}, \quad \mathbf{C}^3 = \begin{pmatrix} 2 & 3 & 2 & 2 & 5 \\ 2 & 2 & 2 & 3 & 5 \\ 3 & 2 & 2 & 2 & 5 \\ 2 & 2 & 3 & 2 & 5 \\ 5 & 5 & 5 & 5 & 4 \end{pmatrix}, \quad \mathbf{C}^4 = \begin{pmatrix} 8 & 7 & 7 & 7 & 9 \\ 7 & 8 & 7 & 7 & 9 \\ 7 & 7 & 8 & 7 & 9 \\ 7 & 7 & 7 & 8 & 9 \\ 9 & 9 & 9 & 9 & 20 \end{pmatrix},$$

and

$$\mathbf{C}+\mathbf{C}^2+\mathbf{C}^3+\mathbf{C}^4=\begin{pmatrix} 11 & 11 & 11 & 11 & 16\\ 11 & 11 & 11 & 11 & 16\\ 11 & 11 & 11 & 11 & 16\\ 11 & 11 & 11 & 11 & 16\\ 16 & 16 & 16 & 16 & 28 \end{pmatrix}.$$

The fact that $[\mathbf{C}^3]_{12}=3$ means there are exactly 3 three-flight routes from city A to city B, and $[\mathbf{C}^4]_{12}=7$ means there are exactly 7 four-flight routes—try to identify them. Furthermore, $[\mathbf{C}+\mathbf{C}^2+\mathbf{C}^3+\mathbf{C}^4]_{12}=11$ means there are 11 routes from city A to city B that require no more than 4 flights.

Exercises for section 3.5

3.5.1. For $\mathbf{A}=\begin{pmatrix} 1 & -2 & 3\\ 0 & -5 & 4\\ 4 & -3 & 8 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 1 & 2\\ 0 & 4\\ 3 & 7 \end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix} 1\\ 2\\ 3 \end{pmatrix}$, compute the following products when possible.
(a) $\mathbf{AB}$, (b) $\mathbf{BA}$, (c) $\mathbf{CB}$, (d) $\mathbf{C}^T\mathbf{B}$, (e) $\mathbf{A}^2$, (f) $\mathbf{B}^2$,
(g) $\mathbf{C}^T\mathbf{C}$, (h) $\mathbf{CC}^T$, (i) $\mathbf{BB}^T$, (j) $\mathbf{B}^T\mathbf{B}$, (k) $\mathbf{C}^T\mathbf{AC}$.

3.5.2. Consider the following system of equations:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3,\\ 4x_1 + 2x_3 &= 10,\\ 2x_1 + 2x_2 &= -2. \end{aligned}$$

(a) Write the system as a matrix equation of the form $\mathbf{Ax}=\mathbf{b}$.
(b) Write the solution of the system as a column $\mathbf{s}$ and verify by matrix multiplication that $\mathbf{s}$ satisfies the equation $\mathbf{Ax}=\mathbf{b}$.
(c) Write $\mathbf{b}$ as a linear combination of the columns in $\mathbf{A}$.

3.5.3. Let $\mathbf{E}=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1 \end{pmatrix}$ and let $\mathbf{A}$ be an arbitrary 3×3 matrix.
(a) Describe the rows of $\mathbf{EA}$ in terms of the rows of $\mathbf{A}$.
(b) Describe the columns of $\mathbf{AE}$ in terms of the columns of $\mathbf{A}$.

3.5.4. Let $\mathbf{e}_j$ denote the j^{th} ***unit column*** that contains a 1 in the j^{th} position and zeros everywhere else. For a general matrix $\mathbf{A}_{n\times n}$, describe the following products. (a) $\mathbf{Ae}_j$ (b) $\mathbf{e}_i^T\mathbf{A}$ (c) $\mathbf{e}_i^T\mathbf{Ae}_j$

3.5.5. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices. If $\mathbf{Ax} = \mathbf{Bx}$ holds for all $n \times 1$ columns $\mathbf{x}$, prove that $\mathbf{A} = \mathbf{B}$. **Hint:** What happens when $\mathbf{x}$ is a unit column?

3.5.6. For $\mathbf{A} = \begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix}$, determine $\lim_{n\to\infty} \mathbf{A}^n$. **Hint:** Compute a few powers of $\mathbf{A}$ and try to deduce the general form of $\mathbf{A}^n$.

3.5.7. If $\mathbf{C}_{m\times 1}$ and $\mathbf{R}_{1\times n}$ are matrices consisting of a single column and a single row, respectively, then the matrix product $\mathbf{P}_{m\times n} = \mathbf{CR}$ is sometimes called the ***outer product*** of $\mathbf{C}$ with $\mathbf{R}$. For conformable matrices $\mathbf{A}$ and $\mathbf{B}$, explain how to write the product $\mathbf{AB}$ as a sum of outer products involving the columns of $\mathbf{A}$ and the rows of $\mathbf{B}$.

3.5.8. A square matrix $\mathbf{U} = [u_{ij}]$ is said to be ***upper triangular*** whenever $u_{ij} = 0$ for $i > j$—i.e., all entries below the main diagonal are 0.

(a) If $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ upper-triangular matrices, explain why the product $\mathbf{AB}$ must also be upper triangular.

(b) If $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times n}$ are upper triangular, what are the diagonal entries of $\mathbf{AB}$?

(c) $\mathbf{L}$ is ***lower triangular*** when $\ell_{ij} = 0$ for $i < j$. Is it true that the product of two $n \times n$ lower-triangular matrices is again lower triangular?

3.5.9. If $\mathbf{A} = [a_{ij}(t)]$ is a matrix whose entries are functions of a variable t, the ***derivative*** of $\mathbf{A}$ with respect to t is defined to be the matrix of derivatives. That is,

$$\frac{d\mathbf{A}}{dt} = \left[\frac{da_{ij}}{dt}\right].$$

Derive the *product rule for differentiation*

$$\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}.$$

3.5.10. Let $\mathbf{C}_{n\times n}$ be the connectivity matrix associated with a network of n nodes such as that described in Example 3.5.2, and let $\mathbf{e}$ be the $n \times 1$ column of all 1's. In terms of the network, describe the entries in each of the following products.

(a) Interpret the product $\mathbf{Ce}$.

(b) Interpret the product $\mathbf{e}^T\mathbf{C}$.

3.5.11. Consider three tanks each containing V gallons of brine. The tanks are connected as shown in Figure 3.5.2, and all spigots are opened at once. As fresh water at the rate of r gal/sec is pumped into the top of the first tank, r gal/sec leaves from the bottom and flows into the next tank, and so on down the line—there are r gal/sec entering at the top and leaving through the bottom of each tank.

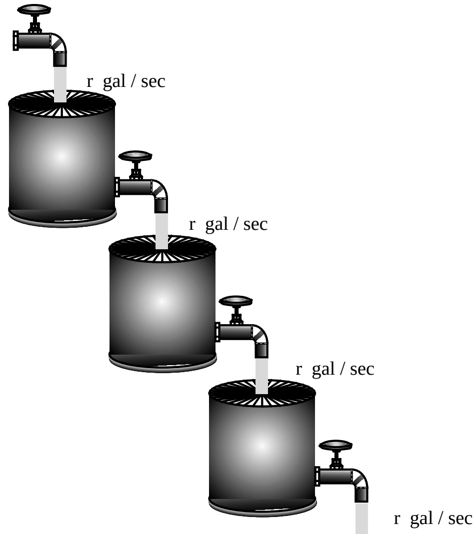


FIGURE 3.5.2

Let $x_i(t)$ denote the number of pounds of salt in tank i at time t, and let

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \quad \text{and} \quad \frac{d\mathbf{x}}{dt} = \begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix}.$$

Assuming that complete mixing occurs in each tank on a continuous basis, show that

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{A} = \frac{r}{V}\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Hint: Use the fact that

$$\frac{dx_i}{dt} = \text{rate of change} = \frac{\text{lbs}}{\text{sec}}\text{ coming in} - \frac{\text{lbs}}{\text{sec}}\text{ going out.}$$

3.6 PROPERTIES OF MATRIX MULTIPLICATION

We saw in the previous section that there are some differences between scalar and matrix algebra—most notable is the fact that matrix multiplication is not commutative, and there is no cancellation law. But there are also some important similarities, and the purpose of this section is to look deeper into these issues.

Although we can adjust to not having the commutative property, the situation would be unbearable if the distributive and associative properties were not available. Fortunately, both of these properties hold for matrix multiplication.

Distributive and Associative Laws

For conformable matrices each of the following is true.

- $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ (left-hand distributive law).
- $(\mathbf{D}+\mathbf{E})\mathbf{F} = \mathbf{DF}+\mathbf{EF}$ (right-hand distributive law).
- $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative law).

Proof. To prove the left-hand distributive property, demonstrate the corresponding entries in the matrices $\mathbf{A}(\mathbf{B}+\mathbf{C})$ and $\mathbf{AB}+\mathbf{AC}$ are equal. To this end, use the definition of matrix multiplication to write

$$\begin{aligned}[\mathbf{A}(\mathbf{B}+\mathbf{C})]_{ij} &= \mathbf{A}_{i*}(\mathbf{B}+\mathbf{C})_{*j} = \sum_k [\mathbf{A}]_{ik}[\mathbf{B}+\mathbf{C}]_{kj} = \sum_k [\mathbf{A}]_{ik}\left([\mathbf{B}]_{kj}+[\mathbf{C}]_{kj}\right) \\ &= \sum_k \left([\mathbf{A}]_{ik}[\mathbf{B}]_{kj}+[\mathbf{A}]_{ik}[\mathbf{C}]_{kj}\right) = \sum_k [\mathbf{A}]_{ik}[\mathbf{B}]_{kj} + \sum_k [\mathbf{A}]_{ik}[\mathbf{C}]_{kj} \\ &= \mathbf{A}_{i*}\mathbf{B}_{*j} + \mathbf{A}_{i*}\mathbf{C}_{*j} = [\mathbf{AB}]_{ij} + [\mathbf{AC}]_{ij} \\ &= [\mathbf{AB}+\mathbf{AC}]_{ij}.\end{aligned}$$

Since this is true for each i and j, it follows that $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$. The proof of the right-hand distributive property is similar and is omitted. To prove the associative law, suppose that $\mathbf{B}$ is $p\times q$ and $\mathbf{C}$ is $q\times n$, and recall from (3.5.7) that the j^{th} column of $\mathbf{BC}$ is a linear combination of the columns in $\mathbf{B}$. That is,

$$[\mathbf{BC}]_{*j} = \mathbf{B}_{*1}c_{1j} + \mathbf{B}_{*2}c_{2j} + \cdots + \mathbf{B}_{*q}c_{qj} = \sum_{k=1}^{q} \mathbf{B}_{*k}c_{kj}.$$

Use this along with the left-hand distributive property to write

$$
\begin{aligned}
[\mathbf{A}(\mathbf{BC})]_{ij} &= \mathbf{A}_{i*}[\mathbf{BC}]_{*j} = \mathbf{A}_{i*}\sum_{k=1}^{q}\mathbf{B}_{*k}c_{kj} = \sum_{k=1}^{q}\mathbf{A}_{i*}\mathbf{B}_{*k}c_{kj} \\
&= \sum_{k=1}^{q}[\mathbf{AB}]_{ik}c_{kj} = [\mathbf{AB}]_{i*}\mathbf{C}_{*j} = [(\mathbf{AB})\mathbf{C}]_{ij}. \quad \blacksquare
\end{aligned}
$$

Example 3.6.1

Linearity of Matrix Multiplication. Let $\mathbf{A}$ be an $m \times n$ matrix, and f be the function defined by matrix multiplication

$$f(\mathbf{X}_{n\times p}) = \mathbf{AX}.$$

The left-hand distributive property guarantees that f is a linear function because for all scalars α and for all $n \times p$ matrices $\mathbf{X}$ and $\mathbf{Y}$,

$$
\begin{aligned}
f(\alpha\mathbf{X}+\mathbf{Y}) = \mathbf{A}(\alpha\mathbf{X}+\mathbf{Y}) &= \mathbf{A}(\alpha\mathbf{X}) + \mathbf{AY} = \alpha\mathbf{AX} + \mathbf{AY} \\
&= \alpha f(\mathbf{X}) + f(\mathbf{Y}).
\end{aligned}
$$

Of course, the linearity of matrix multiplication is no surprise because it was the consideration of linear functions that motivated the definition of the matrix product at the outset.

For scalars, the number 1 is the identity element for multiplication because it has the property that it reproduces whatever it is multiplied by. For matrices, there is an identity element with similar properties.

Identity Matrix

The $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the ***identity matrix*** of order n. For every $m \times n$ matrix $\mathbf{A}$,

$$\mathbf{AI}_n = \mathbf{A} \quad \text{and} \quad \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

The subscript on $\mathbf{I}_n$ is neglected whenever the size is obvious from the context.

Proof. Notice that $\mathbf{I}_{*j}$ has a 1 in the j^{th} position and 0's elsewhere. Recall from Exercise 3.5.4 that such columns were called ***unit columns,*** and they have the property that for any conformable matrix $\mathbf{A}$,

$$\mathbf{A}\mathbf{I}_{*j} = \mathbf{A}_{*j}.$$

Using this together with the fact that $[\mathbf{A}\mathbf{I}]_{*j} = \mathbf{A}\mathbf{I}_{*j}$ produces

$$\mathbf{A}\mathbf{I} = \begin{pmatrix} \mathbf{A}\mathbf{I}_{*1} & \mathbf{A}\mathbf{I}_{*2} & \cdots & \mathbf{A}\mathbf{I}_{*n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{*1} & \mathbf{A}_{*2} & \cdots & \mathbf{A}_{*n} \end{pmatrix} = \mathbf{A}.$$

A similar argument holds when $\mathbf{I}$ appears on the left-hand side of $\mathbf{A}$. ∎

Analogous to scalar algebra, we define the 0^{th} power of a square matrix to be the identity matrix of corresponding size. That is, if $\mathbf{A}$ is $n \times n$, then

$$\mathbf{A}^0 = \mathbf{I}_n.$$

Positive powers of $\mathbf{A}$ are also defined in the natural way. That is,

$$\mathbf{A}^n = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{n \text{ times}}.$$

The associative law guarantees that it makes no difference how matrices are grouped for powering. For example, $\mathbf{A}\mathbf{A}^2$ is the same as $\mathbf{A}^2\mathbf{A}$, so that

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A} = \mathbf{A}\mathbf{A}^2 = \mathbf{A}^2\mathbf{A}.$$

Also, the usual laws of exponents hold. For nonnegative integers r and s,

$$\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s} \quad \text{and} \quad (\mathbf{A}^r)^s = \mathbf{A}^{rs}.$$

We are not yet in a position to define negative or fractional powers, and due to the lack of conformability, powers of nonsquare matrices are never defined.

Example 3.6.2

A Pitfall. For two $n \times n$ matrices, what is $(\mathbf{A}+\mathbf{B})^2$? **Be careful!** Because matrix multiplication is not commutative, the familiar formula from scalar algebra is not valid for matrices. The distributive properties must be used to write

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^2 &= \underbrace{(\mathbf{A}+\mathbf{B})}(\mathbf{A}+\mathbf{B}) = \underbrace{(\mathbf{A}+\mathbf{B})}\mathbf{A} + \underbrace{(\mathbf{A}+\mathbf{B})}\mathbf{B} \\ &= \mathbf{A}^2 + \mathbf{B}\mathbf{A} + \mathbf{A}\mathbf{B} + \mathbf{B}^2,\end{aligned}$$

and this is as far as you can go. The familiar form $\mathbf{A}^2+2\mathbf{A}\mathbf{B}+\mathbf{B}^2$ is obtained only in those rare cases where $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$. To evaluate $(\mathbf{A}+\mathbf{B})^k$, the distributive rules must be applied repeatedly, and the results are a bit more complicated—try it for $k = 3$.

Example 3.6.3

Suppose that the population migration between two geographical regions—say, the North and the South—is as follows. Each year, 50% of the population in the North migrates to the South, while only 25% of the population in the South moves to the North. This situation is depicted by drawing a transition diagram such as that shown in Figure 3.6.1.

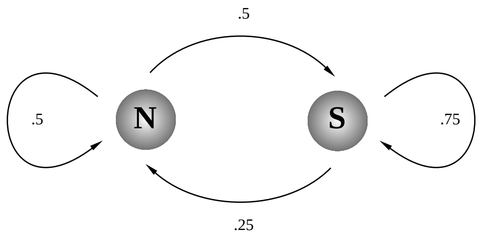


FIGURE 3.6.1

Problem: If this migration pattern continues, will the population in the North continually shrink until the entire population is eventually in the South, or will the population distribution somehow stabilize before the North is completely deserted?

Solution: Let n_k and s_k denote the respective proportions of the total population living in the North and South at the end of year k and assume $n_k + s_k = 1$. The migration pattern dictates that the fractions of the population in each region at the end of year $k+1$ are

$$\begin{aligned} n_{k+1} &= n_k(.5) + s_k(.25), \\ s_{k+1} &= n_k(.5) + s_k(.75). \end{aligned} \tag{3.6.1}$$

If $\mathbf{p}_k^T = (n_k,\, s_k)$ and $\mathbf{p}_{k+1}^T = (n_{k+1},\, s_{k+1})$ denote the respective population distributions at the end of years k and $k+1$, and if

$$\mathbf{T} = \begin{array}{c} \\ \mathrm{N} \\ \mathrm{S} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathrm{N} & \ \ \mathrm{S} \end{array} \\ \begin{pmatrix} .5 & .5 \\ .25 & .75 \end{pmatrix} \end{array}$$

is the associated ***transition matrix,*** then (3.6.1) assumes the matrix form $\mathbf{p}_{k+1}^T = \mathbf{p}_k^T\mathbf{T}$. Inducting on $\mathbf{p}_1^T = \mathbf{p}_0^T\mathbf{T}$, $\mathbf{p}_2^T = \mathbf{p}_1^T\mathbf{T} = \mathbf{p}_0^T\mathbf{T}^2$, $\mathbf{p}_3^T = \mathbf{p}_2^T\mathbf{T} = \mathbf{p}_0^T\mathbf{T}^3$, etc., leads to

$$\mathbf{p}_k^T = \mathbf{p}_0^T\mathbf{T}^k. \tag{3.6.2}$$

Determining the long-run behavior involves evaluating $\lim_{k\to\infty} \mathbf{p}_k^T$, and it's clear from (3.6.2) that this boils down to analyzing $\lim_{k\to\infty} \mathbf{T}^k$. Later, in Example

7.3.5, a more sophisticated approach is discussed, but for now we will use the "brute force" method of successively powering $\mathbf{P}$ until a pattern emerges. The first several powers of $\mathbf{P}$ are shown below with three significant digits displayed.

$$\mathbf{P}^2 = \begin{pmatrix} .375 & .625 \\ .312 & .687 \end{pmatrix} \quad \mathbf{P}^3 = \begin{pmatrix} .344 & .656 \\ .328 & .672 \end{pmatrix} \quad \mathbf{P}^4 = \begin{pmatrix} .328 & .672 \\ .332 & .668 \end{pmatrix}$$
$$\mathbf{P}^5 = \begin{pmatrix} .334 & .666 \\ .333 & .667 \end{pmatrix} \quad \mathbf{P}^6 = \begin{pmatrix} .333 & .667 \\ .333 & .667 \end{pmatrix} \quad \mathbf{P}^7 = \begin{pmatrix} .333 & .667 \\ .333 & .667 \end{pmatrix}$$

This sequence appears to be converging to a limiting matrix of the form

$$\mathbf{P}^\infty = \lim_{k\to\infty} \mathbf{P}^k = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix},$$

so the limiting population distribution is

$$\begin{aligned} \mathbf{p}_\infty^T &= \lim_{k\to\infty} \mathbf{p}_k^T = \lim_{k\to\infty} \mathbf{p}_0^T \mathbf{T}^k = \mathbf{p}_0^T \lim_{k\to\infty} \mathbf{T}^k = \begin{pmatrix} n_0 & s_0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{n_0+s_0}{3} & \dfrac{2(n_0+s_0)}{3} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix}. \end{aligned}$$

Therefore, if the migration pattern continues to hold, then the population distribution will eventually stabilize with 1/3 of the population being in the North and 2/3 of the population in the South. And this is independent of the initial distribution! The powers of $\mathbf{P}$ indicate that the population distribution will be practically stable in no more than 6 years—individuals may continue to move, but the proportions in each region are essentially constant by the sixth year.

The operation of transposition has an interesting effect upon a matrix product—a reversal of order occurs.

Reverse Order Law for Transposition

For conformable matrices $\mathbf{A}$ and $\mathbf{B}$,

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.$$

The case of conjugate transposition is similar. That is,

$$(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*.$$

Proof. By definition,

$$(\mathbf{AB})^T_{ij} = [\mathbf{AB}]_{ji} = \mathbf{A}_{j*}\mathbf{B}_{*i}.$$

Consider the (i,j)-entry of the matrix $\mathbf{B}^T\mathbf{A}^T$ and write

$$\begin{aligned}\left[\mathbf{B}^T\mathbf{A}^T\right]_{ij} &= \left(\mathbf{B}^T\right)_{i*}\left(\mathbf{A}^T\right)_{*j} = \sum_k \left[\mathbf{B}^T\right]_{ik}\left[\mathbf{A}^T\right]_{kj} \\ &= \sum_k [\mathbf{B}]_{ki}[\mathbf{A}]_{jk} = \sum_k [\mathbf{A}]_{jk}[\mathbf{B}]_{ki} \\ &= \mathbf{A}_{j*}\mathbf{B}_{*i}.\end{aligned}$$

Therefore, $(\mathbf{AB})^T_{ij} = \left[\mathbf{B}^T\mathbf{A}^T\right]_{ij}$ for all i and j, and thus $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. The proof for the conjugate transpose case is similar. ∎

Example 3.6.4

For every matrix $\mathbf{A}_{m\times n}$, the products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric matrices because

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\mathbf{A}^{T^T} = \mathbf{A}^T\mathbf{A} \quad \text{and} \quad \left(\mathbf{A}\mathbf{A}^T\right)^T = \mathbf{A}^{T^T}\mathbf{A}^T = \mathbf{A}\mathbf{A}^T.$$

Example 3.6.5

Trace of a Product. Recall from Example 3.3.1 that the trace of a square matrix is the sum of its main diagonal entries. Although matrix multiplication is not commutative, the trace function is one of the few cases where the order of the matrices can be changed without affecting the results.

Problem: For matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times m}$, prove that

$$trace\,(\mathbf{AB}) = trace\,(\mathbf{BA}).$$

Solution:

$$\begin{aligned}trace\,(\mathbf{AB}) &= \sum_i [\mathbf{AB}]_{ii} = \sum_i \mathbf{A}_{i*}\mathbf{B}_{*i} = \sum_i\sum_k a_{ik}b_{ki} = \sum_i\sum_k b_{ki}a_{ik} \\ &= \sum_k\sum_i b_{ki}a_{ik} = \sum_k \mathbf{B}_{k*}\mathbf{A}_{*k} = \sum_k [\mathbf{BA}]_{kk} = trace\,(\mathbf{BA}).\end{aligned}$$

Note: This is true in spite of the fact that $\mathbf{AB}$ is $m\times m$ while $\mathbf{BA}$ is $n\times n$. Furthermore, this result can be extended to say that any product of conformable matrices can be permuted *cyclically* without altering the trace of the product. For example,

$$trace\,(\mathbf{ABC}) = trace\,(\mathbf{BCA}) = trace\,(\mathbf{CAB}).$$

However, a noncyclical permutation may not preserve the trace. For example,

$$trace\,(\mathbf{ABC}) \neq trace\,(\mathbf{BAC}).$$

Executing multiplication between two matrices by partitioning one or both factors into ***submatrices***—a matrix contained within another matrix—can be a useful technique.

Block Matrix Multiplication

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are partitioned into submatrices—often referred to as ***blocks***—as indicated below.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1r} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{s1} & \mathbf{A}_{s2} & \cdots & \mathbf{A}_{sr} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1t} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{r1} & \mathbf{B}_{r2} & \cdots & \mathbf{B}_{rt} \end{pmatrix}.$$

If the pairs $(\mathbf{A}_{ik}, \mathbf{B}_{kj})$ are conformable, then $\mathbf{A}$ and $\mathbf{B}$ are said to be ***conformably partitioned***. For such matrices, the product $\mathbf{AB}$ is formed by combining the blocks exactly the same way as the scalars are combined in ordinary matrix multiplication. That is, the (i, j)-block in $\mathbf{AB}$ is

$$\mathbf{A}_{i1}\mathbf{B}_{1j} + \mathbf{A}_{i2}\mathbf{B}_{2j} + \cdots + \mathbf{A}_{ir}\mathbf{B}_{rj}.$$

Although a completely general proof is possible, looking at some examples better serves the purpose of understanding this technique.

Example 3.6.6

Block multiplication is particularly useful when there are patterns in the matrices to be multiplied. Consider the partitioned matrices

$$\mathbf{A} = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} \mathbf{C} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{B} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{array}\right) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{C} \end{pmatrix},$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Using block multiplication, the product $\mathbf{AB}$ is easily computed to be

$$\mathbf{AB} = \begin{pmatrix} \mathbf{C} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 2\mathbf{C} & \mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 4 & 1 & 2 \\ 6 & 8 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

Example 3.6.7

Reducibility. Suppose that $\mathbf{T}_{n\times n}\mathbf{x} = \mathbf{b}$ represents a system of linear equations in which the coefficient matrix is ***block triangular***. That is, $\mathbf{T}$ can be partitioned as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}, \quad \text{where} \quad \mathbf{A} \text{ is } r \times r \text{ and } \mathbf{C} \text{ is } n-r \times n-r. \tag{3.6.3}$$

If $\mathbf{x}$ and $\mathbf{b}$ are similarly partitioned as $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$, then block multiplication shows that $\mathbf{Tx} = \mathbf{b}$ reduces to two smaller systems

$$\begin{aligned} \mathbf{Ax}_1 + \mathbf{Bx}_2 &= \mathbf{b}_1, \\ \mathbf{Cx}_2 &= \mathbf{b}_2, \end{aligned}$$

so if all systems are consistent, a block version of back substitution is possible—i.e., solve $\mathbf{Cx}_2 = \mathbf{b}_2$ for $\mathbf{x}_2$, and substituted this back into $\mathbf{Ax}_1 = \mathbf{b}_1 - \mathbf{Bx}_2$, which is then solved for $\mathbf{x}_1$. For obvious reasons, block-triangular systems of this type are sometimes referred to as ***reducible systems,*** and $\mathbf{T}$ is said to be a ***reducible matrix.*** Recall that applying Gaussian elimination with back substitution to an $n \times n$ system requires about $n^3/3$ multiplications/divisions and about $n^3/3$ additions/subtractions. This means that it's more efficient to solve two smaller subsystems than to solve one large main system. For example, suppose the matrix $\mathbf{T}$ in (3.6.3) is 100×100 while $\mathbf{A}$ and $\mathbf{C}$ are each 50×50. If $\mathbf{Tx} = \mathbf{b}$ is solved without taking advantage of its reducibility, then about $10^6/3$ multiplications/divisions are needed. But by taking advantage of the reducibility, only about $(250 \times 10^3)/3$ multiplications/divisions are needed to solve both 50×50 subsystems. Another advantage of reducibility is realized when a computer's main memory capacity is not large enough to store the entire coefficient matrix but is large enough to hold the submatrices.

Exercises for section 3.6

3.6.1. For the partitioned matrices

$$\mathbf{A} = \left(\begin{array}{c|cc|ccc} 1 & 0 & 0 & 3 & 3 & 3 \\ 1 & 0 & 0 & 3 & 3 & 3 \\ \hline 1 & 2 & 2 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \mathbf{B} = \left(\begin{array}{cc} -1 & -1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline -1 & -2 \\ -1 & -2 \\ -1 & -2 \end{array}\right),$$

use block multiplication with the indicated partitions to form the product $\mathbf{AB}$.

3.6.2. For all matrices $\mathbf{A}_{n\times k}$ and $\mathbf{B}_{k\times n}$, show that the block matrix

$$\mathbf{L} = \begin{pmatrix} \mathbf{I}-\mathbf{BA} & \mathbf{B} \\ 2\mathbf{A}-\mathbf{ABA} & \mathbf{AB}-\mathbf{I} \end{pmatrix}$$

has the property $\mathbf{L}^2 = \mathbf{I}$. Matrices with this property are said to be ***involutory,*** and they occur in the science of cryptography.

3.6.3. For the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix},$$

determine $\mathbf{A}^{300}$. **Hint:** A square matrix $\mathbf{C}$ is said to be ***idempotent*** when it has the property that $\mathbf{C}^2 = \mathbf{C}$. Make use of idempotent submatrices in $\mathbf{A}$.

3.6.4. For every matrix $\mathbf{A}_{m\times n}$, demonstrate that the products $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ are hermitian matrices.

3.6.5. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices that commute, prove that the product $\mathbf{AB}$ is also symmetric. If $\mathbf{AB} \neq \mathbf{BA}$, is $\mathbf{AB}$ necessarily symmetric?

3.6.6. Prove that the right-hand distributive property is true.

3.6.7. For each matrix $\mathbf{A}_{n\times n}$, explain why it is impossible to find a solution for $\mathbf{X}_{n\times n}$ in the matrix equation

$$\mathbf{AX} - \mathbf{XA} = \mathbf{I}.$$

Hint: Consider the trace function.

3.6.8. Let $\mathbf{y}^T_{1\times m}$ be a row of unknowns, and let $\mathbf{A}_{m\times n}$ and $\mathbf{b}^T_{1\times n}$ be known matrices.

(a) Explain why the matrix equation $\mathbf{y}^T\mathbf{A} = \mathbf{b}^T$ represents a system of n linear equations in m unknowns.

(b) How are the solutions for $\mathbf{y}^T$ in $\mathbf{y}^T\mathbf{A} = \mathbf{b}^T$ related to the solutions for $\mathbf{x}$ in $\mathbf{A}^T\mathbf{x} = \mathbf{b}$?

3.6.9. A particular electronic device consists of a collection of switching circuits that can be either in an ON state or an OFF state. These electronic switches are allowed to change state at regular time intervals called *clock cycles.* Suppose that at the end of each clock cycle, 30% of the switches currently in the OFF state change to ON, while 90% of those in the ON state revert to the OFF state.

(a) Show that the device approaches an equilibrium in the sense that the proportion of switches in each state eventually becomes constant, and determine these equilibrium proportions.

(b) Independent of the initial proportions, about how many clock cycles does it take for the device to become essentially stable?

3.6.10. Write the following system in the form $\mathbf{T}_{n\times n}\mathbf{x} = \mathbf{b}$, where $\mathbf{T}$ is block triangular, and then obtain the solution by solving two small systems as described in Example 3.6.7.

$$\begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 &= -1, \\ 2x_3 + 3x_4 &= 3, \\ x_1 + 2x_2 + 5x_3 + 6x_4 &= -2, \\ x_3 + 2x_4 &= 4. \end{aligned}$$

3.6.11. Prove that each of the following statements is true for conformable matrices.

(a) $\mathit{trace}\,(\mathbf{ABC}) = \mathit{trace}\,(\mathbf{BCA}) = \mathit{trace}\,(\mathbf{CAB})$.

(b) $\mathit{trace}\,(\mathbf{ABC})$ can be different from $\mathit{trace}\,(\mathbf{BAC})$.

(c) $\mathit{trace}\left(\mathbf{A}^T\mathbf{B}\right) = \mathit{trace}\left(\mathbf{A}\mathbf{B}^T\right)$.

3.6.12. Suppose that $\mathbf{A}_{m\times n}$ and $\mathbf{x}_{n\times 1}$ have real entries.

(a) Prove that $\mathbf{x}^T\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(b) Prove that $\mathit{trace}\left(\mathbf{A}^T\mathbf{A}\right) = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

3.7 MATRIX INVERSION

If α is a nonzero scalar, then for each number β the equation $\alpha x = \beta$ has a unique solution given by $x = \alpha^{-1}\beta$. To *prove* that $\alpha^{-1}\beta$ is a solution, write

$$\alpha(\alpha^{-1}\beta) = (\alpha\alpha^{-1})\beta = (1)\beta = \beta. \tag{3.7.1}$$

Uniqueness follows because if x_1 and x_2 are two solutions, then

$$\begin{aligned} \alpha x_1 = \beta = \alpha x_2 \quad &\Longrightarrow \quad \alpha^{-1}(\alpha x_1) = \alpha^{-1}(\alpha x_2) \\ &\Longrightarrow \quad (\alpha^{-1}\alpha)x_1 = (\alpha^{-1}\alpha)x_2 \\ &\Longrightarrow \quad (1)x_1 = (1)x_2 \quad \Longrightarrow \quad x_1 = x_2. \end{aligned} \tag{3.7.2}$$

These observations seem pedantic, but they are important in order to see how to make the transition from scalar equations to matrix equations. In particular, these arguments show that in addition to associativity, the properties

$$\alpha\alpha^{-1} = 1 \quad \text{and} \quad \alpha^{-1}\alpha = 1 \tag{3.7.3}$$

are the key ingredients, so if we want to solve matrix equations in the same fashion as we solve scalar equations, then a matrix analogue of (3.7.3) is needed.

Matrix Inversion

For a given square matrix $\mathbf{A}_{n\times n}$, the matrix $\mathbf{B}_{n\times n}$ that satisfies the conditions

$$\mathbf{AB} = \mathbf{I}_n \quad \text{and} \quad \mathbf{BA} = \mathbf{I}_n$$

is called the ***inverse*** of $\mathbf{A}$ and is denoted by $\mathbf{B} = \mathbf{A}^{-1}$. Not all square matrices are invertible—the zero matrix is a trivial example, but there are also many nonzero matrices that are not invertible. An invertible matrix is said to be ***nonsingular,*** and a square matrix with no inverse is called a ***singular matrix.***

Notice that matrix inversion is defined for square matrices only—the condition $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A}$ rules out inverses of nonsquare matrices.

Example 3.7.1

If

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where} \quad \delta = ad - bc \neq 0,$$

then

$$\mathbf{A}^{-1} = \frac{1}{\delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

because it can be verified that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_2$.

Although not all matrices are invertible, *when an inverse exists, it is unique.* To see this, suppose that $\mathbf{X}_1$ and $\mathbf{X}_2$ are both inverses for a nonsingular matrix $\mathbf{A}$. Then

$$\mathbf{X}_1 = \mathbf{X}_1\mathbf{I} = \mathbf{X}_1(\mathbf{A}\mathbf{X}_2) = (\mathbf{X}_1\mathbf{A})\mathbf{X}_2 = \mathbf{I}\mathbf{X}_2 = \mathbf{X}_2,$$

which implies that only one inverse is possible.

Since matrix inversion was defined analogously to scalar inversion, and since matrix multiplication is associative, exactly the same reasoning used in (3.7.1) and (3.7.2) can be applied to a matrix equation $\mathbf{AX} = \mathbf{B}$, so we have the following statements.

Matrix Equations

- If $\mathbf{A}$ is a nonsingular matrix, then there is a unique solution for $\mathbf{X}$ in the matrix equation $\mathbf{A}_{n\times n}\mathbf{X}_{n\times p} = \mathbf{B}_{n\times p}$, and the solution is

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}. \tag{3.7.4}$$

- A system of n linear equations in n unknowns can be written as a single matrix equation $\mathbf{A}_{n\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{n\times 1}$ (see p. 99), so it follows from (3.7.4) that when $\mathbf{A}$ is nonsingular, the system has a unique solution given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

However, it must be stressed that the representation of the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is mostly a notational or theoretical convenience. *In practice, a nonsingular system* $\mathbf{Ax} = \mathbf{b}$ *is almost never solved by first computing* $\mathbf{A}^{-1}$ *and then the product* $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The reason will be apparent when we learn how much work is involved in computing $\mathbf{A}^{-1}$.

Since not all square matrices are invertible, methods are needed to distinguish between nonsingular and singular matrices. There is a variety of ways to describe the class of nonsingular matrices, but those listed below are among the most important.

Existence of an Inverse

For an $n \times n$ matrix $\mathbf{A}$, the following statements are equivalent.

- $\mathbf{A}^{-1}$ exists ($\mathbf{A}$ is nonsingular). (3.7.5)
- $rank\,(\mathbf{A}) = n$. (3.7.6)
- $\mathbf{A} \xrightarrow{\text{Gauss–Jordan}} \mathbf{I}$. (3.7.7)
- $\mathbf{Ax} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$. (3.7.8)

Proof. The fact that (3.7.6) $\Longleftrightarrow$ (3.7.7) is a direct consequence of the definition of rank, and (3.7.6) $\Longleftrightarrow$ (3.7.8) was established in §2.4. Consequently, statements (3.7.6), (3.7.7), and (3.7.8) are equivalent, so if we establish that (3.7.5) $\Longleftrightarrow$ (3.7.6), then the proof will be complete.

Proof of (3.7.5) $\Longrightarrow$ (3.7.6). Begin by observing that (3.5.5) guarantees that a matrix $\mathbf{X} = [\mathbf{X}_{*1} \,|\, \mathbf{X}_{*2} \,|\, \cdots \,|\, \mathbf{X}_{*n}]$ satisfies the equation $\mathbf{AX} = \mathbf{I}$ if and only if $\mathbf{X}_{*j}$ is a solution of the linear system $\mathbf{Ax} = \mathbf{I}_{*j}$. If $\mathbf{A}$ is nonsingular, then we know from (3.7.4) that there exists a unique solution to $\mathbf{AX} = \mathbf{I}$, and hence each linear system $\mathbf{Ax} = \mathbf{I}_{*j}$ has a unique solution. But in §2.5 we learned that a linear system has a unique solution if and only if the rank of the coefficient matrix equals the number of unknowns, so $rank\,(\mathbf{A}) = n$.

Proof of (3.7.6) $\Longrightarrow$ (3.7.5). If $rank\,(\mathbf{A}) = n$, then (2.3.4) insures that each system $\mathbf{Ax} = \mathbf{I}_{*j}$ is consistent because $rank[\mathbf{A} \,|\, \mathbf{I}_{*j}] = n = rank\,(\mathbf{A})$. Furthermore, the results of §2.5 guarantee that each system $\mathbf{Ax} = \mathbf{I}_{*j}$ has a unique solution, and hence there is a unique solution to the matrix equation $\mathbf{AX} = \mathbf{I}$. We would like to say that $\mathbf{X} = \mathbf{A}^{-1}$, but we cannot jump to this conclusion without first arguing that $\mathbf{XA} = \mathbf{I}$. Suppose this is not true—i.e., suppose that $\mathbf{XA} - \mathbf{I} \neq \mathbf{0}$. Since

$$\mathbf{A}(\mathbf{XA} - \mathbf{I}) = (\mathbf{AX})\mathbf{A} - \mathbf{A} = \mathbf{IA} - \mathbf{A} = \mathbf{0},$$

it follows from (3.5.5) that any nonzero column of $\mathbf{XA} - \mathbf{I}$ is a nontrivial solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$. But this is a contradiction of the fact that (3.7.6) $\Longleftrightarrow$ (3.7.8). Therefore, the supposition that $\mathbf{XA} - \mathbf{I} \neq \mathbf{0}$ must be false, and thus $\mathbf{AX} = \mathbf{I} = \mathbf{XA}$, which means $\mathbf{A}$ is nonsingular. ∎

The definition of matrix inversion says that in order to compute $\mathbf{A}^{-1}$, it is necessary to solve *both* of the matrix equations $\mathbf{AX} = \mathbf{I}$ and $\mathbf{XA} = \mathbf{I}$. These two equations are necessary to rule out the possibility of nonsquare inverses. But when only square matrices are involved, then any one of the two equations will suffice—the following example elaborates.

Example 3.7.2

Problem: If $\mathbf{A}$ and $\mathbf{X}$ are *square* matrices, explain why

$$\mathbf{AX} = \mathbf{I} \quad \Longrightarrow \quad \mathbf{XA} = \mathbf{I}. \tag{3.7.9}$$

In other words, if $\mathbf{A}$ and $\mathbf{X}$ are square and $\mathbf{AX} = \mathbf{I}$, then $\mathbf{X} = \mathbf{A}^{-1}$.

Solution: Notice first that $\mathbf{AX} = \mathbf{I}$ implies $\mathbf{X}$ is nonsingular because if $\mathbf{X}$ is singular, then, by (3.7.8), there is a column vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Xx} = \mathbf{0}$, which is contrary to the fact that $\mathbf{x} = \mathbf{Ix} = \mathbf{AXx} = \mathbf{0}$. Now that we know $\mathbf{X}^{-1}$ exists, we can establish (3.7.9) by writing

$$\mathbf{AX} = \mathbf{I} \quad \Longrightarrow \quad \mathbf{AXX}^{-1} = \mathbf{X}^{-1} \quad \Longrightarrow \quad \mathbf{A} = \mathbf{X}^{-1} \quad \Longrightarrow \quad \mathbf{XA} = \mathbf{I}.$$

Caution! The argument above is not valid for nonsquare matrices. When $m \neq n$, it's possible that $\mathbf{A}_{m\times n}\mathbf{X}_{n\times m} = \mathbf{I}_m$, but $\mathbf{XA} \neq \mathbf{I}_n$.

Although we usually try to avoid computing the inverse of a matrix, there are times when an inverse must be found. To construct an algorithm that will yield $\mathbf{A}^{-1}$ when $\mathbf{A}_{n\times n}$ is nonsingular, recall from Example 3.7.2 that determining $\mathbf{A}^{-1}$ is equivalent to solving the single matrix equation $\mathbf{AX} = \mathbf{I}$, and due to (3.5.5), this in turn is equivalent to solving the n linear systems defined by

$$\mathbf{Ax} = \mathbf{I}_{*j} \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{3.7.10}$$

In other words, if $\mathbf{X}_{*1}, \mathbf{X}_{*2}, \ldots, \mathbf{X}_{*n}$ are the respective solutions to (3.7.10), then $\mathbf{X} = [\mathbf{X}_{*1} \,|\, \mathbf{X}_{*2} \,|\, \cdots \,|\, \mathbf{X}_{*n}]$ solves the equation $\mathbf{AX} = \mathbf{I}$, and hence $\mathbf{X} = \mathbf{A}^{-1}$. If $\mathbf{A}$ is nonsingular, then we know from (3.7.7) that the Gauss–Jordan method reduces the augmented matrix $[\mathbf{A} \,|\, \mathbf{I}_{*j}]$ to $[\mathbf{I} \,|\, \mathbf{X}_{*j}]$, and the results of §1.3 insure that $\mathbf{X}_{*j}$ is the unique solution to $\mathbf{Ax} = \mathbf{I}_{*j}$. That is,

$$[\mathbf{A} \,|\, \mathbf{I}_{*j}] \xrightarrow{\text{Gauss–Jordan}} \left[\mathbf{I} \,\middle|\, [\mathbf{A}^{-1}]_{*j}\right].$$

But rather than solving each system $\mathbf{Ax} = \mathbf{I}_{*j}$ separately, we can solve them simultaneously by taking advantage of the fact that they all have the same coefficient matrix. In other words, applying the Gauss–Jordan method to the larger augmented array $[\mathbf{A} \,|\, \mathbf{I}_{*1} \,|\, \mathbf{I}_{*2} \,|\, \cdots \,|\, \mathbf{I}_{*n}]$ produces

$$[\mathbf{A} \,|\, \mathbf{I}_{*1} \,|\, \mathbf{I}_{*2} \,|\, \cdots \,|\, \mathbf{I}_{*n}] \xrightarrow{\text{Gauss–Jordan}} \left[\mathbf{I} \,\middle|\, [\mathbf{A}^{-1}]_{*1} \,\middle|\, [\mathbf{A}^{-1}]_{*2} \,\middle|\, \cdots \,\middle|\, [\mathbf{A}^{-1}]_{*n}\right],$$

or more compactly,

$$[\mathbf{A} \,|\, \mathbf{I}] \xrightarrow{\text{Gauss–Jordan}} [\mathbf{I} \,|\, \mathbf{A}^{-1}]. \tag{3.7.11}$$

What happens if we try to invert a singular matrix using this procedure? The fact that (3.7.5) $\Longleftrightarrow$ (3.7.6) $\Longleftrightarrow$ (3.7.7) guarantees that a singular matrix $\mathbf{A}$ cannot be reduced to $\mathbf{I}$ by Gauss–Jordan elimination because a zero row will have to emerge in the left-hand side of the augmented array at some point during the process. This means that we do not need to know at the outset whether $\mathbf{A}$ is nonsingular or singular—it becomes self-evident depending on whether or not the reduction (3.7.11) can be completed. A summary is given below.

Computing an Inverse

Gauss–Jordan elimination can be used to invert $\mathbf{A}$ by the reduction

$$[\mathbf{A} \,|\, \mathbf{I}] \xrightarrow{\text{Gauss–Jordan}} [\mathbf{I} \,|\, \mathbf{A}^{-1}]. \tag{3.7.12}$$

The only way for this reduction to fail is for a row of zeros to emerge in the left-hand side of the augmented array, and this occurs if and only if $\mathbf{A}$ is a singular matrix. A different (and somewhat more practical) algorithm is given Example 3.10.3 on p. 148.

Although they are not included in the simple examples of this section, you are reminded that the pivoting and scaling strategies presented in §1.5 need to be incorporated, and the effects of ill-conditioning discussed in §1.6 must be considered whenever matrix inverses are computed using floating-point arithmetic. However, practical applications rarely require an inverse to be computed.

Example 3.7.3

Problem: If possible, find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

Solution:

$$[\mathbf{A} \mid \mathbf{I}] = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|rrr} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|rrr} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

Therefore, the matrix is nonsingular, and $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. If we wish to check this answer, we need only check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. If this holds, then the result of Example 3.7.2 insures that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ will automatically be true.

Earlier in this section it was stated that one almost never solves a nonsingular linear system $\mathbf{Ax} = \mathbf{b}$ by first computing $\mathbf{A}^{-1}$ and then the product $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. To appreciate why this is true, pay attention to how much effort is required to perform one matrix inversion.

Operation Counts for Inversion

Computing $\mathbf{A}^{-1}_{n \times n}$ by reducing $[\mathbf{A}|\mathbf{I}]$ with Gauss–Jordan requires

- n^3 multiplications/divisions,
- $n^3 - 2n^2 + n$ additions/subtractions.

Interestingly, if Gaussian elimination with a back substitution process is applied to $[\mathbf{A}|\mathbf{I}]$ instead of the Gauss–Jordan technique, then exactly the same operation count can be obtained. Although Gaussian elimination with back substitution is more efficient than the Gauss–Jordan method for solving a single linear system, the two procedures are essentially equivalent for inversion.

Solving a nonsingular system $\mathbf{Ax} = \mathbf{b}$ by first computing $\mathbf{A}^{-1}$ and then forming the product $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ requires $n^3 + n^2$ multiplications/divisions and $n^3 - n^2$ additions/subtractions. Recall from §1.5 that Gaussian elimination with back substitution requires only about $n^3/3$ multiplications/divisions and about $n^3/3$ additions/subtractions. In other words, *using* $\mathbf{A}^{-1}$ *to solve a nonsingular system* $\mathbf{Ax} = \mathbf{b}$ *requires about three times the effort as does Gaussian elimination with back substitution.*

To put things in perspective, consider standard matrix multiplication between two $n \times n$ matrices. It is not difficult to verify that n^3 multiplications and $n^3 - n^2$ additions are required. Remarkably, it takes almost exactly as much effort to perform one matrix multiplication as to perform one matrix inversion. This fact always seems to be counter to a novice's intuition—it "feels" like matrix inversion should be a more difficult task than matrix multiplication, but this is not the case.

The remainder of this section is devoted to a discussion of some of the important properties of matrix inversion. We begin with the four basic facts listed below.

Properties of Matrix Inversion

For nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$, the following properties hold.

- $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$. (3.7.13)
- The product $\mathbf{AB}$ is also nonsingular. (3.7.14)
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (the reverse order law for inversion). (3.7.15)
- $\left(\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^T\right)^{-1}$ and $\left(\mathbf{A}^{-1}\right)^* = (\mathbf{A}^*)^{-1}$. (3.7.16)

Proof. Property (3.7.13) follows directly from the definition of inversion. To prove (3.7.14) and (3.7.15), let $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and verify that $(\mathbf{AB})\mathbf{X} = \mathbf{I}$ by writing

$$(\mathbf{AB})\mathbf{X} = (\mathbf{AB})\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{A}(\mathbf{I})\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}.$$

According to the discussion in Example 3.7.2, we are now guaranteed that $\mathbf{X}(\mathbf{AB}) = \mathbf{I}$, and we need not bother to verify it. To prove property (3.7.16), let $\mathbf{X} = \left(\mathbf{A}^{-1}\right)^T$ and verify that $\mathbf{A}^T\mathbf{X} = \mathbf{I}$. Make use of the reverse order law for transposition to write

$$\mathbf{A}^T\mathbf{X} = \mathbf{A}^T\left(\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^{-1}\mathbf{A}\right)^T = \mathbf{I}^T = \mathbf{I}.$$

Therefore, $\left(\mathbf{A}^T\right)^{-1} = \mathbf{X} = \left(\mathbf{A}^{-1}\right)^T$. The proof of the conjugate transpose case is similar. ∎

In general the product of two rank-r matrices does not necessarily have to produce another matrix of rank r. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

each has rank 1, but the product $\mathbf{AB} = \mathbf{0}$ has rank 0. However, we saw in (3.7.14) that the product of two invertible matrices is again invertible. That is, if $rank\,(A_{n\times n}) = n$ and $rank\,(B_{n\times n}) = n$, then $rank\,(\mathbf{AB}) = n$. This generalizes to any number of matrices.

Products of Nonsingular Matrices Are Nonsingular

If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ are each $n \times n$ nonsingular matrices, then the product $\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k$ is also nonsingular, and its inverse is given by the reverse order law. That is,

$$(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}.$$

Proof. Apply (3.7.14) and (3.7.15) inductively. For example, when $k = 3$ you can write

$$(\mathbf{A}_1\{\mathbf{A}_2\mathbf{A}_3\})^{-1} = \{\mathbf{A}_2\mathbf{A}_3\}^{-1}\mathbf{A}_1^{-1} = \mathbf{A}_3^{-1}\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}. \quad \blacksquare$$

Exercises for section 3.7

3.7.1. When possible, find the inverse of each of the following matrices. Check your answer by using matrix multiplication.

$$\text{(a)} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{pmatrix}$$

$$\text{(d)} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{(e)} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

3.7.2. Find the matrix $\mathbf{X}$ such that $\mathbf{X} = \mathbf{AX} + \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}.$$

3.7.3. For a square matrix $\mathbf{A}$, explain why each of the following statements must be true.

(a) If $\mathbf{A}$ contains a zero row or a zero column, then $\mathbf{A}$ is singular.
(b) If $\mathbf{A}$ contains two identical rows or two identical columns, then $\mathbf{A}$ is singular.
(c) If one row (or column) is a multiple of another row (or column), then $\mathbf{A}$ must be singular.

3.7.4. Answer each of the following questions.

(a) Under what conditions is a diagonal matrix nonsingular? Describe the structure of the inverse of a diagonal matrix.
(b) Under what conditions is a triangular matrix nonsingular? Describe the structure of the inverse of a triangular matrix.

3.7.5. If $\mathbf{A}$ is nonsingular and symmetric, prove that $\mathbf{A}^{-1}$ is symmetric.

3.7.6. If $\mathbf{A}$ is a square matrix such that $\mathbf{I}-\mathbf{A}$ is nonsingular, prove that

$$\mathbf{A}(\mathbf{I}-\mathbf{A})^{-1} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{A}.$$

3.7.7. Prove that if $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times m$ such that $\mathbf{AB} = \mathbf{I}_m$ and $\mathbf{BA} = \mathbf{I}_n$, then $m = n$.

3.7.8. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$ are each nonsingular, prove that

$$\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} = \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} = \left(\mathbf{A}^{-1}+\mathbf{B}^{-1}\right)^{-1}.$$

3.7.9. Let $\mathbf{S}$ be a skew-symmetric matrix with real entries.

(a) Prove that $\mathbf{I}-\mathbf{S}$ is nonsingular. **Hint:** $\mathbf{x}^T\mathbf{x} = 0 \implies \mathbf{x} = \mathbf{0}$.
(b) If $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1}$, show that $\mathbf{A}^{-1} = \mathbf{A}^T$.

3.7.10. For matrices $\mathbf{A}_{r\times r}$, $\mathbf{B}_{s\times s}$, and $\mathbf{C}_{r\times s}$ such that $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, verify that each of the following is true.

(a) $\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}$

(b) $\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}$

3.7.11. Consider the block matrix $\begin{pmatrix} \mathbf{A}_{r\times r} & \mathbf{C}_{r\times s} \\ \mathbf{R}_{s\times r} & \mathbf{B}_{s\times s} \end{pmatrix}$. When the indicated inverses exist, the matrices defined by

$$\mathbf{S} = \mathbf{B} - \mathbf{R}\mathbf{A}^{-1}\mathbf{C} \quad \text{and} \quad \mathbf{T} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{R}$$

are called the ***Schur complements***[20] of $\mathbf{A}$ and $\mathbf{B}$, respectively.

(a) If $\mathbf{A}$ and $\mathbf{S}$ are both nonsingular, verify that

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{R} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{R}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{R}\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}.$$

(b) If $\mathbf{B}$ and $\mathbf{T}$ are nonsingular, verify that

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{R} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{R}\mathbf{T}^{-1} & \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{R}\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}.$$

3.7.12. Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are $n \times n$ matrices such that $\mathbf{A}\mathbf{B}^T$ and $\mathbf{C}\mathbf{D}^T$ are each symmetric and $\mathbf{A}\mathbf{D}^T - \mathbf{B}\mathbf{C}^T = \mathbf{I}$. Prove that

$$\mathbf{A}^T\mathbf{D} - \mathbf{C}^T\mathbf{B} = \mathbf{I}.$$

[20] This is named in honor of the German mathematician Issai Schur (1875–1941), who first studied matrices of this type. Schur was a student and collaborator of Ferdinand Georg Frobenius (p. 662). Schur and Frobenius were among the first to study matrix theory as a discipline unto itself, and each made great contributions to the subject. It was Emilie V. Haynsworth (1916–1987)—a mathematical granddaughter of Schur—who introduced the phrase "Schur complement" and developed several important aspects of the concept.

3.8 INVERSES OF SUMS AND SENSITIVITY

The reverse order law for inversion makes the inverse of a product easy to deal with, but the inverse of a sum is much more difficult. To begin with, $(\mathbf{A}+\mathbf{B})^{-1}$ may not exist even if $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ each exist. Moreover, if $(\mathbf{A}+\mathbf{B})^{-1}$ exists, then, with rare exceptions, $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1}+\mathbf{B}^{-1}$. This doesn't even hold for scalars (i.e., 1×1 matrices), so it has no chance of holding in general.

There is no useful general formula for $(\mathbf{A}+\mathbf{B})^{-1}$, but there are some special sums for which something can be said. One of the most easily inverted sums is $\mathbf{I}+\mathbf{c}\mathbf{d}^T$ in which $\mathbf{c}$ and $\mathbf{d}$ are $n\times 1$ nonzero columns such that $1+\mathbf{d}^T\mathbf{c}\neq 0$. It's straightforward to verify by direct multiplication that

$$\left(\mathbf{I}+\mathbf{c}\mathbf{d}^T\right)^{-1} = \mathbf{I} - \frac{\mathbf{c}\mathbf{d}^T}{1+\mathbf{d}^T\mathbf{c}}. \tag{3.8.1}$$

If $\mathbf{I}$ is replaced by a nonsingular matrix $\mathbf{A}$ satisfying $1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\neq 0$, then the reverse order law for inversion in conjunction with (3.8.1) yields

$$\begin{aligned}(\mathbf{A}+\mathbf{c}\mathbf{d}^T)^{-1} &= \left(\mathbf{A}(\mathbf{I}+\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T)\right)^{-1} = (\mathbf{I}+\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T)^{-1}\mathbf{A}^{-1}\\ &= \left(\mathbf{I}-\frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T}{1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}\right)\mathbf{A}^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}}{1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}.\end{aligned}$$

This is often called the Sherman–Morrison[21] rank-one update formula because it can be shown (Exercise 3.9.9, p. 140) that $rank\,(\mathbf{c}\mathbf{d}^T)=1$ when $\mathbf{c}\neq\mathbf{0}\neq\mathbf{d}$.

Sherman–Morrison Formula

- If $\mathbf{A}_{n\times n}$ is nonsingular and if $\mathbf{c}$ and $\mathbf{d}$ are $n\times 1$ columns such that $1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\neq 0$, then the sum $\mathbf{A}+\mathbf{c}\mathbf{d}^T$ is nonsingular, and

$$\left(\mathbf{A}+\mathbf{c}\mathbf{d}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}}{1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}. \tag{3.8.2}$$

- The *Sherman–Morrison–Woodbury formula* is a generalization. If $\mathbf{C}$ and $\mathbf{D}$ are $n\times k$ such that $(\mathbf{I}+\mathbf{D}^T\mathbf{A}^{-1}\mathbf{C})^{-1}$ exists, then

$$(\mathbf{A}+\mathbf{C}\mathbf{D}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{I}+\mathbf{D}^T\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{D}^T\mathbf{A}^{-1}. \tag{3.8.3}$$

[21] This result appeared in the 1949–1950 work of American statisticians J. Sherman and W. J. Morrison, but they were not the first to discover it. The formula was independently presented by the English mathematician W. J. Duncan in 1944 and by American statisticians L. Guttman (1946), Max Woodbury (1950), and M. S. Bartlett (1951). Since its derivation is so natural, it almost certainly was discovered by many others along the way. Recognition and fame are often not afforded simply for introducing an idea, but rather for applying the idea to a useful end.

The Sherman–Morrison–Woodbury formula (3.8.3) can be verified with direct multiplication, or it can be derived as indicated in Exercise 3.8.6.

To appreciate the utility of the Sherman–Morrison formula, suppose $\mathbf{A}^{-1}$ is known from a previous calculation, but now one entry in $\mathbf{A}$ needs to be changed or updated—say we need to add α to a_{ij}. It's not necessary to start from scratch to compute the new inverse because Sherman–Morrison shows how the previously computed information in $\mathbf{A}^{-1}$ can be updated to produce the new inverse. Let $\mathbf{c} = \mathbf{e}_i$ and $\mathbf{d} = \alpha\mathbf{e}_j$, where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the i^{th} and j^{th} unit columns, respectively. The matrix $\mathbf{cd}^T$ has α in the (i, j)-position and zeros elsewhere so that

$$\mathbf{B} = \mathbf{A} + \mathbf{cd}^T = \mathbf{A} + \alpha\mathbf{e}_i\mathbf{e}_j^T$$

is the updated matrix. According to the Sherman–Morrison formula,

$$\begin{aligned}\mathbf{B}^{-1} = \left(\mathbf{A} + \alpha\mathbf{e}_i\mathbf{e}_j^T\right)^{-1} &= \mathbf{A}^{-1} - \alpha\frac{\mathbf{A}^{-1}\mathbf{e}_i\mathbf{e}_j^T\mathbf{A}^{-1}}{1 + \alpha\mathbf{e}_j^T\mathbf{A}^{-1}\mathbf{e}_i} \\ &= \mathbf{A}^{-1} - \alpha\frac{[\mathbf{A}^{-1}]_{*i}[\mathbf{A}^{-1}]_{j*}}{1 + \alpha[\mathbf{A}^{-1}]_{ji}} \quad \text{(recall Exercise 3.5.4).}\end{aligned} \tag{3.8.4}$$

This shows how $\mathbf{A}^{-1}$ changes when a_{ij} is perturbed, and it provides a useful algorithm for updating $\mathbf{A}^{-1}$.

Example 3.8.1

Problem: Start with $\mathbf{A}$ and $\mathbf{A}^{-1}$ given below. Update $\mathbf{A}$ by adding 1 to a_{21}, and then use the Sherman–Morrison formula to update $\mathbf{A}^{-1}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}.$$

Solution: The updated matrix is

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \mathbf{A} + \mathbf{e}_2\mathbf{e}_1^T.$$

Applying the Sherman–Morrison formula yields the updated inverse

$$\begin{aligned}\mathbf{B}^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{e}_2\mathbf{e}_1^T\mathbf{A}^{-1}}{1 + \mathbf{e}_1^T\mathbf{A}^{-1}\mathbf{e}_2} &= \mathbf{A}^{-1} - \frac{[\mathbf{A}^{-1}]_{*2}[\mathbf{A}^{-1}]_{1*}}{1 + [\mathbf{A}^{-1}]_{12}} \\ = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} - \frac{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \end{pmatrix}}{1 - 2} &= \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}.\end{aligned}$$

Another sum that often requires inversion is $\mathbf{I}-\mathbf{A}$, but we have to be careful because $(\mathbf{I}-\mathbf{A})^{-1}$ need not always exist. However, we are safe when the entries in $\mathbf{A}$ are sufficiently small. In particular, if the entries in $\mathbf{A}$ are small enough in magnitude to insure that $\lim_{n\to\infty}\mathbf{A}^n=\mathbf{0}$, then, analogous to scalar algebra,

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{n-1})=\mathbf{I}-\mathbf{A}^n\to\mathbf{I}\quad\text{as}\quad n\to\infty,$$

so we have the following matrix version of a geometric series.

Neumann Series

If $\lim_{n\to\infty}\mathbf{A}^n=\mathbf{0}$, then $\mathbf{I}-\mathbf{A}$ is nonsingular and

$$(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots=\sum_{k=0}^{\infty}\mathbf{A}^k. \tag{3.8.5}$$

This is the ***Neumann series.*** It provides approximations of $(\mathbf{I}-\mathbf{A})^{-1}$ when $\mathbf{A}$ has entries of small magnitude. For example, a first-order approximation is $(\mathbf{I}-\mathbf{A})^{-1}\approx\mathbf{I}+\mathbf{A}$. More on the Neumann series appears in Example 7.3.1, p. 527, and the complete statement is developed on p. 618.

While there is no useful formula for $(\mathbf{A}+\mathbf{B})^{-1}$ in general, the Neumann series allows us to say something when $\mathbf{B}$ has small entries relative to $\mathbf{A}$, or vice versa. For example, if $\mathbf{A}^{-1}$ exists, and if the entries in $\mathbf{B}$ are small enough in magnitude to insure that $\lim_{n\to\infty}\left(\mathbf{A}^{-1}\mathbf{B}\right)^n=\mathbf{0}$, then

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^{-1}&=\Big(\mathbf{A}\left(\mathbf{I}-\left[-\mathbf{A}^{-1}\mathbf{B}\right]\right)\Big)^{-1}=\Big(\mathbf{I}-\left[-\mathbf{A}^{-1}\mathbf{B}\right]\Big)^{-1}\mathbf{A}^{-1}\\&=\left(\sum_{k=0}^{\infty}\left[-\mathbf{A}^{-1}\mathbf{B}\right]^k\right)\mathbf{A}^{-1},\end{aligned}$$

and a first-order approximation is

$$(\mathbf{A}+\mathbf{B})^{-1}\approx\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{3.8.6}$$

Consequently, if $\mathbf{A}$ is perturbed by a small matrix $\mathbf{B}$, possibly resulting from errors due to inexact measurements or perhaps from roundoff error, then the resulting change in $\mathbf{A}^{-1}$ is about $\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$. In other words, the effect of a small perturbation (or error) $\mathbf{B}$ is magnified by multiplication (on both sides) with $\mathbf{A}^{-1}$, so if $\mathbf{A}^{-1}$ has large entries, small perturbations (or errors) in $\mathbf{A}$ can produce large perturbations (or errors) in the resulting inverse. You can reach essentially the same conclusion from (3.8.4) when only a single entry is perturbed and from Exercise 3.8.2 when a single column is perturbed.

This discussion resolves, at least in part, an issue raised in §1.6—namely, "What mechanism determines the extent to which a nonsingular system $\mathbf{Ax} = \mathbf{b}$ is ill-conditioned?" To see how, an aggregate measure of the magnitude of the entries in $\mathbf{A}$ is needed, and one common measure is

$$\|\mathbf{A}\| = \max_i \sum_j |a_{ij}| = \text{the maximum absolute row sum.} \tag{3.8.7}$$

This is one example of a *matrix norm,* a detailed discussion of which is given in §5.1. Theoretical properties specific to (3.8.7) are developed on pp. 280 and 283, and one property established there is the fact that $\|\mathbf{XY}\| \le \|\mathbf{X}\|\,\|\mathbf{Y}\|$ for all conformable matrices $\mathbf{X}$ and $\mathbf{Y}$. But let's keep things on an intuitive level for the time being and defer the details. Using the norm (3.8.7), the approximation (3.8.6) insures that if $\|\mathbf{B}\|$ is sufficiently small, then

$$\left\|\mathbf{A}^{-1} - (\mathbf{A}+\mathbf{B})^{-1}\right\| \approx \left\|\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\right\| \le \left\|\mathbf{A}^{-1}\right\|\,\|\mathbf{B}\|\,\left\|\mathbf{A}^{-1}\right\|,$$

so, if we interpret $x \lesssim y$ to mean that x is bounded above by something not far from y, we can write

$$\frac{\left\|\mathbf{A}^{-1} - (\mathbf{A}+\mathbf{B})^{-1}\right\|}{\left\|\mathbf{A}^{-1}\right\|} \lesssim \left\|\mathbf{A}^{-1}\right\|\,\|\mathbf{B}\| = \left\|\mathbf{A}^{-1}\right\|\,\|\mathbf{A}\|\left\{\frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}\right\}.$$

The term on the left is the relative change in the inverse, and $\|\mathbf{B}\|\,/\,\|\mathbf{A}\|$ is the relative change in $\mathbf{A}$. The number $\kappa = \left\|\mathbf{A}^{-1}\right\|\,\|\mathbf{A}\|$ is therefore the "magnification factor" that dictates how much the relative change in $\mathbf{A}$ is magnified. This magnification factor κ is called a ***condition number*** for $\mathbf{A}$. In other words, if κ is small relative to 1 (i.e., if $\mathbf{A}$ is *well conditioned*), then a small relative change (or error) in $\mathbf{A}$ cannot produce a large relative change (or error) in the inverse, but if κ is large (i.e., if $\mathbf{A}$ is *ill conditioned*), then a small relative change (or error) in $\mathbf{A}$ can possibly (but not necessarily) result in a large relative change (or error) in the inverse.

The situation for linear systems is similar. If the coefficients in a nonsingular system $\mathbf{Ax} = \mathbf{b}$ are slightly perturbed to produce the system $(\mathbf{A}+\mathbf{B})\tilde{\mathbf{x}} = \mathbf{b}$, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ and $\tilde{\mathbf{x}} = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{b}$ so that (3.8.6) implies

$$\mathbf{x} - \tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{b} - (\mathbf{A}+\mathbf{B})^{-1}\mathbf{b} \approx \mathbf{A}^{-1}\mathbf{b} - \left(\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\right)\mathbf{b} = \mathbf{A}^{-1}\mathbf{B}\mathbf{x}.$$

For column vectors, (3.8.7) reduces to $\|\mathbf{x}\| = \max_i |x_i|$, and we have

$$\|\mathbf{x} - \tilde{\mathbf{x}}\| \lesssim \left\|\mathbf{A}^{-1}\right\|\,\|\mathbf{B}\|\,\|\mathbf{x}\|,$$

so the relative change in the solution is

$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \lesssim \left\|\mathbf{A}^{-1}\right\| \|\mathbf{B}\| = \left\|\mathbf{A}^{-1}\right\| \|\mathbf{A}\| \left\{\frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}\right\} = \kappa \left\{\frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}\right\}. \tag{3.8.8}$$

Again, the condition number κ is pivotal because when κ is small, a small relative change in $\mathbf{A}$ cannot produce a large relative change in $\mathbf{x}$, but for larger values of κ, a small relative change in $\mathbf{A}$ can possibly result in a large relative change in $\mathbf{x}$. Below is a summary of these observations.

Sensitivity and Conditioning

- A nonsingular matrix $\mathbf{A}$ is said to be ***ill conditioned*** if a small relative change in $\mathbf{A}$ can cause a large relative change in $\mathbf{A}^{-1}$. The degree of ill-conditioning is gauged by a ***condition number*** $\kappa = \|\mathbf{A}\| \left\|\mathbf{A}^{-1}\right\|$, where $\|\star\|$ is a matrix norm.
- The sensitivity of the solution of $\mathbf{Ax} = \mathbf{b}$ to perturbations (or errors) in $\mathbf{A}$ is measured by the extent to which $\mathbf{A}$ is an ill-conditioned matrix. More is said in Example 5.12.1 on p. 414.

Example 3.8.2

It was demonstrated in Example 1.6.1 that the system

$$\begin{aligned} .835x + .667y &= .168, \\ .333x + .266y &= .067, \end{aligned}$$

is sensitive to small perturbations. We can understand this in the current context by examining the condition number of the coefficient matrix. If the matrix norm (3.8.7) is employed with

$$\mathbf{A} = \begin{pmatrix} .835 & .667 \\ .333 & .266 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{pmatrix} -266000 & 667000 \\ 333000 & -835000 \end{pmatrix},$$

then the condition number for $\mathbf{A}$ is

$$\kappa = \kappa = \|\mathbf{A}\| \left\|\mathbf{A}^{-1}\right\| = (1.502)(1168000) = 1,754,336 \approx 1.7 \times 10^6.$$

Since the right-hand side of (3.8.8) is only an estimate of the relative error in the solution, the exact value of κ is not as important as its order of magnitude. Because κ is of order 10^6, (3.8.8) holds the possibility that the relative change (or error) in the solution can be about a million times larger than the relative change (or error) in $\mathbf{A}$. Therefore, we must consider $\mathbf{A}$ and the associated linear system to be ill conditioned.

A Rule of Thumb. If Gaussian elimination with partial pivoting is used to solve a well-scaled nonsingular system $\mathbf{Ax} = \mathbf{b}$ using t-digit floating-point arithmetic, then, assuming no other source of error exists, it can be argued that when κ is of order 10^p, the computed solution is expected to be accurate to at least $t - p$ significant digits, more or less. In other words, one expects to lose roughly p significant figures. For example, if Gaussian elimination with 8-digit arithmetic is used to solve the 2×2 system given above, then only about $t - p = 8 - 6 = 2$ significant figures of accuracy should be expected. This doesn't preclude the possibility of getting lucky and attaining a higher degree of accuracy—it just says that you shouldn't bet the farm on it.

The complete story of conditioning has not yet been told. As pointed out earlier, it's about three times more costly to compute $\mathbf{A}^{-1}$ than to solve $\mathbf{Ax} = \mathbf{b}$, so it doesn't make sense to compute $\mathbf{A}^{-1}$ just to estimate the condition of $\mathbf{A}$. Questions concerning condition estimation without explicitly computing an inverse still need to be addressed. Furthermore, liberties allowed by using the $\approx$ and $\lesssim$ symbols produce results that are intuitively correct but not rigorous. Rigor will eventually be attained—see Example 5.12.1on p. 414.

Exercises for section 3.8

3.8.1. Suppose you are given that

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

(a) Use the Sherman–Morrison formula to determine the inverse of the matrix $\mathbf{B}$ that is obtained by changing the $(3, 2)$-entry in $\mathbf{A}$ from 0 to 2.

(b) Let $\mathbf{C}$ be the matrix that agrees with $\mathbf{A}$ except that $c_{32} = 2$ and $c_{33} = 2$. Use the Sherman–Morrison formula to find $\mathbf{C}^{-1}$.

3.8.2. Suppose $\mathbf{A}$ and $\mathbf{B}$ are nonsingular matrices in which $\mathbf{B}$ is obtained from $\mathbf{A}$ by replacing $\mathbf{A}_{*j}$ with another column $\mathbf{b}$. Use the Sherman–Morrison formula to derive the fact that

$$\mathbf{B}^{-1} = \mathbf{A}^{-1} - \frac{\left(\mathbf{A}^{-1}\mathbf{b} - \mathbf{e}_j\right)[\mathbf{A}^{-1}]_{j*}}{[\mathbf{A}^{-1}]_{j*}\mathbf{b}}.$$

3.8.3. Suppose the coefficient matrix of a nonsingular system $\mathbf{Ax} = \mathbf{b}$ is updated to produce another nonsingular system $(\mathbf{A} + \mathbf{cd}^T)\mathbf{z} = \mathbf{b}$, where $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \Re^{n\times 1}$, and let $\mathbf{y}$ be the solution of $\mathbf{Ay} = \mathbf{c}$. Show that $\mathbf{z} = \mathbf{x} - \mathbf{y}\mathbf{d}^T\mathbf{x}/(1 + \mathbf{d}^T\mathbf{y})$.

3.8.4. (a) Use the Sherman–Morrison formula to prove that if $\mathbf{A}$ is nonsingular, then $\mathbf{A} + \alpha\mathbf{e}_i\mathbf{e}_j^T$ is nonsingular for a sufficiently small α.

(b) Use part (a) to prove that $\mathbf{I} + \mathbf{E}$ is nonsingular when all ϵ_{ij} 's are sufficiently small in magnitude. This is an alternative to using the Neumann series argument.

3.8.5. For given matrices $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ is nonsingular, explain why $\mathbf{A} + \epsilon\mathbf{B}$ is also nonsingular when the real number ϵ is constrained to a sufficiently small interval about the origin. In other words, prove that small perturbations of nonsingular matrices are also nonsingular.

3.8.6. Derive the Sherman–Morrison–Woodbury formula. **Hint:** Recall Exercise 3.7.11, and consider the product $\begin{pmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{D}^T & -\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}^T & \mathbf{I} \end{pmatrix}$.

3.8.7. Using the norm (3.8.7), rank the following matrices according to their degree of ill-conditioning:

$$\mathbf{A} = \begin{pmatrix} 100 & 0 & -100 \\ 0 & 100 & -100 \\ -100 & -100 & 300 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 8 & -1 \\ -9 & -71 & 11 \\ 1 & 17 & 18 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & 22 & -42 \\ 0 & 1 & -45 \\ -45 & -948 & 1 \end{pmatrix}.$$

3.8.8. Suppose that the entries in $\mathbf{A}(t)$, $\mathbf{x}(t)$, and $\mathbf{b}(t)$ are differentiable functions of a real variable t such that $\mathbf{A}(t)\mathbf{x}(t) = \mathbf{b}(t)$.

(a) Assuming that $\mathbf{A}(t)^{-1}$ exists, explain why

$$\frac{d\mathbf{A}(t)^{-1}}{dt} = -\mathbf{A}(t)^{-1}\mathbf{A}'(t)\mathbf{A}(t)^{-1}.$$

(b) Derive the equation

$$\mathbf{x}'(t) = \mathbf{A}(t)^{-1}\mathbf{b}'(t) - \mathbf{A}(t)^{-1}\mathbf{A}'(t)\mathbf{x}(t).$$

This shows that $\mathbf{A}^{-1}$ magnifies both the change in $\mathbf{A}$ and the change in $\mathbf{b}$, and thus it confirms the observation derived from (3.8.8) saying that the sensitivity of a nonsingular system to small perturbations is directly related to the magnitude of the entries in $\mathbf{A}^{-1}$.

3.9 ELEMENTARY MATRICES AND EQUIVALENCE

A common theme in mathematics is to break complicated objects into more elementary components, such as factoring large polynomials into products of smaller polynomials. The purpose of this section is to lay the groundwork for similar ideas in matrix algebra by considering how a general matrix might be factored into a product of more "elementary" matrices.

Elementary Matrices

Matrices of the form $\mathbf{I}-\mathbf{u}\mathbf{v}^T$, where $\mathbf{u}$ and $\mathbf{v}$ are $n\times 1$ columns such that $\mathbf{v}^T\mathbf{u} \neq 1$ are called ***elementary matrices***, and we know from (3.8.1) that all such matrices are nonsingular and

$$\left(\mathbf{I}-\mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{I} - \frac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u}-1}. \tag{3.9.1}$$

Notice that inverses of elementary matrices are elementary matrices.

We are primarily interested in the elementary matrices associated with the three elementary row (or column) operations hereafter referred to as follows.

- Type I is interchanging rows (columns) i and j.
- Type II is multiplying row (column) i by $\alpha \neq 0$.
- Type III is adding a multiple of row (column) i to row (column) j.

An elementary matrix of Type I, II, or III is created by performing an elementary operation of Type I, II, or III to an identity matrix. For example, the matrices

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \tag{3.9.2}$$

are elementary matrices of Types I, II, and III, respectively, because $\mathbf{E}_1$ arises by interchanging rows 1 and 2 in $\mathbf{I}_3$, whereas $\mathbf{E}_2$ is generated by multiplying row 2 in $\mathbf{I}_3$ by α, and $\mathbf{E}_3$ is constructed by multiplying row 1 in $\mathbf{I}_3$ by α and adding the result to row 3. The matrices in (3.9.2) also can be generated by column operations. For example, $\mathbf{E}_3$ can be obtained by adding α times the third column of $\mathbf{I}_3$ to the first column. The fact that $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are of the form (3.9.1) follows by using the unit columns $\mathbf{e}_i$ to write

$$\mathbf{E}_1 = \mathbf{I}-\mathbf{u}\mathbf{u}^T, \text{ where } \mathbf{u} = \mathbf{e}_1-\mathbf{e}_2, \quad \mathbf{E}_2 = \mathbf{I}-(1-\alpha)\mathbf{e}_2\mathbf{e}_2^T, \quad \text{and} \quad \mathbf{E}_3 = \mathbf{I}+\alpha\mathbf{e}_3\mathbf{e}_1^T.$$

These observations generalize to matrices of arbitrary size.

One of our objectives is to remove the arrows from Gaussian elimination because the inability to do "arrow algebra" limits the theoretical analysis. For example, while it makes sense to add two equations together, there is no meaningful analog for arrows—reducing $\mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \to \mathbf{D}$ by row operations does not guarantee that $\mathbf{A} + \mathbf{C} \to \mathbf{B} + \mathbf{D}$ is possible. The following properties are the mechanisms needed to remove the arrows from elimination processes.

Properties of Elementary Matrices

- When used as a *left-hand* multiplier, an elementary matrix of Type I, II, or III executes the corresponding *row* operation.
- When used as a *right-hand* multiplier, an elementary matrix of Type I, II, or III executes the corresponding *column* operation.

Proof. A proof for Type III operations is given—the other two cases are left to the reader. Using $\mathbf{I} + \alpha\mathbf{e}_j\mathbf{e}_i^T$ as a left-hand multiplier on an arbitrary matrix $\mathbf{A}$ produces

$$\left(\mathbf{I} + \alpha\mathbf{e}_j\mathbf{e}_i^T\right)\mathbf{A} = \mathbf{A} + \alpha\mathbf{e}_j\mathbf{A}_{i*} = \mathbf{A} + \alpha \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{array}{l} \\ \\ \longleftarrow j^{th} \text{ row}. \\ \\ \\ \end{array}$$

This is exactly the matrix produced by a Type III row operation in which the i^{th} row of $\mathbf{A}$ is multiplied by α and added to the j^{th} row. When $\mathbf{I} + \alpha\mathbf{e}_j\mathbf{e}_i^T$ is used as a right-hand multiplier on $\mathbf{A}$, the result is

$$\mathbf{A}\left(\mathbf{I} + \alpha\mathbf{e}_j\mathbf{e}_i^T\right) = \mathbf{A} + \alpha\mathbf{A}_{*j}\mathbf{e}_i^T = \mathbf{A} + \alpha \begin{array}{c} \qquad\quad i^{th} \text{ col} \\ \qquad\quad \downarrow \\ \begin{pmatrix} 0 & \cdots & a_{1j} & \cdots & 0 \\ 0 & \cdots & a_{2j} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{nj} & \cdots & 0 \end{pmatrix} \end{array}.$$

This is the result of a Type III column operation in which the j^{th} column of $\mathbf{A}$ is multiplied by α and then added to the i^{th} column. ∎

Example 3.9.1

The sequence of row operations used to reduce $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 13 \end{pmatrix}$ to $\mathbf{E_A}$ is indicated below.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 13 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\textit{Interchange } R_2 \textit{ and } R_3} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 4R_2 \\ \\ \\ \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{E_A}.$$

The reduction can be accomplished by a sequence of left-hand multiplications with the corresponding elementary matrices as shown below.

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} = \mathbf{E_A}.$$

The product of these elementary matrices is $\mathbf{P} = \begin{pmatrix} 13 & 0 & -4 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$, and you can verify that it is indeed the case that $\mathbf{PA} = \mathbf{E_A}$. Thus the arrows are eliminated by replacing them with a product of elementary matrices.

We are now in a position to understand why nonsingular matrices are precisely those matrices that can be factored as a product of elementary matrices.

Products of Elementary Matrices

- $\mathbf{A}$ is a nonsingular matrix if and only if $\mathbf{A}$ is the product of elementary matrices of Type I, II, or III. (3.9.3)

Proof. If $\mathbf{A}$ is nonsingular, then the Gauss–Jordan technique reduces $\mathbf{A}$ to $\mathbf{I}$ by row operations. If $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k$ is the sequence of elementary matrices that corresponds to the elementary row operations used, then

$$\mathbf{G}_k \cdots \mathbf{G}_2 \mathbf{G}_1 \mathbf{A} = \mathbf{I} \text{ or, equivalently, } \mathbf{A} = \mathbf{G}_1^{-1} \mathbf{G}_2^{-1} \cdots \mathbf{G}_k^{-1}.$$

Since the inverse of an elementary matrix is again an elementary matrix of the same type, this proves that $\mathbf{A}$ is the product of elementary matrices of Type I, II, or III. Conversely, if $\mathbf{A} = \mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_k$ is a product of elementary matrices, then $\mathbf{A}$ must be nonsingular because the $\mathbf{E}_i$'s are nonsingular, and a product of nonsingular matrices is also nonsingular. ■

Equivalence

- Whenever $\mathbf{B}$ can be derived from $\mathbf{A}$ by a combination of elementary row and column operations, we write $\mathbf{A} \sim \mathbf{B}$, and we say that $\mathbf{A}$ and $\mathbf{B}$ are ***equivalent matrices***. Since elementary row and column operations are left-hand and right-hand multiplication by elementary matrices, respectively, and in view of (3.9.3), we can say that
$$\mathbf{A} \sim \mathbf{B} \iff \mathbf{PAQ} = \mathbf{B} \quad \text{for nonsingular } \mathbf{P} \text{ and } \mathbf{Q}.$$
- Whenever $\mathbf{B}$ can be obtained from $\mathbf{A}$ by performing a sequence of elementary *row* operations only, we write $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, and we say that $\mathbf{A}$ and $\mathbf{B}$ are ***row equivalent***. In other words,
$$\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \iff \mathbf{PA} = \mathbf{B} \quad \text{for a nonsingular } \mathbf{P}.$$
- Whenever $\mathbf{B}$ can be obtained from $\mathbf{A}$ by performing a sequence of *column* operations only, we write $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$, and we say that $\mathbf{A}$ and $\mathbf{B}$ are ***column equivalent***. In other words,
$$\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \iff \mathbf{AQ} = \mathbf{B} \quad \text{for a nonsingular } \mathbf{Q}.$$

If it's possible to go from $\mathbf{A}$ to $\mathbf{B}$ by elementary row and column operations, then clearly it's possible to start with $\mathbf{B}$ and get back to $\mathbf{A}$ because elementary operations are reversible—i.e., $\mathbf{PAQ} = \mathbf{B} \implies \mathbf{P}^{-1}\mathbf{B}\mathbf{Q}^{-1} = \mathbf{A}$. It therefore makes sense to talk about the equivalence of a pair of matrices without regard to order. In other words, $\mathbf{A} \sim \mathbf{B} \iff \mathbf{B} \sim \mathbf{A}$. Furthermore, it's not difficult to see that each type of equivalence is ***transitive*** in the sense that

$$\mathbf{A} \sim \mathbf{B} \quad \text{and} \quad \mathbf{B} \sim \mathbf{C} \implies \mathbf{A} \sim \mathbf{C}.$$

In §2.2 it was stated that each matrix $\mathbf{A}$ possesses a unique reduced row echelon form $\mathbf{E_A}$, and we accepted this fact because it is intuitively evident. However, we are now in a position to understand a rigorous proof.

Example 3.9.2

Problem: Prove that $\mathbf{E_A}$ is uniquely determined by $\mathbf{A}$.

Solution: Without loss of generality, we may assume that $\mathbf{A}$ is square—otherwise the appropriate number of zero rows or columns can be adjoined to $\mathbf{A}$ without affecting the results. Suppose that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E}_1$ and $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E}_2$, where $\mathbf{E}_1$ and $\mathbf{E}_2$ are both in reduced row echelon form. Consequently, $\mathbf{E}_1 \overset{\text{row}}{\sim} \mathbf{E}_2$, and hence there is a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{PE}_1 = \mathbf{E}_2. \tag{3.9.4}$$

Furthermore, by permuting the rows of $\mathbf{E}_1$ and $\mathbf{E}_2$ to force the pivotal 1's to occupy the diagonal positions, we see that

$$\mathbf{E}_1 \overset{\text{row}}{\sim} \mathbf{T}_1 \quad \text{and} \quad \mathbf{E}_2 \overset{\text{row}}{\sim} \mathbf{T}_2, \tag{3.9.5}$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are upper-triangular matrices in which the basic columns in each $\mathbf{T}_i$ occupy the same positions as the basic columns in $\mathbf{E}_i$. For example, if

$$\mathbf{E} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ then } \mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Each $\mathbf{T}_i$ has the property that $\mathbf{T}_i^2 = \mathbf{T}_i$ because there is a ***permutation matrix*** $\mathbf{Q}_i$ (a product of elementary interchange matrices of Type I) such that

$$\mathbf{Q}_i\mathbf{T}_i\mathbf{Q}_i^T = \begin{pmatrix} \mathbf{I}_{r_i} & \mathbf{J}_i \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ or, equivalently, } \mathbf{T}_i = \mathbf{Q}_i^T \begin{pmatrix} \mathbf{I}_{r_i} & \mathbf{J}_i \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{Q}_i,$$

and $\mathbf{Q}_i^T = \mathbf{Q}_i^{-1}$ (see Exercise 3.9.4) implies $\mathbf{T}_i^2 = \mathbf{T}_i$. It follows from (3.9.5) that $\mathbf{T}_1 \overset{\text{row}}{\sim} \mathbf{T}_2$, so there is a nonsingular matrix $\mathbf{R}$ such that $\mathbf{R}\mathbf{T}_1 = \mathbf{T}_2$. Thus

$$\mathbf{T}_2 = \mathbf{R}\mathbf{T}_1 = \mathbf{R}\mathbf{T}_1\mathbf{T}_1 = \mathbf{T}_2\mathbf{T}_1 \quad \text{and} \quad \mathbf{T}_1 = \mathbf{R}^{-1}\mathbf{T}_2 = \mathbf{R}^{-1}\mathbf{T}_2\mathbf{T}_2 = \mathbf{T}_1\mathbf{T}_2.$$

Because $\mathbf{T}_1$ and $\mathbf{T}_2$ are both upper triangular, $\mathbf{T}_1\mathbf{T}_2$ and $\mathbf{T}_2\mathbf{T}_1$ have the same diagonal entries, and hence $\mathbf{T}_1$ and $\mathbf{T}_2$ have the same diagonal. Therefore, the positions of the basic columns (i.e., the pivotal positions) in $\mathbf{T}_1$ agree with those in $\mathbf{T}_2$, and hence $\mathbf{E}_1$ and $\mathbf{E}_2$ have basic columns in exactly the same positions. This means there is a permutation matrix $\mathbf{Q}$ such that

$$\mathbf{E}_1\mathbf{Q} = \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{E}_2\mathbf{Q} = \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Using (3.9.4) yields $\mathbf{P}\mathbf{E}_1\mathbf{Q} = \mathbf{E}_2\mathbf{Q}$, or

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

which in turn implies that $\mathbf{P}_{11} = \mathbf{I}_r$ and $\mathbf{P}_{11}\mathbf{J}_1 = \mathbf{J}_2$. Consequently, $\mathbf{J}_1 = \mathbf{J}_2$, and it follows that $\mathbf{E}_1 = \mathbf{E}_2$.

In passing, notice that the uniqueness of $\mathbf{E}_\mathbf{A}$ implies the uniqueness of the pivot positions in any other row echelon form derived from $\mathbf{A}$. If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{U}_1$ and $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{U}_2$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are row echelon forms with different pivot positions, then Gauss–Jordan reduction applied to $\mathbf{U}_1$ and $\mathbf{U}_2$ would lead to two different reduced echelon forms, which is impossible.

In §2.2 we observed the fact that the column relationships in a matrix $\mathbf{A}$ are exactly the same as the column relationships in $\mathbf{E}_\mathbf{A}$. This observation is a special case of the more general result presented below.

Column and Row Relationships

- If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, then linear relationships existing among columns of $\mathbf{A}$ also hold among corresponding columns of $\mathbf{B}$. That is,

$$\mathbf{B}_{*k} = \sum_{j=1}^{n} \alpha_j \mathbf{B}_{*j} \quad \text{if and only if} \quad \mathbf{A}_{*k} = \sum_{j=1}^{n} \alpha_j \mathbf{A}_{*j}. \tag{3.9.6}$$

- In particular, the column relationships in $\mathbf{A}$ and $\mathbf{E_A}$ must be identical, so the nonbasic columns in $\mathbf{A}$ must be linear combinations of the basic columns in $\mathbf{A}$ as described in (2.2.3).
- If $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$, then linear relationships existing among rows of $\mathbf{A}$ must also hold among corresponding rows of $\mathbf{B}$.
- **Summary.** *Row* equivalence preserves *column* relationships, and *column* equivalence preserves *row* relationships.

Proof. If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, then $\mathbf{PA} = \mathbf{B}$ for some nonsingular $\mathbf{P}$. Recall from (3.5.5) that the j^{th} column in $\mathbf{B}$ is given by

$$\mathbf{B}_{*j} = (\mathbf{PA})_{*j} = \mathbf{PA}_{*j}.$$

Therefore, if $\mathbf{A}_{*k} = \sum_j \alpha_j \mathbf{A}_{*j}$, then multiplication by $\mathbf{P}$ on the left produces $\mathbf{B}_{*k} = \sum_j \alpha_j \mathbf{B}_{*j}$. Conversely, if $\mathbf{B}_{*k} = \sum_j \alpha_j \mathbf{B}_{*j}$, then multiplication on the left by $\mathbf{P}^{-1}$ produces $\mathbf{A}_{*k} = \sum_j \alpha_j \mathbf{A}_{*j}$. The statement concerning column equivalence follows by considering transposes. ■

The reduced row echelon form $\mathbf{E_A}$ is as far as we can go in reducing $\mathbf{A}$ by using only row operations. However, if we are allowed to use row operations in conjunction with column operations, then, as described below, the end result of a complete reduction is much simpler.

Rank Normal Form

If $\mathbf{A}$ is an $m \times n$ matrix such that $rank\,(\mathbf{A}) = r$, then

$$\mathbf{A} \sim \mathbf{N}_r = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{3.9.7}$$

$\mathbf{N}_r$ is called the ***rank normal form*** for $\mathbf{A}$, and it is the end product of a complete reduction of $\mathbf{A}$ by using both row and column operations.

Proof. It is always true that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_A}$ so that there is a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{E_A}$. If $rank(\mathbf{A}) = r$, then the basic columns in $\mathbf{E_A}$ are the r unit columns. Apply column interchanges to $\mathbf{E_A}$ so as to move these r unit columns to the far left-hand side. If $\mathbf{Q}_1$ is the product of the elementary matrices corresponding to these column interchanges, then $\mathbf{PAQ}_1$ has the form

$$\mathbf{PAQ}_1 = \mathbf{E_A Q}_1 = \begin{pmatrix} \mathbf{I}_r & \mathbf{J} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Multiplying both sides of this equation on the right by the nonsingular matrix

$$\mathbf{Q}_2 = \begin{pmatrix} \mathbf{I}_r & -\mathbf{J} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \text{ produces } \mathbf{PAQ}_1\mathbf{Q}_2 = \begin{pmatrix} \mathbf{I}_r & \mathbf{J} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I}_r & -\mathbf{J} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Thus $\mathbf{A} \sim \mathbf{N}_r$. because $\mathbf{P}$ and $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$ are nonsingular. ∎

Example 3.9.3

Problem: Explain why $rank\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = rank(\mathbf{A}) + rank(\mathbf{B})$.

Solution: If $rank(\mathbf{A}) = r$ and $rank(\mathbf{B}) = s$, then $\mathbf{A} \sim \mathbf{N}_r$ and $\mathbf{B} \sim \mathbf{N}_s$. Consequently,

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \sim \begin{pmatrix} \mathbf{N}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_s \end{pmatrix} \implies rank\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = rank\begin{pmatrix} \mathbf{N}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_s \end{pmatrix} = r + s.$$

Given matrices $\mathbf{A}$ and $\mathbf{B}$, how do we decide whether or not $\mathbf{A} \sim \mathbf{B}$, $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, or $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$? We could use a trial-and-error approach by attempting to reduce $\mathbf{A}$ to $\mathbf{B}$ by elementary operations, but this would be silly because there are easy tests, as described below.

Testing for Equivalence

For $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ the following statements are true.

- $\mathbf{A} \sim \mathbf{B}$ if and only if $rank(\mathbf{A}) = rank(\mathbf{B})$. (3.9.8)
- $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ if and only if $\mathbf{E_A} = \mathbf{E_B}$. (3.9.9)
- $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$ if and only if $\mathbf{E}_{\mathbf{A}^T} = \mathbf{E}_{\mathbf{B}^T}$. (3.9.10)

Corollary. Multiplication by nonsingular matrices cannot change rank.

Proof. To establish the validity of (3.9.8), observe that $rank(\mathbf{A}) = rank(\mathbf{B})$ implies $\mathbf{A} \sim \mathbf{N}_r$ and $\mathbf{B} \sim \mathbf{N}_r$. Therefore, $\mathbf{A} \sim \mathbf{N}_r \sim \mathbf{B}$. Conversely, if $\mathbf{A} \sim \mathbf{B}$, where $rank(\mathbf{A}) = r$ and $rank(\mathbf{B}) = s$, then $\mathbf{A} \sim \mathbf{N}_r$ and $\mathbf{B} \sim \mathbf{N}_s$, and hence $\mathbf{N}_r \sim \mathbf{A} \sim \mathbf{B} \sim \mathbf{N}_s$. Clearly, $\mathbf{N}_r \sim \mathbf{N}_s$ implies $r = s$. To prove (3.9.9), suppose first that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. Because $\mathbf{B} \overset{\text{row}}{\sim} \mathbf{E_B}$, it follows that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_B}$. Since a matrix has a uniquely determined reduced echelon form, it must be the case that $\mathbf{E_B} = \mathbf{E_A}$. Conversely, if $\mathbf{E_A} = \mathbf{E_B}$, then

$$\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_A} = \mathbf{E_B} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}.$$

The proof of (3.9.10) follows from (3.9.9) by considering transposes because

$$\begin{aligned} \mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} &\iff \mathbf{AQ} = \mathbf{B} \iff (\mathbf{AQ})^T = \mathbf{B}^T \\ &\iff \mathbf{Q}^T\mathbf{A}^T = \mathbf{B}^T \iff \mathbf{A}^T \overset{\text{row}}{\sim} \mathbf{B}^T. \quad \blacksquare \end{aligned}$$

Example 3.9.4

Problem: Are the relationships that exist among the columns in $\mathbf{A}$ the same as the column relationships in $\mathbf{B}$, and are the row relationships in $\mathbf{A}$ the same as the row relationships in $\mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -3 & -1 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 2 & 1 & -1 \end{pmatrix}?$$

Solution: Straightforward computation reveals that

$$\mathbf{E_A} = \mathbf{E_B} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. Therefore, the column relationships in $\mathbf{A}$ and $\mathbf{B}$ must be identical, and they must be the same as those in $\mathbf{E_A}$. Examining $\mathbf{E_A}$ reveals that $\mathbf{E}_{*3} = -2\mathbf{E}_{*1} + 3\mathbf{E}_{*2}$, so it must be the case that

$$\mathbf{A}_{*3} = -2\mathbf{A}_{*1} + 3\mathbf{A}_{*2} \quad \text{and} \quad \mathbf{B}_{*3} = -2\mathbf{B}_{*1} + 3\mathbf{B}_{*2}.$$

The row relationships in $\mathbf{A}$ and $\mathbf{B}$ are different because $\mathbf{E}_{\mathbf{A}^T} \neq \mathbf{E}_{\mathbf{B}^T}$.

On the surface, it may not seem plausible that a matrix and its transpose should have the same rank. After all, if $\mathbf{A}$ is 3×100, then $\mathbf{A}$ can have as many as 100 basic columns, but $\mathbf{A}^T$ can have at most three. Nevertheless, we can now demonstrate that $rank(\mathbf{A}) = rank(\mathbf{A}^T)$.

Transposition and Rank

Transposition does not change the rank—i.e., for all $m \times n$ matrices,

$$rank\,(\mathbf{A}) = rank\left(\mathbf{A}^T\right) \quad \text{and} \quad rank\,(\mathbf{A}) = rank\,(\mathbf{A}^*). \tag{3.9.11}$$

Proof. Let $rank\,(\mathbf{A}) = r$, and let $\mathbf{P}$ and $\mathbf{Q}$ be nonsingular matrices such that

$$\mathbf{PAQ} = \mathbf{N}_r = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times n-r} \\ \mathbf{0}_{m-r\times r} & \mathbf{0}_{m-r\times n-r} \end{pmatrix}.$$

Applying the reverse order law for transposition produces $\mathbf{Q}^T\mathbf{A}^T\mathbf{P}^T = \mathbf{N}_r^T$. Since $\mathbf{Q}^T$ and $\mathbf{P}^T$ are nonsingular, it follows that $\mathbf{A}^T \sim \mathbf{N}_r^T$, and therefore

$$rank\left(\mathbf{A}^T\right) = rank\left(\mathbf{N}_r^T\right) = rank\begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times m-r} \\ \mathbf{0}_{n-r\times r} & \mathbf{0}_{n-r\times m-r} \end{pmatrix} = r = rank\,(\mathbf{A}).$$

To prove $rank\,(\mathbf{A}) = rank\,(\mathbf{A}^*)$, write $\mathbf{N}_r = \overline{\mathbf{N}_r} = \overline{\mathbf{PAQ}} = \bar{\mathbf{P}}\bar{\mathbf{A}}\bar{\mathbf{Q}}$, and use the fact that the conjugate of a nonsingular matrix is again nonsingular (because $\bar{\mathbf{K}}^{-1} = \overline{\mathbf{K}^{-1}}$) to conclude that $\mathbf{N}_r \sim \bar{\mathbf{A}}$, and hence $rank\,(\mathbf{A}) = rank\left(\bar{\mathbf{A}}\right)$. It now follows from $rank\,(\mathbf{A}) = rank\left(\mathbf{A}^T\right)$ that

$$rank\,(\mathbf{A}^*) = rank\left(\bar{\mathbf{A}}^T\right) = rank\left(\bar{\mathbf{A}}\right) = rank\,(\mathbf{A}). \quad \blacksquare$$

Exercises for section 3.9

3.9.1. Suppose that $\mathbf{A}$ is an $m \times n$ matrix.

(a) If $[\mathbf{A}|\mathbf{I}_m]$ is row reduced to a matrix $[\mathbf{B}|\mathbf{P}]$, explain why $\mathbf{P}$ must be a nonsingular matrix such that $\mathbf{PA} = \mathbf{B}$.

(b) If $\left[\frac{\mathbf{A}}{\mathbf{I}_n}\right]$ is column reduced to $\left[\frac{\mathbf{C}}{\mathbf{Q}}\right]$, explain why $\mathbf{Q}$ must be a nonsingular matrix such that $\mathbf{AQ} = \mathbf{C}$.

(c) Find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{E}_{\mathbf{A}}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 6 \end{pmatrix}.$$

(d) Find nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PAQ}$ is in rank normal form.

3.9.2. Consider the two matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 & -1 \\ 3 & -1 & 4 & 0 \\ 0 & -8 & 8 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -6 & 8 & 2 \\ 5 & 1 & 4 & -1 \\ 3 & -9 & 12 & 3 \end{pmatrix}.$$

(a) Are $\mathbf{A}$ and $\mathbf{B}$ equivalent?
(b) Are $\mathbf{A}$ and $\mathbf{B}$ row equivalent?
(c) Are $\mathbf{A}$ and $\mathbf{B}$ column equivalent?

3.9.3. If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, explain why the basic columns in $\mathbf{A}$ occupy exactly the same positions as the basic columns in $\mathbf{B}$.

3.9.4. A product of elementary interchange matrices—i.e., elementary matrices of Type I—is called a ***permutation matrix.*** If $\mathbf{P}$ is a permutation matrix, explain why $\mathbf{P}^{-1} = \mathbf{P}^T$.

3.9.5. If $\mathbf{A}_{n\times n}$ is a nonsingular matrix, which (if any) of the following statements are true?

(a) $\mathbf{A} \sim \mathbf{A}^{-1}$. (b) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{A}^{-1}$. (c) $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{A}^{-1}$.
(d) $\mathbf{A} \sim \mathbf{I}$. (e) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{I}$. (f) $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{I}$.

3.9.6. Which (if any) of the following statements are true?

(a) $\mathbf{A} \sim \mathbf{B} \implies \mathbf{A}^T \sim \mathbf{B}^T$. (b) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A}^T \overset{\text{row}}{\sim} \mathbf{B}^T$.
(c) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A}^T \overset{\text{col}}{\sim} \mathbf{B}^T$. (d) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A} \sim \mathbf{B}$.
(e) $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \implies \mathbf{A} \sim \mathbf{B}$. (f) $\mathbf{A} \sim \mathbf{B} \implies \mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$.

3.9.7. Show that every elementary matrix of Type I can be written as a product of elementary matrices of Types II and III. **Hint:** Recall Exercise 1.2.12 on p. 14.

3.9.8. If $rank\,(\mathbf{A}_{m\times n}) = r$, show that there exist matrices $\mathbf{B}_{m\times r}$ and $\mathbf{C}_{r\times n}$ such that $\mathbf{A} = \mathbf{BC}$, where $rank\,(\mathbf{B}) = rank\,(\mathbf{C}) = r$. Such a factorization is called a ***full-rank factorization.*** **Hint:** Consider the basic columns of $\mathbf{A}$ and the nonzero rows of $\mathbf{E_A}$.

3.9.9. Prove that $rank\,(\mathbf{A}_{m\times n}) = 1$ if and only if there are nonzero columns $\mathbf{u}_{m\times 1}$ and $\mathbf{v}_{n\times 1}$ such that

$$\mathbf{A} = \mathbf{uv}^T.$$

3.9.10. Prove that if $rank\,(\mathbf{A}_{n\times n}) = 1$, then $\mathbf{A}^2 = \tau\mathbf{A}$, where $\tau = trace\,(\mathbf{A})$.

3.10 THE LU FACTORIZATION

We have now come full circle, and we are back to where the text began—solving a nonsingular system of linear equations using Gaussian elimination with back substitution. This time, however, the goal is to describe and understand the process in the context of matrices.

If $\mathbf{Ax} = \mathbf{b}$ is a nonsingular system, then the object of Gaussian elimination is to reduce $\mathbf{A}$ to an upper-triangular matrix using elementary row operations. If no zero pivots are encountered, then row interchanges are not necessary, and the reduction can be accomplished by using only elementary row operations of Type III. For example, consider reducing the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$$

to upper-triangular form as shown below:

$$\begin{aligned} \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} &\longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{pmatrix} \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix} \\ &\longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{U}. \end{aligned} \tag{3.10.1}$$

We learned in the previous section that each of these Type III operations can be executed by means of a left-hand multiplication with the corresponding elementary matrix $\mathbf{G}_i$, and the product of all of these $\mathbf{G}_i$'s is

$$\mathbf{G}_3\mathbf{G}_2\mathbf{G}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}.$$

In other words, $\mathbf{G}_3\mathbf{G}_2\mathbf{G}_1\mathbf{A} = \mathbf{U}$, so that $\mathbf{A} = \mathbf{G}_1^{-1}\mathbf{G}_2^{-1}\mathbf{G}_3^{-1}\mathbf{U} = \mathbf{LU}$, where $\mathbf{L}$ is the lower-triangular matrix

$$\mathbf{L} = \mathbf{G}_1^{-1}\mathbf{G}_2^{-1}\mathbf{G}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}.$$

Thus $\mathbf{A} = \mathbf{LU}$ is a product of a lower-triangular matrix $\mathbf{L}$ and an upper-triangular matrix $\mathbf{U}$. Naturally, this is called an $\mathbf{LU}$ ***factorization*** of $\mathbf{A}$.

Observe that $\mathbf{U}$ is the end product of Gaussian elimination and has the pivots on its diagonal, while $\mathbf{L}$ has 1's on its diagonal. Moreover, $\mathbf{L}$ has the remarkable property that below its diagonal, *each entry* ℓ_{ij} *is precisely the multiplier used in the elimination* (3.10.1) *to annihilate the* (i,j)*-position.*

This is characteristic of what happens in general. To develop the general theory, it's convenient to introduce the concept of an ***elementary lower-triangular matrix,*** which is defined to be an $n \times n$ triangular matrix of the form

$$\mathbf{T}_k = \mathbf{I} - \mathbf{c}_k \mathbf{e}_k^T,$$

where $\mathbf{c}_k$ is a column with zeros in the first k positions. In particular, if

$$\mathbf{c}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \mu_{k+1} \\ \vdots \\ \mu_n \end{pmatrix}, \quad \text{then} \quad \mathbf{T}_k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -\mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\mu_n & 0 & \cdots & 1 \end{pmatrix}. \tag{3.10.2}$$

By observing that $\mathbf{e}_k^T \mathbf{c}_k = 0,$ the formula for the inverse of an elementary matrix given in (3.9.1) produces

$$\mathbf{T}_k^{-1} = \mathbf{I} + \mathbf{c}_k \mathbf{e}_k^T = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n & 0 & \cdots & 1 \end{pmatrix}, \tag{3.10.3}$$

which is also an elementary lower-triangular matrix. The utility of elementary lower-triangular matrices lies in the fact that all of the Type III row operations needed to annihilate the entries below the k^{th} pivot can be accomplished with one multiplication by $\mathbf{T}_k$. If

$$\mathbf{A}_{k-1} = \begin{pmatrix} * & * & \cdots & \alpha_1 & * & \cdots & * \\ 0 & * & \cdots & \alpha_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_k & * & \cdots & * \\ 0 & 0 & \cdots & \alpha_{k+1} & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n & * & \cdots & * \end{pmatrix}$$

is the partially triangularized result after $k-1$ elimination steps, then

$$\mathbf{T}_k\mathbf{A}_{k-1} = \left(\mathbf{I} - \mathbf{c}_k\mathbf{e}_k^T\right)\mathbf{A}_{k-1} = \mathbf{A}_{k-1} - \mathbf{c}_k\mathbf{e}_k^T\mathbf{A}_{k-1}$$

$$= \begin{pmatrix} * & * & \cdots & \alpha_1 & * & \cdots & * \\ 0 & * & \cdots & \alpha_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_k & * & \cdots & * \\ 0 & 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}, \qquad \text{where} \quad \mathbf{c}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_{k+1}/\alpha_k \\ \vdots \\ \alpha_n/\alpha_k \end{pmatrix}$$

contains the multipliers used to annihilate those entries below α_k. Notice that $\mathbf{T}_k$ does not alter the first $k-1$ columns of $\mathbf{A}_{k-1}$ because $\mathbf{e}_k^T\left[\mathbf{A}_{k-1}\right]_{*j} = 0$ whenever $j \le k-1$. Therefore, if no row interchanges are required, then reducing $\mathbf{A}$ to an upper-triangular matrix $\mathbf{U}$ by Gaussian elimination is equivalent to executing a sequence of $n-1$ left-hand multiplications with elementary lower-triangular matrices. That is, $\mathbf{T}_{n-1}\cdots\mathbf{T}_2\mathbf{T}_1\mathbf{A} = \mathbf{U}$, and hence

$$\mathbf{A} = \mathbf{T}_1^{-1}\mathbf{T}_2^{-1}\cdots\mathbf{T}_{n-1}^{-1}\mathbf{U}. \tag{3.10.4}$$

Making use of the fact that $\mathbf{e}_j^T\mathbf{c}_k = 0$ whenever $j \le k$ and applying (3.10.3) reveals that

$$\begin{aligned}\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}\cdots\mathbf{T}_{n-1}^{-1} &= \left(\mathbf{I} + \mathbf{c}_1\mathbf{e}_1^T\right)\left(\mathbf{I} + \mathbf{c}_2\mathbf{e}_2^T\right)\cdots\left(\mathbf{I} + \mathbf{c}_{n-1}\mathbf{e}_{n-1}^T\right) \\ &= \mathbf{I} + \mathbf{c}_1\mathbf{e}_1^T + \mathbf{c}_2\mathbf{e}_2^T + \cdots + \mathbf{c}_{n-1}\mathbf{e}_{n-1}^T.\end{aligned} \tag{3.10.5}$$

By observing that

$$\mathbf{c}_k\mathbf{e}_k^T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \ell_{k+1,k} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \ell_{nk} & 0 & \cdots & 0 \end{pmatrix},$$

where the ℓ_{ik} 's are the multipliers used at the k^{th} stage to annihilate the entries below the k^{th} pivot, it now follows from (3.10.4) and (3.10.5) that

$$\mathbf{A} = \mathbf{L}\mathbf{U},$$

where

$$\mathbf{L} = \mathbf{I} + \mathbf{c}_1\mathbf{e}_1^T + \mathbf{c}_2\mathbf{e}_2^T + \cdots + \mathbf{c}_{n-1}\mathbf{e}_{n-1}^T = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{pmatrix} \tag{3.10.6}$$

is the lower-triangular matrix with 1's on the diagonal, and where ℓ_{ij} is precisely the multiplier used to annihilate the (i,j)-position during Gaussian elimination. Thus the factorization $\mathbf{A} = \mathbf{LU}$ can be viewed as the matrix formulation of Gaussian elimination, with the understanding that no row interchanges are used.

LU Factorization

If $\mathbf{A}$ is an $n \times n$ matrix such that a zero pivot is never encountered when applying Gaussian elimination with Type III operations, then $\mathbf{A}$ can be factored as the product $\mathbf{A} = \mathbf{LU}$, where the following hold.

- $\mathbf{L}$ is lower triangular and $\mathbf{U}$ is upper triangular. (3.10.7)
- $\ell_{ii} = 1$ and $u_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. (3.10.8)
- Below the diagonal of $\mathbf{L}$, the entry ℓ_{ij} is the multiple of row j that is subtracted from row i in order to annihilate the (i,j)-position during Gaussian elimination.
- $\mathbf{U}$ is the final result of Gaussian elimination applied to $\mathbf{A}$.
- The matrices $\mathbf{L}$ and $\mathbf{U}$ are uniquely determined by properties (3.10.7) and (3.10.8).

The decomposition of $\mathbf{A}$ into $\mathbf{A} = \mathbf{LU}$ is called the ***LU factorization of*** $\mathbf{A}$, and the matrices $\mathbf{L}$ and $\mathbf{U}$ are called the ***LU factors of*** $\mathbf{A}$.

Proof. Except for the statement concerning the uniqueness of the LU factors, each point has already been established. To prove uniqueness, observe that LU factors must be nonsingular because they have nonzero diagonals. If $\mathbf{L}_1\mathbf{U}_1 = \mathbf{A} = \mathbf{L}_2\mathbf{U}_2$ are two LU factorizations for $\mathbf{A}$, then

$$\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{U}_2\mathbf{U}_1^{-1}. \tag{3.10.9}$$

Notice that $\mathbf{L}_2^{-1}\mathbf{L}_1$ is lower triangular, while $\mathbf{U}_2\mathbf{U}_1^{-1}$ is upper triangular because the inverse of a matrix that is upper (lower) triangular is again upper (lower) triangular, and because the product of two upper (lower) triangular matrices is also upper (lower) triangular. Consequently, (3.10.9) implies $\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{D} = \mathbf{U}_2\mathbf{U}_1^{-1}$ must be a diagonal matrix. However, $[\mathbf{L}_2]_{ii} = 1 = [\mathbf{L}_2^{-1}]_{ii}$, so it must be the case that $\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{I} = \mathbf{U}_2\mathbf{U}_1^{-1}$, and thus $\mathbf{L}_1 = \mathbf{L}_2$ and $\mathbf{U}_1 = \mathbf{U}_2$. ∎

Example 3.10.1

Once $\mathbf{L}$ and $\mathbf{U}$ are known, there is usually no need to manipulate with $\mathbf{A}$. This together with the fact that the multipliers used in Gaussian elimination occur in just the right places in $\mathbf{L}$ means that $\mathbf{A}$ can be successively overwritten with the information in $\mathbf{L}$ and $\mathbf{U}$ as Gaussian elimination evolves. The rule is to store the multiplier ℓ_{ij} in the position it annihilates—namely, the (i,j)-position of the array. For a 3×3 matrix, the result looks like this:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\text{Type III operations}} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21} & u_{22} & u_{23} \\ \ell_{31} & \ell_{32} & u_{33} \end{pmatrix}.$$

For example, generating the LU factorization of

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$$

by successively overwriting a single 3×3 array would evolve as shown below:

$$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 12 & 16 \end{pmatrix} R_3 - 4R_2 \longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \end{pmatrix}.$$

Thus

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

This is an important feature in practical computation because it guarantees that an LU factorization requires no more computer memory than that required to store the original matrix $\mathbf{A}$.

Once the LU factors for a nonsingular matrix $\mathbf{A}_{n\times n}$ have been obtained, it's relatively easy to solve a linear system $\mathbf{Ax} = \mathbf{b}$. By rewriting $\mathbf{Ax} = \mathbf{b}$ as

$$\mathbf{L}(\mathbf{Ux}) = \mathbf{b} \quad \text{and setting} \quad \mathbf{y} = \mathbf{Ux},$$

we see that $\mathbf{Ax} = \mathbf{b}$ is equivalent to the two triangular systems

$$\mathbf{Ly} = \mathbf{b} \quad \text{and} \quad \mathbf{Ux} = \mathbf{y}.$$

First, the lower-triangular system $\mathbf{Ly} = \mathbf{b}$ is solved for $\mathbf{y}$ by ***forward substitution.*** That is, if

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix},$$

set

$$y_1 = b_1, \quad y_2 = b_2 - \ell_{21}y_1, \quad y_3 = b_3 - \ell_{31}y_1 - \ell_{32}y_2, \quad \text{etc.}$$

The forward substitution algorithm can be written more concisely as

$$y_1 = b_1 \quad \text{and} \quad y_i = b_i - \sum_{k=1}^{i-1} \ell_{ik}y_k \quad \text{for} \quad i = 2, 3, \ldots, n. \tag{3.10.10}$$

After $\mathbf{y}$ is known, the upper-triangular system $\mathbf{Ux} = \mathbf{y}$ is solved using the standard back substitution procedure by starting with $x_n = y_n/u_{nn}$, and setting

$$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{k=i+1}^{n} u_{ik}x_k\right) \quad \text{for} \quad i = n-1,\ n-2,\ \ldots, 1. \tag{3.10.11}$$

It can be verified that only n^2 multiplications/divisions and $n^2 - n$ additions/subtractions are required when (3.10.10) and (3.10.11) are used to solve the two triangular systems $\mathbf{Ly} = \mathbf{b}$ and $\mathbf{Ux} = \mathbf{y}$, so it's relatively cheap to solve $\mathbf{Ax} = \mathbf{b}$ once $\mathbf{L}$ and $\mathbf{U}$ are known—recall from §1.2 that these operation counts are about $n^3/3$ when we start from scratch.

If only one system $\mathbf{Ax} = \mathbf{b}$ is to be solved, then there is no significant difference between the technique of reducing the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to a row echelon form and the LU factorization method presented here. However, suppose it becomes necessary to later solve other systems $\mathbf{Ax} = \tilde{\mathbf{b}}$ with the same coefficient matrix but with different right-hand sides, which is frequently the case in applied work. If the LU factors of $\mathbf{A}$ were computed and saved when the original system was solved, then they need not be recomputed, and the solutions to all subsequent systems $\mathbf{Ax} = \tilde{\mathbf{b}}$ are therefore relatively cheap to obtain. That is, the operation counts for each subsequent system are on the order of n^2, whereas these counts would be on the order of $n^3/3$ if we would start from scratch each time.

Summary

- To solve a nonsingular system $\mathbf{Ax} = \mathbf{b}$ using the LU factorization $\mathbf{A} = \mathbf{LU}$, first solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ with the forward substitution algorithm (3.10.10), and then solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$ with the back substitution procedure (3.10.11).

- The advantage of this approach is that once the LU factors for $\mathbf{A}$ have been computed, any other linear system $\mathbf{Ax} = \tilde{\mathbf{b}}$ can be solved with only n^2 multiplications/divisions and $n^2 - n$ additions/subtractions.

Example 3.10.2

Problem 1: Use the LU factorization of $\mathbf{A}$ to solve $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix}.$$

Problem 2: Suppose that after solving the original system new information is received that changes $\mathbf{b}$ to

$$\tilde{\mathbf{b}} = \begin{pmatrix} 6 \\ 24 \\ 70 \end{pmatrix}.$$

Use the LU factors of $\mathbf{A}$ to solve the updated system $\mathbf{Ax} = \tilde{\mathbf{b}}$.

Solution 1: The LU factors of the coefficient matrix were determined in Example 3.10.1 to be

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

The strategy is to set $\mathbf{Ux} = \mathbf{y}$ and solve $\mathbf{Ax} = \mathbf{L}(\mathbf{Ux}) = \mathbf{b}$ by solving the two triangular systems

$$\mathbf{Ly} = \mathbf{b} \quad \text{and} \quad \mathbf{Ux} = \mathbf{y}.$$

First solve the lower-triangular system $\mathbf{Ly} = \mathbf{b}$ by using forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} y_1 &= 12, \\ y_2 &= 24 - 2y_1 = 0, \\ y_3 &= 12 - 3y_1 - 4y_2 = -24. \end{aligned}$$

Now use back substitution to solve the upper-triangular system $\mathbf{Ux} = \mathbf{y}$:

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -24 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} x_1 &= (12 - 2x_2 - 2x_3)/2 = 6, \\ x_2 &= (0 - 3x_3)/3 = 6, \\ x_3 &= -24/4 = -6. \end{aligned}$$

Solution 2: To solve the updated system $\mathbf{Ax} = \tilde{\mathbf{b}}$, simply repeat the forward and backward substitution steps with $\mathbf{b}$ replaced by $\tilde{\mathbf{b}}$. Solving $\mathbf{Ly} = \tilde{\mathbf{b}}$ with forward substitution gives the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \\ 70 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} y_1 &= 6, \\ y_2 &= 24 - 2y_1 = 12, \\ y_3 &= 70 - 3y_1 - 4y_2 = 4. \end{aligned}$$

Using back substitution to solve $\mathbf{Ux} = \mathbf{y}$ gives the following updated solution:

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ 4 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} x_1 &= (6 - 2x_2 - 2x_3)/2 = -1, \\ x_2 &= (12 - 3x_3)/3 = 3, \\ x_3 &= 4/4 = 1. \end{aligned}$$

Example 3.10.3

Computing $\mathbf{A}^{-1}$. Although matrix inversion is not used for solving $\mathbf{Ax} = \mathbf{b}$, there are a few applications where explicit knowledge of $\mathbf{A}^{-1}$ is desirable.

Problem: Explain how to use the LU factors of a nonsingular matrix $\mathbf{A}_{n\times n}$ to compute $\mathbf{A}^{-1}$ efficiently.

Solution: The strategy is to solve the matrix equation $\mathbf{AX} = \mathbf{I}$. Recall from (3.5.5) that $\mathbf{AA}^{-1} = \mathbf{I}$ implies $\mathbf{A}[\mathbf{A}^{-1}]_{*j} = \mathbf{e}_j$, so the j^{th} column of $\mathbf{A}^{-1}$ is the solution of a system $\mathbf{Ax}_j = \mathbf{e}_j$. Each of these n systems has the same coefficient matrix, so, once the LU factors for $\mathbf{A}$ are known, each system $\mathbf{Ax}_j = \mathbf{LUx}_j = \mathbf{e}_j$ can be solved by the standard two-step process.

(1) Set $\mathbf{y}_j = \mathbf{Ux}_j$, and solve $\mathbf{Ly}_j = \mathbf{e}_j$ for $\mathbf{y}_j$ by forward substitution.

(2) Solve $\mathbf{Ux}_j = \mathbf{y}_j$ for $\mathbf{x}_j = [\mathbf{A}^{-1}]_{*j}$ by back substitution.

This method has at least two advantages: it's efficient, and any code written to solve $\mathbf{Ax} = \mathbf{b}$ can also be used to compute $\mathbf{A}^{-1}$.

Note: A tempting alternate solution might be to use the fact $\mathbf{A}^{-1} = (\mathbf{LU})^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$. But computing $\mathbf{U}^{-1}$ and $\mathbf{L}^{-1}$ explicitly and then multiplying the results is not as computationally efficient as the method just described.

Not all nonsingular matrices possess an LU factorization. For example, there is clearly no nonzero value of u_{11} that will satisfy

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \ell_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}.$$

The problem here is the zero pivot in the (1,1)-position. Our development of the LU factorization using elementary lower-triangular matrices shows that if no zero pivots emerge, then no row interchanges are necessary, and the LU factorization can indeed be carried to completion. The converse is also true (its proof is left as an exercise), so we can say that *a nonsingular matrix* $\mathbf{A}$ *has an LU factorization if and only if a zero pivot does not emerge during row reduction to upper-triangular form with Type III operations.*

Although it is a bit more theoretical, there is another interesting way to characterize the existence of LU factors. This characterization is given in terms of the ***leading principal submatrices of*** $\mathbf{A}$ that are defined to be those submatrices taken from the upper-left-hand corner of $\mathbf{A}$. That is,

$$\mathbf{A}_1 = \begin{pmatrix} a_{11} \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ldots, \mathbf{A}_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}, \ldots.$$

Existence of LU Factors

Each of the following statements is equivalent to saying that a nonsingular matrix $\mathbf{A}_{n\times n}$ possesses an LU factorization.

- A zero pivot does not emerge during row reduction to upper-triangular form with Type III operations.
- Each leading principal submatrix $\mathbf{A}_k$ is nonsingular. (3.10.12)

Proof. We will prove the statement concerning the leading principal submatrices and leave the proof concerning the nonzero pivots as an exercise. Assume first that $\mathbf{A}$ possesses an LU factorization and partition $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{11}\mathbf{U}_{11} & * \\ * & * \end{pmatrix},$$

where $\mathbf{L}_{11}$ and $\mathbf{U}_{11}$ are each $k \times k$. Thus $\mathbf{A}_k = \mathbf{L}_{11}\mathbf{U}_{11}$ must be nonsingular because $\mathbf{L}_{11}$ and $\mathbf{U}_{11}$ are each nonsingular—they are triangular with nonzero diagonal entries. Conversely, suppose that each leading principal submatrix in $\mathbf{A}$ is nonsingular. Use induction to prove that each $\mathbf{A}_k$ possesses an LU factorization. For $k = 1$, this statement is clearly true because if $\mathbf{A}_1 = (a_{11})$ is nonsingular, then $\mathbf{A}_1 = (1)(a_{11})$ is its LU factorization. Now assume that $\mathbf{A}_k$ has an LU factorization and show that this together with the nonsingularity condition implies $\mathbf{A}_{k+1}$ must also possess an LU factorization. If $\mathbf{A}_k = \mathbf{L}_k\mathbf{U}_k$ is the LU factorization for $\mathbf{A}_k$, then $\mathbf{A}_k^{-1} = \mathbf{U}_k^{-1}\mathbf{L}_k^{-1}$ so that

$$\mathbf{A}_{k+1} = \begin{pmatrix} \mathbf{A}_k & \mathbf{b} \\ \mathbf{c}^T & \alpha_{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_k & \mathbf{0} \\ \mathbf{c}^T\mathbf{U}_k^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{U}_k & \mathbf{L}_k^{-1}\mathbf{b} \\ \mathbf{0} & \alpha_{k+1} - \mathbf{c}^T\mathbf{A}_k^{-1}\mathbf{b} \end{pmatrix}, \quad (3.10.13)$$

where $\mathbf{c}^T$ and $\mathbf{b}$ contain the first k components of $\mathbf{A}_{k+1*}$ and $\mathbf{A}_{*k+1}$, respectively. Observe that this is the LU factorization for $\mathbf{A}_{k+1}$ because

$$\mathbf{L}_{k+1} = \begin{pmatrix} \mathbf{L}_k & \mathbf{0} \\ \mathbf{c}^T\mathbf{U}_k^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U}_{k+1} = \begin{pmatrix} \mathbf{U}_k & \mathbf{L}_k^{-1}\mathbf{b} \\ \mathbf{0} & \alpha_{k+1} - \mathbf{c}^T\mathbf{A}_k^{-1}\mathbf{b} \end{pmatrix}$$

are lower- and upper-triangular matrices, respectively, and $\mathbf{L}$ has 1's on its diagonal while the diagonal entries of $\mathbf{U}$ are nonzero. The fact that

$$\alpha_{k+1} - \mathbf{c}^T\mathbf{A}_k^{-1}\mathbf{b} \neq 0$$

follows because $\mathbf{A}_{k+1}$ and $\mathbf{L}_{k+1}$ are each nonsingular, so $\mathbf{U}_{k+1} = \mathbf{L}_{k+1}^{-1}\mathbf{A}_{k+1}$ must also be nonsingular. Therefore, the nonsingularity of the leading principal submatrices implies that each $\mathbf{A}_k$ possesses an LU factorization, and hence $\mathbf{A}_n = \mathbf{A}$ must have an LU factorization. ■

Up to this point we have avoided dealing with row interchanges because if a row interchange is needed to remove a zero pivot, then no LU factorization is possible. However, we know from the discussion in §1.5 that practical computation necessitates row interchanges in the form of partial pivoting. So even if no zero pivots emerge, it is usually the case that we must still somehow account for row interchanges.

To understand the effects of row interchanges in the framework of an LU decomposition, let $\mathbf{T}_k = \mathbf{I} - \mathbf{c}_k\mathbf{e}_k^T$ be an elementary lower-triangular matrix as described in (3.10.2), and let $\mathbf{E} = \mathbf{I} - \mathbf{u}\mathbf{u}^T$ with $\mathbf{u} = \mathbf{e}_{k+i} - \mathbf{e}_{k+j}$ be the Type I elementary interchange matrix associated with an interchange of rows $k+i$ and $k+j$. Notice that $\mathbf{e}_k^T\mathbf{E} = \mathbf{e}_k^T$ because $\mathbf{e}_k^T$ has 0's in positions $k+i$ and $k+j$. This together with the fact that $\mathbf{E}^2 = \mathbf{I}$ guarantees

$$\mathbf{E}\mathbf{T}_k\mathbf{E} = \mathbf{E}^2 - \mathbf{E}\mathbf{c}_k\mathbf{e}_k^T\mathbf{E} = \mathbf{I} - \tilde{\mathbf{c}}_k\mathbf{e}_k^T, \quad \text{where} \quad \tilde{\mathbf{c}}_k = \mathbf{E}\mathbf{c}_k.$$

In other words, the matrix

$$\tilde{\mathbf{T}}_k = \mathbf{E}\mathbf{T}_k\mathbf{E} = \mathbf{I} - \tilde{\mathbf{c}}_k\mathbf{e}_k^T \tag{3.10.14}$$

is also an elementary lower-triangular matrix, and $\tilde{\mathbf{T}}_k$ agrees with $\mathbf{T}_k$ in all positions except that the multipliers μ_{k+i} and μ_{k+j} have traded places. As before, assume we are row reducing an $n \times n$ nonsingular matrix $\mathbf{A}$, but suppose that an interchange of rows $k+i$ and $k+j$ is necessary immediately after the k^{th} stage so that the sequence of left-hand multiplications $\mathbf{E}\mathbf{T}_k\mathbf{T}_{k-1}\cdots\mathbf{T}_1$ is applied to $\mathbf{A}$. Since $\mathbf{E}^2 = \mathbf{I}$, we may insert $\mathbf{E}^2$ to the right of each $\mathbf{T}$ to obtain

$$\begin{aligned}
\mathbf{E}\mathbf{T}_k\mathbf{T}_{k-1}\cdots\mathbf{T}_1 &= \mathbf{E}\mathbf{T}_k\mathbf{E}^2\mathbf{T}_{k-1}\mathbf{E}^2\cdots\mathbf{E}^2\mathbf{T}_1\mathbf{E}^2 \\
&= \left(\mathbf{E}\mathbf{T}_k\mathbf{E}\right)\left(\mathbf{E}\mathbf{T}_{k-1}\mathbf{E}\right)\cdots\left(\mathbf{E}\mathbf{T}_1\mathbf{E}\right)\mathbf{E} \\
&= \tilde{\mathbf{T}}_k\tilde{\mathbf{T}}_{k-1}\cdots\tilde{\mathbf{T}}_1\mathbf{E}.
\end{aligned}$$

In such a manner, the necessary interchange matrices $\mathbf{E}$ can be "factored" to the far-right-hand side, and the matrices $\tilde{\mathbf{T}}$ retain the desirable feature of being elementary lower-triangular matrices. Furthermore, (3.10.14) implies that $\tilde{\mathbf{T}}_k\tilde{\mathbf{T}}_{k-1}\cdots\tilde{\mathbf{T}}_1$ differs from $\mathbf{T}_k\mathbf{T}_{k-1}\cdots\mathbf{T}_1$ only in the sense that the multipliers in rows $k+i$ and $k+j$ have traded places. Therefore, row interchanges in Gaussian elimination can be accounted for by writing $\tilde{\mathbf{T}}_{n-1}\cdots\tilde{\mathbf{T}}_2\tilde{\mathbf{T}}_1\mathbf{P}\mathbf{A} = \mathbf{U}$, where $\mathbf{P}$ is the product of all elementary interchange matrices used during the reduction and where the $\tilde{\mathbf{T}}_k$'s are elementary lower-triangular matrices in which the multipliers have been permuted according to the row interchanges that were implemented. Since all of the $\tilde{\mathbf{T}}_k$'s are elementary lower-triangular matrices, we may proceed along the same lines discussed in (3.10.4)—(3.10.6) to obtain

$$\mathbf{P}\mathbf{A} = \mathbf{L}\mathbf{U}, \quad \text{where} \quad \mathbf{L} = \tilde{\mathbf{T}}_1^{-1}\tilde{\mathbf{T}}_2^{-1}\cdots\tilde{\mathbf{T}}_{n-1}^{-1}. \tag{3.10.15}$$

When row interchanges are allowed, zero pivots can always be avoided when the original matrix $\mathbf{A}$ is nonsingular. Consequently, we may conclude that *for every nonsingular matrix* $\mathbf{A}$, *there exists a* ***permutation matrix*** $\mathbf{P}$ *(a product of elementary interchange matrices) such that* $\mathbf{PA}$ *has an LU factorization.* Furthermore, because of the observation in (3.10.14) concerning how the multipliers in $\mathbf{T}_k$ and $\tilde{\mathbf{T}}_k$ trade places when a row interchange occurs, and because

$$\tilde{\mathbf{T}}_k^{-1} = \left(\mathbf{I} - \tilde{\mathbf{c}}_k\mathbf{e}_k^T\right)^{-1} = \mathbf{I} + \tilde{\mathbf{c}}_k\mathbf{e}_k^T,$$

it is not difficult to see that the same line of reasoning used to arrive at (3.10.6) can be applied to conclude that the multipliers in the matrix $\mathbf{L}$ in (3.10.15) are permuted according to the row interchanges that are executed. More specifically, *if rows* k *and* $k+i$ *are interchanged to create the* k^{th} *pivot, then the multipliers*

$$\begin{pmatrix} \ell_{k1} & \ell_{k2} & \cdots & \ell_{k,k-1} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \ell_{k+i,1} & \ell_{k+i,2} & \cdots & \ell_{k+i,k-1} \end{pmatrix}$$

trade places in the formation of $\mathbf{L}$.

This means that we can proceed just as in the case when no interchanges are used and successively overwrite the array originally containing $\mathbf{A}$ with each multiplier replacing the position it annihilates. Whenever a row interchange occurs, the corresponding multipliers will be correctly interchanged as well. The permutation matrix $\mathbf{P}$ is simply the cumulative record of the various interchanges used, and the information in $\mathbf{P}$ is easily accounted for by a simple technique that is illustrated in the following example.

Example 3.10.4

Problem: Use partial pivoting on the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{pmatrix}$$

and determine the LU decomposition $\mathbf{PA} = \mathbf{LU}$, where $\mathbf{P}$ is the associated permutation matrix.

Solution: As explained earlier, the strategy is to successively overwrite the array $\mathbf{A}$ with components from $\mathbf{L}$ and $\mathbf{U}$. For the sake of clarity, the multipliers ℓ_{ij} are shown in boldface type. Adjoin a "permutation counter column" $\mathbf{p}$ that is initially set to the natural order 1,2,3,4. Permuting components of $\mathbf{p}$ as the various row interchanges are executed will accumulate the desired permutation. The matrix $\mathbf{P}$ is obtained by executing the final permutation residing in $\mathbf{p}$ to the rows of an appropriate size identity matrix:

$$[\mathbf{A}|\mathbf{p}] = \left(\begin{array}{cccc|c} 1 & 2 & -3 & 4 & 1 \\ 4 & 8 & 12 & -8 & 2 \\ 2 & 3 & 2 & 1 & 3 \\ -3 & -1 & 1 & -4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 4 & 8 & 12 & -8 & 2 \\ 1 & 2 & -3 & 4 & 1 \\ 2 & 3 & 2 & 1 & 3 \\ -3 & -1 & 1 & -4 & 4 \end{array}\right)$$

$$\longrightarrow\left(\begin{array}{rrrr|r} 4 & 8 & 12 & -8 & 2 \\ \mathbf{1/4} & 0 & -6 & 6 & 1 \\ \mathbf{1/2} & -1 & -4 & 5 & 3 \\ \mathbf{-3/4} & 5 & 10 & -10 & 4 \end{array}\right) \longrightarrow\left(\begin{array}{rrrr|r} 4 & 8 & 12 & -8 & 2 \\ \mathbf{-3/4} & 5 & 10 & -10 & 4 \\ \mathbf{1/2} & -1 & -4 & 5 & 3 \\ \mathbf{1/4} & 0 & -6 & 6 & 1 \end{array}\right)$$

$$\longrightarrow\left(\begin{array}{rrrr|r} 4 & 8 & 12 & -8 & 2 \\ \mathbf{-3/4} & 5 & 10 & -10 & 4 \\ \mathbf{1/2} & \mathbf{-1/5} & -2 & 3 & 3 \\ \mathbf{1/4} & \mathbf{0} & -6 & 6 & 1 \end{array}\right) \longrightarrow\left(\begin{array}{rrrr|r} 4 & 8 & 12 & -8 & 2 \\ \mathbf{-3/4} & 5 & 10 & -10 & 4 \\ \mathbf{1/4} & \mathbf{0} & -6 & 6 & 1 \\ \mathbf{1/2} & \mathbf{-1/5} & -2 & 3 & 3 \end{array}\right)$$

$$\longrightarrow\left(\begin{array}{rrrr|r} 4 & 8 & 12 & -8 & 2 \\ \mathbf{-3/4} & 5 & 10 & -10 & 4 \\ \mathbf{1/4} & \mathbf{0} & -6 & 6 & 1 \\ \mathbf{1/2} & \mathbf{-1/5} & \mathbf{1/3} & 1 & 3 \end{array}\right).$$

Therefore,

$$\mathbf{L}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & -1/5 & 1/3 & 1 \end{pmatrix},\ \mathbf{U}=\begin{pmatrix} 4 & 8 & 12 & -8 \\ 0 & 5 & 10 & -10 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathbf{P}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is easy to combine the advantages of partial pivoting with the LU decomposition in order to solve a nonsingular system $\mathbf{Ax} = \mathbf{b}$. Because permutation matrices are nonsingular, the system $\mathbf{Ax} = \mathbf{b}$ is equivalent to

$$\mathbf{PAx} = \mathbf{Pb},$$

and hence we can employ the LU solution techniques discussed earlier to solve this permuted system. That is, if we have already performed the factorization $\mathbf{PA} = \mathbf{LU}$—as illustrated in Example 3.10.4—then we can solve $\mathbf{Ly} = \mathbf{Pb}$ for $\mathbf{y}$ by forward substitution, and then solve $\mathbf{Ux} = \mathbf{y}$ by back substitution.

It should be evident that the permutation matrix $\mathbf{P}$ is not really needed. All that is necessary is knowledge of the LU factors along with the final permutation contained in the permutation counter column $\mathbf{p}$ illustrated in Example 3.10.4. The column $\tilde{\mathbf{b}} = \mathbf{Pb}$ is simply a rearrangement of the components of $\mathbf{b}$ according to the final permutation shown in $\mathbf{p}$. In other words, the strategy is to first permute $\mathbf{b}$ into $\tilde{\mathbf{b}}$ according to the permutation $\mathbf{p}$, and then solve $\mathbf{Ly} = \tilde{\mathbf{b}}$ followed by $\mathbf{Ux} = \mathbf{y}$.

Example 3.10.5

Problem: Use the LU decomposition obtained with partial pivoting to solve the system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 60 \\ 1 \\ 5 \end{pmatrix}.$$

Solution: The LU decomposition with partial pivoting was computed in Example 3.10.4. Permute the components in $\mathbf{b}$ according to the permutation $\mathbf{p} = (\,2\quad 4\quad 1\quad 3\,)$, and call the result $\tilde{\mathbf{b}}$. Now solve $\mathbf{Ly} = \tilde{\mathbf{b}}$ by applying forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & -1/5 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 60 \\ 5 \\ 3 \\ 1 \end{pmatrix} \implies \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 60 \\ 50 \\ -12 \\ -15 \end{pmatrix}.$$

Then solve $\mathbf{Ux} = \mathbf{y}$ by applying back substitution:

$$\begin{pmatrix} 4 & 8 & 12 & -8 \\ 0 & 5 & 10 & -10 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 60 \\ 50 \\ -12 \\ -15 \end{pmatrix} \implies \mathbf{x} = \begin{pmatrix} 12 \\ 6 \\ -13 \\ -15 \end{pmatrix}.$$

LU Factorization with Row Interchanges

- For each nonsingular matrix $\mathbf{A}$, there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{PA}$ possesses an LU factorization $\mathbf{PA} = \mathbf{LU}$.

- To compute $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{P}$, successively overwrite the array originally containing $\mathbf{A}$. Replace each entry being annihilated with the multiplier used to execute the annihilation. Whenever row interchanges such as those used in partial pivoting are implemented, the multipliers in the array will automatically be interchanged in the correct manner.

- Although the entire permutation matrix $\mathbf{P}$ is rarely called for, it can be constructed by permuting the rows of the identity matrix $\mathbf{I}$ according to the various interchanges used. These interchanges can be accumulated in a "permutation counter column" $\mathbf{p}$ that is initially in natural order $(\,1, 2, \ldots, n\,)$—see Example 3.10.4.

- To solve a nonsingular linear system $\mathbf{Ax} = \mathbf{b}$ using the LU decomposition with partial pivoting, permute the components in $\mathbf{b}$ to construct $\tilde{\mathbf{b}}$ according to the sequence of interchanges used—i.e., according to $\mathbf{p}$—and then solve $\mathbf{Ly} = \tilde{\mathbf{b}}$ by forward substitution followed by the solution of $\mathbf{Ux} = \mathbf{y}$ using back substitution.

Example 3.10.6

The LDU factorization. There's some asymmetry in an LU factorization because the lower factor has 1's on its diagonal while the upper factor has a nonunit diagonal. This is easily remedied by factoring the diagonal entries out of the upper factor as shown below:

$$\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & \cdots & u_{2n}/u_{22} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Setting $\mathbf{D} = \operatorname{diag}(u_{11}, u_{22}, \ldots, u_{nn})$ (the diagonal matrix of pivots) and redefining $\mathbf{U}$ to be the rightmost upper-triangular matrix shown above allows any LU factorization to be written as $\mathbf{A} = \mathbf{LDU}$, where $\mathbf{L}$ and $\mathbf{U}$ are lower- and upper-triangular matrices with 1's on both of their diagonals. This is called the ***LDU factorization*** of $\mathbf{A}$. It is uniquely determined, and when $\mathbf{A}$ is symmetric, the LDU factorization is $\mathbf{A} = \mathbf{LDL}^T$ (Exercise 3.10.9).

Example 3.10.7

The Cholesky Factorization.[22] A symmetric matrix $\mathbf{A}$ possessing an LU factorization in which each pivot is positive is said to be ***positive definite***.

Problem: Prove that $\mathbf{A}$ is positive definite if and only if $\mathbf{A}$ can be uniquely factored as $\mathbf{A} = \mathbf{R}^T\mathbf{R}$, where $\mathbf{R}$ is an upper-triangular matrix with positive diagonal entries. This is known as the *Cholesky factorization* of $\mathbf{A}$, and $\mathbf{R}$ is called the *Cholesky factor* of $\mathbf{A}$.

Solution: If $\mathbf{A}$ is positive definite, then, as pointed out in Example 3.10.6, it has an LDU factorization $\mathbf{A} = \mathbf{LDL}^T$ in which $\mathbf{D} = \operatorname{diag}(p_1, p_2, \ldots, p_n)$ is the diagonal matrix containing the pivots $p_i > 0$. Setting $\mathbf{R} = \mathbf{D}^{1/2}\mathbf{L}^T$ where $\mathbf{D}^{1/2} = \operatorname{diag}\left(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}\right)$ yields the desired factorization because $\mathbf{A} = \mathbf{LD}^{1/2}\mathbf{D}^{1/2}\mathbf{L}^T = \mathbf{R}^T\mathbf{R}$, and $\mathbf{R}$ is upper triangular with positive diagonal

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[22] This is named in honor of the French military officer Major André-Louis Cholesky (1875–1918). Although originally assigned to an artillery branch, Cholesky later became attached to the Geodesic Section of the Geographic Service in France where he became noticed for his extraordinary intelligence and his facility for mathematics. From 1905 to 1909 Cholesky was involved with the problem of adjusting the triangularization grid for France. This was a huge computational task, and there were arguments as to what computational techniques should be employed. It was during this period that Cholesky invented the ingenious procedure for solving a positive definite system of equations that is the basis for the matrix factorization that now bears his name. Unfortunately, Cholesky's mathematical talents were never allowed to flower. In 1914 war broke out, and Cholesky was again placed in an artillery group—but this time as the commander. On August 31, 1918, Major Cholesky was killed in battle. Cholesky never had time to publish his clever computational methods—they were carried forward by word-of-mouth. Issues surrounding the Cholesky factorization have been independently rediscovered several times by people who were unaware of Cholesky, and, in some circles, the Cholesky factorization is known as the *square root method.*

entries. Conversely, if $\mathbf{A} = \mathbf{R}\mathbf{R}^T$, where $\mathbf{R}$ is lower triangular with a positive diagonal, then factoring the diagonal entries out of $\mathbf{R}$ as illustrated in Example 3.10.6 produces $\mathbf{R} = \mathbf{L}\mathbf{D}$, where $\mathbf{L}$ is lower triangular with a unit diagonal and $\mathbf{D}$ is the diagonal matrix whose diagonal entries are the r_{ii}'s. Consequently, $\mathbf{A} = \mathbf{L}\mathbf{D}^2\mathbf{L}^T$ is the LDU factorization for $\mathbf{A}$, and thus the pivots must be positive because they are the diagonal entries in $\mathbf{D}^2$. We have now proven that $\mathbf{A}$ is positive definite if and only if it has a Cholesky factorization. To see why such a factorization is unique, suppose $\mathbf{A} = \mathbf{R}_1\mathbf{R}_1^T = \mathbf{R}_2\mathbf{R}_2^T$, and factor out the diagonal entries as illustrated in Example 3.10.6 to write $\mathbf{R}_1 = \mathbf{L}_1\mathbf{D}_1$ and $\mathbf{R}_2 = \mathbf{L}_2\mathbf{D}_2$, where each $\mathbf{R}_i$ is lower triangular with a unit diagonal and $\mathbf{D}_i$ contains the diagonal of $\mathbf{R}_i$ so that $\mathbf{A} = \mathbf{L}_1\mathbf{D}_1^2\mathbf{L}_1^T = \mathbf{L}_2\mathbf{D}_2^2\mathbf{L}_2^T$. The uniqueness of the LDU factors insures that $\mathbf{L}_1 = \mathbf{L}_2$ and $\mathbf{D}_1 = \mathbf{D}_2$, so $\mathbf{R}_1 = \mathbf{R}_2$. **Note:** More is said about the Cholesky factorization and positive definite matrices on pp. 313, 345, and 559.

Exercises for section 3.10

3.10.1. Let $\mathbf{A} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{pmatrix}$.

(a) Determine the LU factors of $\mathbf{A}$.

(b) Use the LU factors to solve $\mathbf{A}\mathbf{x}_1 = \mathbf{b}_1$ as well as $\mathbf{A}\mathbf{x}_2 = \mathbf{b}_2$, where

$$\mathbf{b}_1 = \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}.$$

(c) Use the LU factors to determine $\mathbf{A}^{-1}$.

3.10.2. Let $\mathbf{A}$ and $\mathbf{b}$ be the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & 17 \\ 3 & 6 & -12 & 3 \\ 2 & 3 & -3 & 2 \\ 0 & 2 & -2 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 17 \\ 3 \\ 3 \\ 4 \end{pmatrix}.$$

(a) Explain why $\mathbf{A}$ does not have an LU factorization.

(b) Use partial pivoting and find the permutation matrix $\mathbf{P}$ as well as the LU factors such that $\mathbf{PA} = \mathbf{LU}$.

(c) Use the information in $\mathbf{P}$, $\mathbf{L}$, and $\mathbf{U}$ to solve $\mathbf{Ax} = \mathbf{b}$.

3.10.3. Determine all values of ξ for which $\mathbf{A} = \begin{pmatrix} \xi & 2 & 0 \\ 1 & \xi & 1 \\ 0 & 1 & \xi \end{pmatrix}$ fails to have an LU factorization.

3.10.4. If $\mathbf{A}$ is a nonsingular matrix that possesses an LU factorization, prove that the pivot that emerges after $(k+1)$ stages of standard Gaussian elimination using only Type III operations is given by

$$p_{k+1} = a_{k+1,k+1} - \mathbf{c}^T \mathbf{A}_k^{-1} \mathbf{b},$$

where $\mathbf{A}_k$ and

$$\mathbf{A}_{k+1} = \begin{pmatrix} \mathbf{A}_k & \mathbf{b} \\ \mathbf{c}^T & a_{k+1,k+1} \end{pmatrix}$$

are the leading principal submatrices of orders k and $k+1$, respectively. Use this to deduce that all pivots must be nonzero when an LU factorization for $\mathbf{A}$ exists.

3.10.5. If $\mathbf{A}$ is a matrix that contains only integer entries and all of its pivots are 1, explain why $\mathbf{A}^{-1}$ must also be an integer matrix. **Note:** This fact can be used to construct random integer matrices that possess integer inverses by randomly generating integer matrices $\mathbf{L}$ and $\mathbf{U}$ with unit diagonals and then constructing the product $\mathbf{A} = \mathbf{LU}$.

3.10.6. Consider the tridiagonal matrix $\mathbf{T} = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & 0 \\ \alpha_1 & \beta_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{pmatrix}$.

(a) Assuming that $\mathbf{T}$ possesses an LU factorization, verify that it is given by

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_1/\pi_1 & 1 & 0 & 0 \\ 0 & \alpha_2/\pi_2 & 1 & 0 \\ 0 & 0 & \alpha_3/\pi_3 & 1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ 0 & \pi_2 & \gamma_2 & 0 \\ 0 & 0 & \pi_3 & \gamma_3 \\ 0 & 0 & 0 & \pi_4 \end{pmatrix},$$

where the π_i 's are generated by the recursion formula

$$\pi_1 = \beta_1 \quad \text{and} \quad \pi_{i+1} = \beta_{i+1} - \frac{\alpha_i \gamma_i}{\pi_i}.$$

Note: This holds for tridiagonal matrices of arbitrary size thereby making the LU factors of these matrices very easy to compute.

(b) Apply the recursion formula given above to obtain the LU factorization of

$$\mathbf{T} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

3.10.7. $\mathbf{A}_{n\times n}$ is called a ***band matrix*** if $a_{ij} = 0$ whenever $|i-j| > w$ for some positive integer w, called the ***bandwidth.*** In other words, the nonzero entries of $\mathbf{A}$ are constrained to be in a band of w diagonal lines above and below the main diagonal. For example, tridiagonal matrices have bandwidth one, and diagonal matrices have bandwidth zero. If $\mathbf{A}$ is a nonsingular matrix with bandwidth w, and if $\mathbf{A}$ has an LU factorization $\mathbf{A} = \mathbf{LU}$, then $\mathbf{L}$ inherits the lower band structure of $\mathbf{A}$, and $\mathbf{U}$ inherits the upper band structure in the sense that $\mathbf{L}$ has "lower bandwidth" w, and $\mathbf{U}$ has "upper bandwidth" w. Illustrate why this is true by using a generic 5×5 matrix with a bandwidth of $w = 2$.

3.10.8. (a) Construct an example of a nonsingular symmetric matrix that fails to possess an LU (or LDU) factorization.

(b) Construct an example of a nonsingular symmetric matrix that has an LU factorization but is not positive definite.

3.10.9. (a) Determine the LDU factors for $\mathbf{A} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{pmatrix}$ (this is the same matrix used in Exercise 3.10.1).

(b) Prove that if a matrix has an LDU factorization, then the LDU factors are uniquely determined.

(c) If $\mathbf{A}$ is symmetric and possesses an LDU factorization, explain why it must be given by $\mathbf{A} = \mathbf{LDL}^T$.

3.10.10. Explain why $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 12 \\ 3 & 12 & 27 \end{pmatrix}$ is positive definite, and then find the Cholesky factor $\mathbf{R}$.

As for everything else, so for a mathematical theory:
beauty can be perceived but not explained.
— *Arthur Cayley* (1821–1895)