Chapter 1

Generalities

The present chapter serves as an introduction. The first section contains several historical comments, while the second one is dedicated to a general presentation of the discipline. The third section reviews the most representative differential equations which can be solved by elementary methods. In the fourth section we gathered several mathematical models which illustrate the applicative power of the discipline. The fifth section is dedicated to some integral inequalities which will prove useful later, while the last sixth section contains several exercises and problems (whose proofs can be found at the end of the book).

1.1 Brief History

1.1.1 *The Birth of the Discipline*

The name of "*equatio differentialis*" has been used for the first time in 1676 by Gottfried Wilhelm von Leibniz in order to designate the determination of a function to satisfy together with one or more of its derivatives a given relation. This concept arose as a necessity to handle into a unitary and abstract frame a wide variety of problems in Mathematical Analysis and Mathematical Modelling formulated (and some of them even solved) by the middle of the XVII century. One of the first problems belonging to the domain of differential equations is the so-called *problem of inverse tangents* consisting in the determination of a plane curve by knowing the properties of its tangent at any point of it. The first who has tried to reduce this problem to quadratures [1] was Isaac Barrow[2] (1630–1677) who, using a geometric procedure invented by himself (in fact a substitute of the method of separation of variables), has solved several problems of this sort. In 1687 Sir Isaac Newton has integrated a linear differential equation and, in 1694, Jean Bernoulli (1667–1748) has used the *integrand factor method* in order to solve some n^{th}-order linear differential equations. In 1693 Leibniz has employed the substitution $y = tx$ in order to solve homogeneous equations, and, in 1697, Jean Bernoulli has succeeded to integrate the homonymous equation in the particular case of constant coefficients. Eighteen years later, Jacopo Riccati (1676–1754) has presented a procedure of reduction of the order of a second-order differential equation containing only one of the variables and has begun a systematic study of the equation which inherited his name. In 1760 Leonhard Euler (1707–1783) has observed that, whenever a particular solution of the Riccati equation is known, the latter can be reduced, by means of a substitution, to a linear equation. More than this, he has remarked that, if one knows two particular solutions of the same equation, its solving reduces to a single quadrature. By the systematic study of this kind of equation, Euler was one of the first important forerunners of this discipline. It is the merit of Jean le Rond D'Alembert (1717–1783) to have had observed that an n^{th}-order differential equation is equivalent to a system of n first-order differential equations. In 1775 Joseph Louis de Lagrange (1736–1813) has introduced the *variation of constants method*, which, as we can deduce from a letter to Daniel Bernoulli (1700–1782) in 1739, was been already invented by Euler. The equations of the form $Pdx + Qdy + Rdz = 0$ were for a long time considered absurd whenever the left-hand side was not an exact differential, although they were studied by Newton. It was Gaspard Monge (1746–1816) who, in 1787, has given their geometric interpretation and has rehabilitated them in the mathematical world. The notion of *singular solution* was introduced in 1715 by Brook Taylor (1685–1731) and was studied in 1736 by Alexis Clairaut (1713–1765). However, it is the merit of Lagrange who, in 1801, has defined the concept of singular solution in its nowadays acceptation, making a net

[1] By quadrature we mean the method of reducing a given problem to the computation of an integral, defined or not. The name comes from the homonymous procedure, known from the early times of Greek Geometry, which consists in finding the area of a plane figure by constructing, only by means of the ruler and compass, of a square with the same area.

[2] Professor of Sir Isaac Newton (1642–1727), Isaac Barrow is considered one of the forerunners of the Differential Calculus independently invented by two brilliant mathematicians: his former student and Gottfried Wilhelm von Leibniz (1646–1716).

distinction between this kind of solution and that of particular solution. The scientists have realized soon that many classes of differential equations cannot be solved explicitly and therefore they have been led to develop a wide variety of approximating methods, one more effective than another. Newton' statement, in the treatise on *fluxional equations* written in 1671 but published in 1736, that: *all differential equations can be solved by using power series with undetermined coefficients*, has had a deep influence on the mathematical thinking of the XVIII$^{\text{th}}$ century. So, in 1768, Euler has imaged such kind of approximation methods based on the development of the solution in power series. It is interesting to notice that, during this research process, Euler has defined the *cylindric functions* which have been baptized subsequently by the name of whom has succeeded to use them very efficiently: the astronomer Friedrich Wilhelm Bessel (1784–1846). We emphasize that, at this stage, the mathematicians have not questioned on the convergence of the power series used, and even less on the existence of the "solution to be approximated".

1.1.2 *Major Themes*

In all what follows we confine ourselves to a very brief presentation of the most important steps in the study of the *initial-value problem*, called also *Cauchy problem*. This consists in the determination of a solution x, of a differential equation $x' = f(t, x)$, which for a preassigned value a of the argument takes a preassigned value ξ, i.e. $x(a) = \xi$. We deliberately do not touch upon some other problems, as for instance the boundary-value problems, very important in fact, but which do not belong to the proposed topic of this book.

As we have already mentioned, the mathematicians have realized soon that many differential equations can not be solved explicitly. This situation has faced them several major, but quite difficult problems which have had to be solved. A problem of this kind consists in finding general sufficient conditions on the data of an initial-value problem in order that this have at least one solution. The first who has established a notable result in this respect was the Baron Augustin Cauchy[3] who, in 1820, has employed the *polygonal lines method* in order to prove the local existence for the initial-value problem associated to a differential equation whose right-hand side

[3] French mathematician (1789–1857). He is the founder of Complex Analysis and the author of the first modern course in Mathematical Analysis (1821). He has observed the link between convergent and fundamental sequences of real numbers.

is of class C^1. The method, improved in 1876 by Rudolf Otto Sigismund Lipschitz (1832–1903), has been definitively imposed in 1890 in its most general and natural frame by Giuseppe Peano[4]. This explains why, in many monographs, this is referred to as the *Cauchy–Lipschitz–Peano's method*.

As in other cases, rather frequent in mathematics, in the domain of differential equations, the method of proof has preceded and finally has eclipsed the result to whose proof has had a decisive role. So, as we have already mentioned, the method of power series, one of the most in vogue among the equationists of both XVII and XVIII centuries, has become soon the favorite approach in the approximation of the solutions of certain initial-value problems. This method has circumvented its class of applicability (that class for which the right-hand side is an analytic function) only at the middle of the XIX century, almost at the same time with the development of the modern Complex Function Theory. This might explain why, the first rigorous existence result concerning analytic solutions for an initial-value problem has referred to a class of differential equations in the complex field $\mathbb{C}$ and not, as we could expect, in the real field $\mathbb{R}$. More precisely, in 1842, Cauchy, reanalyzing in a critical manner Newton' statement referring to the possibility of solving all differential equations in $\mathbb{R}$ by means of power series, has placed this problem within its most natural frame (for the time being): the Theory of Complex Functions of Several Complex Variables. In this context, in order to prove the convergence of the power series whose partial sum defines the approximate solution for an initial-value problem, he was led to invent the so-called *method of majorant series*. This method consists in the construction of a convergent series with positive terms, with the property that its general term is a majorant for the absolute value of the general term of the approximate solution' series. Such a series is called a majorant for the initial one. The method has been refined by Ernst Lindelöff who, in 1896, has proposed a majorant series, better than that one used by Cauchy, and who has shown that the very subtle arguments of Cauchy, based on the Theory of Complex Functions of Several Complex Variables, are also at hand in the real field, and more than this, *even by using simpler arguments*.

Another important step concerning the approximation of the solutions of an initial-value problem is due to Emile Picard (1856–1941) who, in 1890,

[4]Italian mathematician (1858–1932) with notable contributions in Mathematical Logic. He has formulated the axiomatic system of natural numbers and the Axiom of Choice. However, his excessive formalism was very often a real brake in the process of understanding his contributions.

in a paper mainly dedicated to partial differential equations, has introduced *the method of successive approximations*. This method, who has became well-known very soon, has its roots in Newton's *method of tangents*, and has constituted the starting point for several fundamental results in Functional Analysis as Banach's fixed point theorem.

In the very same period was born the so-called *Qualitative Theory of Differential Equations* by the fundamental contributions of Henri Poincaré.[5] As we have already noticed, the main preoccupation of the equationists of the XVII and XVIII centuries was to find efficient methods, either to solve explicitly a given initial-value problem, or at least to approximate its solutions as accurate as possible. Unfortunately, none of these objectives were realizable, and for that reason, they have been soon abandoned. Without any doubt, it is the great merit of Poincaré for being the first who has caught the fact that, in all these cases in which the quantitative arguments are not efficient, one can however obtain crucial information on a solution which can be neither expressed explicitly, nor approximated accurately.[6] More precisely, he put the problem of finding, at a first stage, of the "allure" of the curve, associated with the solution in question, leaving aside any continuous transformation which could modify it. For instance, in Poincaré's acceptation, the two curves in $\mathbb{R}^3$ illustrated in Figure 1.1.1 (a) and (b) can be identified modulo "allure", while the other two, i.e. (c) and (d) in the same Figure 1.1.1, can not. At the same time it was the birthday of the modern *Theory of Stability.* The fundamental contributions of Poincaré, of James Clerk Maxwell[7] to the study of the planets' motions, but especially

[5]French mathematician (1854–1912), the initiator of the Dynamical System Theory (an abstract version of the Theory of Differential Equations which is mainly concerned with the qualitative aspects of solutions) and that of Algebraic Topology. In *Les méthodes nouvelles de la mécanique celéste*, Volumes I, II, III, Gauthier-Villars, 1892–1893–1899, enunciates and applies several stability results to the study of the planets' motions.

[6]In his address to the International Congress of Mathematicians in 1908, Poincaré said: "In the past an equation was only considered to be solved when one had expressed the solutions with the aid of a finite number of known functions; but this is hardly possible one time in one hundred. What we can always do, or rather what we should always try to do, is to solve the *qualitative* problem so to speak, that is to try to find the general form of the curve representing the unknown function." (M. W. Hirsch's translation.)

[7]British physicist and mathematician (1831–1879) who has succeeded to unify the general theories referring the electricity and magnetism establishing the general laws of electromagnetism on whose basis he has predicted the existence of the electromagnetic field. This prediction has been confirmed later by the experiments of Heinrich Hertz (1857–1894). At the same time, he was the first who has applied the general concepts and results of stability in the study of the evolution of the rings of Saturn.

those of Alexsandr Mihailovici Lyapunov[8], have emerged into a tremendous stream of a new theory of great practical interest. A similar moment, from the viewpoint of its importance for the Stability Theory, will come only after seven decades, with the first results of Vasile M. Popov concerning the stability of the automatic controlled systems.

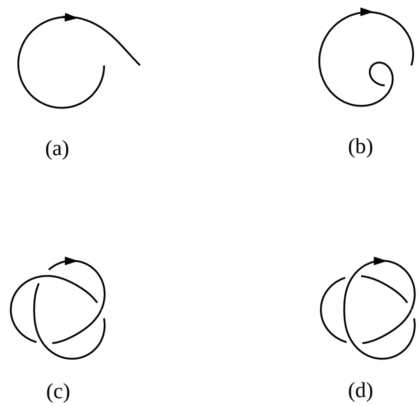

Figure 1.1.1

The last years of the XIX century were, for sure, the most prolific from the viewpoint of Differential Equations. In those golden times there have been proved the fundamental results concerning: the local existence of at least one solution (Peano 1890), the approximation of the solutions (Picard 1890), the analyticity of the solutions as functions of parameters (Poincaré 1890), the simple or asymptotic stability of solutions (Lyapunov 1892), (Poincaré 1892), the uniqueness of the solution of a given initial-value problem (William Fogg Osgood 1898). Also in the last two decades of the XIX century, Poincaré has outlined the concept of *dynamical system* in its nowadays meaning and has begun a systematic study of one of the most important and, at the same time most fascinating problems belonging to the Qualitative Theory of Differential Equations: the classification of the solutions according to their intrinsic topological properties. These referential moments have been the starting points of two new mathematical disciplines: the *Dynamical System Theory* and the *Algebraic Topology* which have developed by their own even from the first years of the XX

[8]Russian mathematician (1857–1918) who, in his doctoral thesis defended in 1892, has defined the main concepts of stability as known nowadays. He also has introduced two fundamental methods of study of the stability problems.

century. It should be also mentioned that, starting from an astrophysical problem he has raised in 1885, again Poincaré was the founder of a new discipline: *Bifurcation Theory.* Among the most representatives contributors are: Lyapunov, Erhald Schmidt, Mark Alexsandrovici Krasnoselski, David H. Sattinger and by Paul Rabinowitz, to list only a few. Also in the last decade of the XIX century, another fundamental result referring to the differentiability of the solution with respect to the initial data has been discovered. Namely, in 1896, Ivar Bendixon has proved the above mentioned result for the scalar differential equation, in 1897 Peano has extended it to the case of a system of differential equations, but it was the merit of Thomas H. Gronwall who, in 1919, using the homonymous integral inequality he has proved just to this aim, has given the most elegant proof and, therefore the most frequently used by now.

The beginning of the XX century was been deeply influenced by Poincaré's innovating ideas. Namely, in 1920, Garret David Birkhoff has rigorously founded the *Dynamical System Theory.* At this point, one should mention that the subsequent fundamental contributions are due mainly to Andrej Nikolaevich Kolmogorov[9], Vladimir Igorevich Arnold, Jürgen Kurt Moser, Joseph Pierre LaSalle (1916–1983), Morris W. Hirsch, Stephen Smale and George Sell. A special mention in this respect deserves the so-called KAM Theory, i.e. Kolmogorov–Arnold–Moser Theory. Coming back to the third decade of the XX century, at that time, a very important step was made toward a functional approach for such kind of problems. Birkhoff, together with Oliver Dimon Kellogg were the first who, in 1922, have used fixed point topological arguments in order to prove some existence and uniqueness results for certain classes of differential equations. These topological methods were initiated by Luitzen Egbertus Jan Brouwer[10], extended and generalized subsequently by Solomon Lefschetz (1984–1972), and refined in 1934 by Jean Leray and Juliusz Schauder who have expressed them into a very general abstract and elegant form, known nowadays under the name of *Leray–Schauder Topological Degree.* Renato Cacciopoli was the first who, in 1930, has employed the *Contraction Principle* as a method of proof for an existence and uniqueness theorem. However, it is

[9]Russian mathematician (1903–1987). He is the founder of the modern Probability Theory. He has remarkable contributions in Dynamical System Theory with application to Hamiltonian systems.

[10]Dutch mathematician and philosopher (1881–1966). He is one of the founders of the Intuitionists School. His famous fixed point theorem says that *every continuous function f, from a nonempty convex compact set $K \subset \mathbb{R}^n$ into K, has at least one fixed point $x \in K$, i.e. $f(x) = x$.*

the merit of Stefan Banach who, even earlier, i.e. in 1922, has given its general abstract form known, as a result under the name of *Banach's fixed point theorem*, and as a method of proof under the name of *the method of successive approximations*.

Concerning the qualitative properties of solutions the mathematicians have focused their attention on the study of the so-called *ergodic behavior* beginning with Birkhoff (1931) and continuing with John von Neumann[11] (1932), Kôsaku Yosida (1938), Yosida and Shizuo Kakutani (1938), *etc.* Due mainly to their applications in Chemistry, Electricity and Biology, the existence and properties of the so-called *limit cycles*, whose study was initiated also by Poincaré (1881), became another subject of great interest. Motivated by the study of self-sustained oscillations in nonlinear electric circuits, the theory of limit cycles grew up rapidly since the 1920s and 1930s with the contributions of G. Duffing, M. H. Dulac, B. Van der Pol and A. A. Andronov. Notable contributions in this topic (especially to the study of some specific classes of quadratic systems) are mostly due to Chinese, Russian and Ukraïnean mathematicians as N. N. Bautin, A. N. Sharkovskij, S.-L. Shi, S. I. Yashenko, Y. C. Ye, and others.

In this period Erich Kamke has established the classical theorem on the continuous dependence of the solution of an initial-value problem on the data and on the parameters, theorem extended in 1957 by Jaroslav Kurzweil. Also Kamke, following Paul Montel, Enrico Bompiani, Leonida Tonelli and Oscar Perron, has introduced the so-called *comparison method* in order to obtain sharp uniqueness results. This method proved useful in the study of some stability problems and, surprisingly, as subsequently observed by Felix E. Browder, even in the proof of existence theorems.

Concerning the concept of solution, the new type of integral defined in 1904 by Henri Lebesgue, has offered the possibility to extend the classical theory of differential equations based on the Riemann (in fact Cauchy) integral to another theory resting heavily upon the *Lebesgue integral*. This major step was made in 1918 by Constantin Carathéodory. Subsequent extensions, based on another type of integral, more general than that of Lebesgue, and known as the *Kurzweil–Henstock integral*, have been initiated in 1957 by Kurzweil.

With the same idea in mind, i.e. to enlarge the class of candidates to the title of solution, but from a completely different perspective, a new

[11] American mathematician born in Budapest (1903–1957). He is the creator of the Game Theory and has notable contributions in Functional Analysis and in Information Theory.

discipline was born: the *Theory of Distributions* initiated in 1936 by Serghei Sobolev and definitely founded in 1950–1951 by Laurent Schwartz. Initially thought as a theory exclusively useful in the linear case, the Theory of Distributions has proved its efficiency in the study of various nonlinear problems as well.

Other types of generalized solutions on which to rebuild an effective theory, especially in the nonlinear case, the so-called *viscosity solutions*, were introduced in 1950 by Eberhard Hopf and subsequently studied by Olga Oleinik and Paul Lax (1957), Stanislav Kružkov (1970), Michael G. Crandall and Pierre–Louis Lions (1983) and Daniel Tătaru (1990), among others. Notable results on the uniqueness problem, very important but at the same time extremely difficult in this context, have been obtained in 1987 by Michael G. Crandall, Hitoshi Ishii and Pierre–Louis Lions.

Since 1950, with the publication of the famous counter-example due to Jean Dieudonné, one has realized that, on some infinite dimensional spaces, as for instance c_0[12], only the continuity of the right-hand side is not enough to ensure the local existence for an initial-value problem. This strange, but not unexpected situation, was been completely elucidated in 1975 by Alexsandr Nicolaevici Godunov, who has proved that, *for every infinite dimensional Banach space X there exist a continuous function $f : X \to X$ and $\xi \in X$ such that the Cauchy problem $x' = f(x)$, $x(0) = \xi$ has no local solution.* Maybe from these reasons, starting with the end of the fifties, one has observed a growing interest in the study of the local existence problem in infinite dimensional Banach spaces and of some qualitative problems. In this respect we mention the results of Constantin Corduneanu and Aristide Halanay.

The development of a functional calculus based on the Theory of Functions of a Complex Variable taking values into a Banach algebra was accomplished in parallel with the study of the "Abstract Theory of Differential Equations". So, in 1935, Nelson Dunford has introduced the curvilinear integral of an analytic function with values in a Banach algebra and has proved a Cauchy type representation formula for the exponential as a function of an operator. In 1948, Einar Hille and Kôsaku Yosida, starting from the study of some partial differential equations, has introduced and studied independently an abstract class of linear differential equations, with possible discontinuous right hand-side, and have proved the famous generation theorem concerning C_0-semigroups, known as *the Hille–Yosida Theorem.*

[12] We recall that c_0 is the space of all real sequences approaching 0 as n tends to ∞. Endowed with the sup norm this is an infinite dimensional real Banach space.

The necessary and sufficient condition expressed in this theorem has been extended in 1967 to the fully nonlinear case, but only in a Hilbert space frame, by Yukio Kõmura, while the sufficiency part, by far the most interesting, has been proved in the general Banach space frame in 1971 by Michael G. Crandall and Thomas M. Liggett. This result[13] is known as the *Crandall–Liggett Generation Theorem*, while the formula established in the proof as the *Exponential Formula*.

In parallel with the extension of the differential equations' framework to infinite dimensional spaces via the already mentioned contributions, but also through those of Philippe Bénilan (1940–2000), Haïm Brezis, Toshio Kato, Jaques–Louis Lions (1928–2001), Amnon Pazy, one has reconsidered the study of some problems of major interest in this new and fairly general context. So, in 1979, Ciprian Foiaş and Roger Temam have obtained one of the first deepest results concerning the existence of the inertial manifolds and have estimated the dimension of such manifolds in the case of the Navier–Stokes system in fluid dynamics. Results of this kind essentially state that, some infinite-dimensional systems have, for large values of the time variable, a "finite-dimensional-type" behavior.

The systematic study of optimal control problems in $\mathbb{R}^n$, initiated in the fifties by Lev Pontriaghin (1908–1988), Revaz Valerianovici Gamkrélidze and Vladimir Grigorievici Boltianski, has been continued in the sixties and seventies by: Lamberto Cesari, Richard Bellman, Rudolf Emil Kalman, Wendell Helms Fleming, Jaques–Louis Lions, Hector O. Fattorini, among others. We notice that Lions was the first who has extended this theory to the framework of linear differential equations in infinite-dimensional spaces in order to handle control problems governed by partial differential equations as well. Notable results in this direction, but in the fully nonlinear case, have been obtained subsequently by Viorel Barbu.

We conclude these brief historical considerations which reflect rather a subjective viewpoint of the author and which are far from being complete[14], by emphasizing that the Theory of Differential Equations is a continuously growing discipline, whose by now classical results are very often extended and generalized in order to handle new cases suggested by practice and even who is permanently enriched by completely new results having no direct

[13] A simplified version of this fundamental result is presented in Section 3.4 of this book.

[14] The interested reader willing to get additional information concerning the evolution of this discipline is referred to [Wieleitner (1964)], [Hirsch (1984)] and [Piccinini *et al.* (1984)].

correspondence within its classical counterpart. For this reason, all those interested in mathematical research may found in this domain a wealth of various open problems waiting to be solved, or even more, they may formulate and solve by themselves new and interesting problems.

1.2 Introduction

Differential Equations and Systems. *Differential Equations* have their roots as a "by its own" discipline in the natural interest of scientists to predict, as accurate as possible, the future evolution of a certain physical, biological, chemical, sociological, *etc.* system. It is easy to realize that, in order to get a fairly acceptable prediction close enough to the reality, we need fairly precise data on the present state of the system, as well as, sound knowledge on the law(s) according to which the instantaneous state of the system affects its instantaneous rate of change. *Mathematical Modelling* is that discipline which comes into play at this point, offering the scientist the description of such laws in a mathematical language, laws which, in many specific situations, take the form of differential equations, or even of systems of differential equations.

The goal of the present section is to define the concept of differential equation, as well as that of system of differential equations, and to give a brief review of the main problems to be studied in this book.

Roughly speaking, a *scalar differential equation* represents a functional dependence relationship between the values of a real valued function, called *unknown function*, some, but at least one of its ordinary (partial) derivatives up to a given order n, and the independent variable(s).

The highest order of differentiation of the unknown function involved in the equation is called the *order of the equation*.

A differential equation whose unknown function depends on one real variable is called *ordinary differential equation*, while a differential equation whose unknown function depends on two, or more, real independent variables is called a *partial differential equation*. For instance the equation

$$x'' + x = \sin t,$$

whose unknown function x depends on one real variable t, is an ordinary differential equation of second order, while the equation

$$\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial u}{\partial y} = 0,$$

whose unknown function u depends on two independent real variables x and y, is a third-order partial differential equation.

In the present book we will focus our attention mainly on the study of ordinary differential equations which from now on, whenever no confusion may occur, we simply refer to as differential equations. However, we will touch upon on passing some problems referring to a special class of partial differential equations whose most appropriate and natural approach is offered by the ordinary differential equations' frame.

The general form of an n^{th}*-order scalar differential equation* with the unknown function x is

$$F(t, x, x', \dots, x^{(n)}) = 0, \tag{$\mathcal{E}$}$$

where F is a function defined on a subset $D(F)$ in $\mathbb{R}^{n+2}$ and taking values in $\mathbb{R}$, which is not constant with respect to the last variable.

Under usual regularity assumptions on the function F (required by the applicability of the *Implicit Functions Theorem*), ($\mathcal{E}$) may be rewritten as

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}), \tag{$\mathcal{N}$}$$

where f is a function defined on a subset $D(f)$ in $\mathbb{R}^{n+1}$ with values in $\mathbb{R}$, which explicitly defines $x^{(n)}$ (at least locally) as a function of $t, x, x', \dots, x^{(n-1)}$, by means of the relation $F(t, x, x', \dots, x^{(n)}) = 0$. An equation of the form ($\mathcal{N}$) is called n^{th}*-order scalar differential equation in normal form*. With few exceptions, in all what follows, we will focus our attention on the study of *first-order differential equations in normal form*, i.e. on the study of differential equations of the form

$$x' = f(t, x), \tag{$\mathcal{O}$}$$

where f is a function defined on $D(f) \subseteq \mathbb{R}^2$ taking values in $\mathbb{R}$.

By analogy, if $g : D(g) \to \mathbb{R}^n$ is a given function, $g = (g_1, g_2, \dots, g_n)$, where $D(g)$ is included in $\mathbb{R} \times \mathbb{R}^n$, we may define a *system of n first-order differential equations* with n unknown functions: $y_1, y_2, \dots, y_n$, as a system of the form

$$\begin{cases} y_i' = g_i(t, y_1, y_2, \dots, y_n) \\ i = 1, 2, \dots, n, \end{cases} \tag{$\mathcal{S}$}$$

which, in its turn, represents the componentwise expression of a *first-order vector differential equation*

$$y' = g(t, y). \tag{$\mathcal{V}$}$$

By means of the transformations[15]

$$\begin{cases} y = (y_1, y_2, \ldots, y_n) = (x, x', \ldots, x^{(n-1)}) \\ g(t, y) = (y_2, y_3, \ldots, y_n,\ f(t, y_1, y_2, \ldots, y_n)), \end{cases} \tag{$\mathcal{T}$}$$

$(\mathcal{N})$ can be equivalently rewritten as system of n scalar differential equations with n unknown functions:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = f(t, y_1, y_2, \ldots, y_n), \end{cases}$$

or, in other words, as a first-order vector differential equation $(\mathcal{V})$, with g defined by $(\mathcal{T})$. This way, the study of the equation $(\mathcal{N})$ reduces to the study of an equation of the type $(\mathcal{V})$ or, equivalently, to the study of a first-order differential system. This explains why, in all what follows, we will merely study the equation $(\mathcal{V})$, noticing only, whenever necessary, how to transcribe the results referring to $(\mathcal{V})$ in terms of $(\mathcal{N})$ by means of the transformations $(\mathcal{T})$.

We notice that, when the function g in $(\mathcal{V})$ does not depend explicitly on t, the equation $(\mathcal{V})$ is called *autonomous*. Under similar circumstances, the system $(\mathcal{S})$ is called *autonomous*. For instance, the equation

$$y' = 2y$$

is autonomous, while the equation

$$y' = 2y + t$$

is not. We emphasize however that every non-autonomous equation of the form $(\mathcal{V})$ may be equivalently rewritten as an autonomous one:

$$z' = h(z), \tag{$\mathcal{V}'$}$$

where the unknown function z has an extra-component (than y). More precisely, setting $z = (z_1, z_2, \ldots, z_{n+1}) = (t, y_1, y_2, \ldots, y_n)$ and defining $h : D(g) \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$h(z) = (1, g_1(z_1, z_2, \ldots, z_{n+1}), \ldots, g_n(z_1, z_2, \ldots, z_{n+1}))$$

[15]Transformations proposed by Jean Le Rond D'Alembert.

for each $z \in D(g)$, we observe that $(\mathcal{V}')$ represents the equivalent writing of $(\mathcal{V})$. So, the first-order scalar differential equation $y' = 2y + t$ may be rewritten as a first-order vector differential equation in $\mathbb{R}^2$, of the form $z' = h(z)$, where $z = (z_1, z_2) = (t, y)$ and $h(z) = (1, 2z_2 + z_1)$. Similar considerations are in effect for the differential system $(\mathcal{S})$ too.

Type of Solutions. As defined by now, somehow descriptive and far from being rigorous, the concept of differential equation is ambiguous because we have not specified what is the sense in which the equality $(\mathcal{E})$ should be understood[16]. Namely, let us observe from the very beginning that anyone of the two formal equalities $(\mathcal{E})$, or $(\mathcal{N})$ may be thought as being satisfied in at least one of the next three particular meanings described below:

(i) for every t in the domain $\mathbb{I}_x$ of the unknown function x;
(ii) for every t in $\mathbb{I}_x \setminus \mathbb{E}$, with $\mathbb{E}$ an exceptional set (finite, countable, negligible, *etc.*);
(iii) in a generalized sense which might have nothing to do with the usual point-wise equality.

It becomes now clear that a crucial problem arising at this stage is that of how to define the concept of solution for $(\mathcal{E})$ by specifying what is the precise meaning of the equality $(\mathcal{E})$. It should be noted that any construction of a rigorous theory of Differential Equations is very sensitive on the manner in which we solve this starting problem. The following examples are of some help in order to understand the importance, and to evaluate the exact "dimension" of this challenge.

Example 1.2.1 Let us consider the so-called *eikonal equation*

$$|x'| = 1. \tag{1.2.1}$$

It is easy to see that the only C^1 functions, $x : \mathbb{R} \to \mathbb{R}$, satisfying (1.2.1) for each $t \in \mathbb{R}$ are of the form $x(t) = t + c$, or $x(t) = -t + c$, with $c \in \mathbb{R}$ and conversely. On the other hand, if we ask that (1.2.1) be satisfied for each $t \in \mathbb{R}$, with the possible exception of those points in a finite subset, besides the functions specified above, we may easily see that any function having the graph as in Figure 1.2.1 is a solution of (1.2.1) in this new acceptation.

[16] In fact, we indicated only a formal relation which could define a predicate (the differential equation) but we did not specify the domain on which it acts (it is defined).

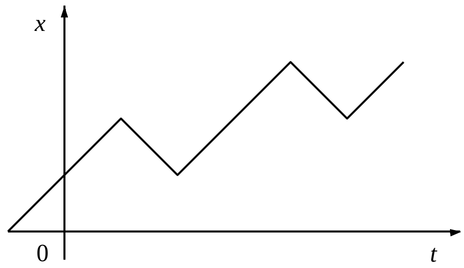

Figure 1.2.1

Example 1.2.2 Now, let us consider the differential equation

$$x' = h,$$

where $h : \mathbb{I} \to \mathbb{R}$ is a given function. It is obvious that if h is continuous, then x is of class C^1, while if h is discontinuous, the equation above cannot have C^1 solutions defined on the whole interval $\mathbb{I}$.

These examples emphasize the importance of the class of functions in which we agree to accept the candidates to the title of solution. So, if this class is too narrow, the chance to have ensured the existence of at least one solution is very small, while, if this class is too broad, this chance, which is obviously increasing, is drastically counterbalanced by the price paid by the lack of several regularity properties of solutions. Therefore, the concept of solution for a differential equation has to be defined having in mind a compromise, namely that on one hand to let have at least one solution and, on the other one, each solution to let have sufficient regularity properties in order to be of some use in practice. From the examples previously analyzed, it is easy to see that the definition of this concept should take into account firstly the regularity properties of the function F. Throughout, we shall say that an interval is *nontrivial* if it has nonempty interior. So, assuming that F is of class C^n, it is natural to adopt:

Definition 1.2.1 A *solution* of the n^{th}-order scalar differential equation $(\mathcal{E})$ is a function $x : \mathbb{I}_x \to \mathbb{R}$ of class C^n on the nontrivial interval $\mathbb{I}_x$, which satisfies $(t, x(t), x'(t), \ldots, x^{(n)}(t)) \in D(F)$ and

$$F(t, x(t), x'(t), \ldots x^{(n)}(t)) = 0$$

for each $t \in \mathbb{I}_x$.

Definition 1.2.2 A *solution* of the n^{th}-order scalar differential equation in the normal form $(\mathcal{N})$ is a function $x : \mathbb{I}_x \to \mathbb{R}$ of class C^n on the nontrivial interval $\mathbb{I}_x$, which satisfies $(t, x(t), x'(t), \dots, x^{(n-1)}(t)) \in D(f)$ and

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots x^{(n-1)}(t))$$

for each $t \in \mathbb{I}_x$.

Definition 1.2.3 A *solution* of the system of first-order differential equations $(\mathcal{S})$ is an n-tuple of functions $(y_1, y_2, \dots, y_n) : \mathbb{I}_y \to \mathbb{R}^n$ of class C^1 on the nontrivial interval $\mathbb{I}_y$, which satisfies $(t, y_1(t), y_2(t), \dots, y_n(t)) \in D(g)$ and $y_i'(t) = g_i(t, y_1(t), y_2(t), \dots, y_n(t))$, $i = 1, 2, \dots, n$, for each $t \in \mathbb{I}_y$. *The trajectory* corresponding to the solution y is the set $\tau(y) = \{y(t);\ t \in \mathbb{I}_y\}$.

The trajectory corresponding to a given solution $y = (y_1, y_2)$ of a differential system in $\mathbb{R}^2$ is illustrated in Figure 1.2.2 (a), while the graph of the solution in Figure 1.2.2 (b).

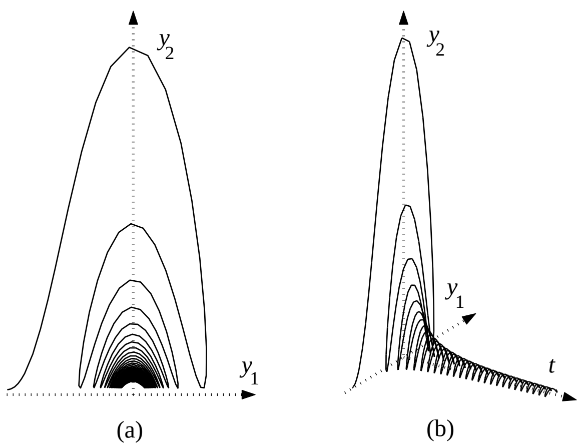

Figure 1.2.2

Definition 1.2.4 A *solution* of the first-order vector differential equation $(\mathcal{V})$ is a function $y : \mathbb{I}_y \to \mathbb{R}^n$ of class C^1 on the nontrivial interval $\mathbb{I}_y$, which satisfies $(t, y(t)) \in D(g)$ and $y'(t) = g(t, y(t))$ for each $t \in \mathbb{I}_y$. *The trajectory* corresponding to the solution y is the set $\tau(y) = \{y(t);\ t \in \mathbb{I}_y\}$.

Let us observe that the problem of finding the antiderivatives of a continuous function h on a given interval $\mathbb{I}$ may be embedded into a first-order differential equation of the form $x' = h$ for which, from the set of solutions given by Definition 1.2.1, we keep only those defined on $\mathbb{I}$, the "maximal domain" of the function h.

Definition 1.2.5 A family $\{x(\cdot, c) : \mathbb{I}_{x,c} \to \mathbb{R};\ c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n\}$ of functions, implicitly defined by a relation of the form

$$G(t, x, c_1, c_2, \ldots, c_n) = 0, \tag{$\mathcal{G}$}$$

where $G : D(G) \subseteq \mathbb{R}^{n+2} \to \mathbb{R}$, is a function of class C^n with respect to the first two variables, with the property that, by eliminating the constants $c_1, c_2, \ldots, c_n$ from the system

$$\begin{cases} \dfrac{d}{dt}\left[G(\cdot, x(\cdot), c_1, c_2, \ldots, c_n)\right](t) = 0 \\[2ex] \dfrac{d^2}{dt^2}\left[G(\cdot, x(\cdot), c_1, c_2, \ldots, c_n)\right](t) = 0 \\ \vdots \\ \dfrac{d^n}{dt^n}\left[G(\cdot, x(\cdot), c_1, c_2, \ldots, c_n)\right](t) = 0 \end{cases}$$

and substituting these in $(\mathcal{G})$ one gets exactly $(\mathcal{E})$, is called *the general integral*, or *the general solution* of $(\mathcal{E})$.

Usually, we identify the general solution by its relation of definition saying that $(\mathcal{G})$ is *the general solution*, or *the general integral* of $(\mathcal{E})$.

Example 1.2.3 The general integral of the second-order differential equation

$$x'' + a^2 x = 0,$$

with $a > 0$, is $\{x(\cdot, c_1, c_2);\ (c_1, c_2) \in \mathbb{R}^2\}$, where

$$x(t, c_1, c_2) = c_1 \sin at + c_2 \cos at$$

for $t \in \mathbb{I}_{x,c}$[17]. Indeed, it is easy to see that the equation is obtained by eliminating the constants c_1, c_2 from the system

$$\begin{cases} (x - c_1 \sin at - c_2 \cos at)' = 0 \\ (x - c_1 \sin at - c_2 \cos at)'' = 0. \end{cases}$$

[17]We mention that, in this case, the general integral contains also functions defined on the whole set $\mathbb{R}$, i.e. for which $\mathbb{I}_{x,c} = \mathbb{R}$.

In this case, $G : \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$G(t, x, c_1, c_2) = x - c_1 \sin at - c_2 \cos at$$

for each $(t, x, c_1, c_2) \in \mathbb{R}^4$, and $(\mathcal{G})$ may be equivalently rewritten as

$$x = c_1 \sin at + c_2 \cos at,$$

relation which defines explicitly the general integral. As we shall see later, in many other specific cases too, in which from $(\mathcal{G})$ one can get the explicit form of x as a function of $t, c_1, c_2, \ldots, c_n$, the general integral of $(\mathcal{E})$ can be expressed in an explicit form as $x(t, c_1, c_2, \ldots, c_n) = H(t, c_1, c_2, \ldots, c_n)$, with $H : D(H) \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ a function of class C^n.

Problems to be Studied. Next, we shall list several problems which we shall approach in the study of the equation $(\mathcal{V})$. We begin by noticing that the main problem we are going to treat is the so-called *Cauchy problem*, or *initial value problem* associated to $(\mathcal{V})$. More precisely, given $(a, \xi) \in D(g)$, the *Cauchy problem* for $(\mathcal{V})$ with data a and ξ consists in finding of a particular solution $y : \mathbb{I}_y \to \mathbb{R}^n$ of $(\mathcal{V})$, with $a \in \mathbb{I}_y$ and satisfying the *initial condition* $y(a) = \xi$. Customarily a is called the *initial time*, while ξ the *initial state*.

In the study of this problem we shall encounter the following subproblems of obvious importance: (1) the existence problem which consists in finding reasonable sufficient conditions on the function g so that, for each $(a, \xi) \in D(g)$, the Cauchy problem for the equation $(\mathcal{V})$, with a and ξ as data, have at least one solution[18]; (2) the uniqueness problem which consists in finding sufficient conditions on the function g so that, for each $(a, \xi) \in D(g)$, the Cauchy problem for the equation $(\mathcal{V})$, with a and ξ as data, have at most one solution defined on a given interval containing a; (3) the problem of continuation of the solutions; (4) the problem of the behavior of the non-continuable solution at the end(s) of the maximal interval of definition; (5) the problem of approximation of a given solution; (6) the problem of continuous dependence of the solution on both the initial

[18] In many circumstances, in the process of establishing a mathematical model, one deliberately ignores the contribution of certain "parameters" whose influence on the evolution of the system in question is considered irrelevant. For this reason, almost all mathematical models are not at all identical copies of the reality and, accordingly, a first problem of great importance we face in this context (problem which is superfluous in the case of the real phenomenon) is that of the consistency of the model. But this consists in showing that the model in question has at least one solution.

datum ξ and the right-hand side g; (7) the problem of differentiability of the solution with respect to the initial datum ξ; (8) the problem of getting additional information in the particular case in which $g : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ and, for each $t \in \mathbb{I}$, $g(t, \cdot)$ is a linear function; (9) the study of the behavior of the solutions as t approaches $+\infty$.

1.3 Elementary Equations

The goal of this section is to collect several types of differential equations whose general solutions can be found by means of a finite number of integration procedures. Since the integration of real functions of one real variable is also called *quadrature*, these equations are known under the name of *equations solved by quadratures*.

1.3.1 *Equations with Separable Variables*

An equation of the form

$$x' = f(t)g(x), \tag{1.3.1}$$

where $f : \mathbb{I} \to \mathbb{R}$ and $g : \mathbb{J} \to \mathbb{R}$ are two continuous functions with $g(y) \neq 0$ for each $y \in \mathbb{J}$, is called *with separable variables.*

Theorem 1.3.1 *Let $\mathbb{I}$ and $\mathbb{J}$ be two nontrivial intervals in $\mathbb{R}$ and let $f : \mathbb{I} \to \mathbb{R}$ and $g : \mathbb{J} \to \mathbb{R}$ be two continuous functions with $g(y) \neq 0$ for each $y \in \mathbb{J}$. Then, the general solution of the equation* (1.3.1) *is given by*

$$x(t) = G^{-1}\left(\int_{t_0}^{t} f(s)\,ds\right) \tag{1.3.2}$$

for each $t \in \mathrm{Dom}(x)$, where t_0 is a fixed point in $\mathbb{I}$, and $G : \mathbb{J} \to \mathbb{R}$ is defined by

$$G(y) = \int_{\xi}^{y} \frac{d\tau}{g(\tau)}$$

for each $y \in \mathbb{J}$, with $\xi \in \mathbb{J}$.

Proof. Since g does not vanish on $\mathbb{J}$ and is continuous, it preserves constant sign on $\mathbb{J}$. Changing the sign of the function f if necessary, we may assume that $g(y) > 0$ for each $y \in \mathbb{J}$. Then, the function G is well-defined and strictly increasing on $\mathbb{J}$.

We begin by observing that the function x defined by means of the relation (1.3.2) is a solution of the equation (1.3.1) which satisfies $x(t_0) = \xi$. Namely,

$$x'(t) = \left[G^{-1}\left(\int_{t_0}^{t} f(s)\,ds\right)\right]' = \frac{1}{G'\left(G^{-1}\left(\int_{t_0}^{t} f(s)\,ds\right)\right)} f(t) = g(x(t))f(t)$$

for each t in the domain of the function x. In addition, from the definition of G, it follows that $x(t_0) = \xi$.

To complete the proof it suffices to show that every solution of the equation (1.3.1) is of the form (1.3.2). To this aim, let $x : \mathrm{Dom}(x) \to \mathbb{J}$ be a solution of the equation (1.3.1) and let us observe that this may be equivalently rewritten as

$$\frac{x'(t)}{g(x(t))} = f(t)$$

for each $t \in \mathrm{Dom}(x)$. Integrating this equality both sides over $[\,t_0, t\,]$, we get

$$\int_{t_0}^{t} \frac{x'(s)\,ds}{g(x(s))} = \int_{t_0}^{t} f(s)\,ds$$

for each $t \in \mathrm{Dom}(x)$. Consequently we have

$$G(x(t)) = \int_{t_0}^{t} f(s)\,ds,$$

where G is defined as above with $\xi = x(t_0)$. Recalling that G is strictly increasing on $\mathbb{J}$, we conclude that it is invertible from its range $G(\mathbb{J})$ into $\mathbb{J}$. From this remark and the last equality we deduce (1.3.2). $\square$

1.3.2 *Linear Equations*

A *linear equation* is an equation of the form

$$x' = a(t)x + b(t), \tag{1.3.3}$$

where $a, b : \mathbb{I} \to \mathbb{R}$ are continuous functions on $\mathbb{I}$. If $b \equiv 0$ on $\mathbb{I}$ the equation is called *linear and homogeneous*, otherwise *linear and non-homogeneous.*

Theorem 1.3.2 *If a and b are continuous on $\mathbb{I}$ then the general solution of the equation* (1.3.3) *is given by the so-called variation of constants* *formula*

$$x(t) = \exp\left(\int_{t_0}^{t} a(s)\,ds\right)\xi + \int_{t_0}^{t} \exp\left(\int_{s}^{t} a(\tau)\,d\tau\right) b(s)\,ds \tag{1.3.4}$$

for each $t \in \mathrm{Dom}(x)$, *where* $t_0 \in \mathrm{Dom}(x)$ *is fixed,* $\xi \in \mathbb{R}$ *and* $\exp(y) = e^y$ *for each* $y \in \mathbb{R}$.

Proof. A simple computational argument shows that x defined by (1.3.4) is a solution of (1.3.3) which satisfies $x(t_0) = \xi$. So, we have merely to show that each solution of (1.3.3) is of the form (1.3.4) on its interval of definition. To this aim, let $x : \mathbb{I}_0 \to \mathbb{R}$ be a solution of the equation (1.3.3), where $\mathbb{I}_0$ is a nontrivial interval included in $\mathbb{I}$. Fix $t_0 \in \mathbb{I}_0$ and multiply both sides in (1.3.3) (with t substituted by s) by

$$\exp\left(-\int_{t_0}^{s} a(\tau)\,d\tau\right)$$

where $s \in \mathbb{I}_0$. After some obvious rearrangements, we obtain

$$\frac{d}{ds}\left(x(s)\exp\left(-\int_{t_0}^{s} a(\tau)\,d\tau\right)\right) = b(s)\exp\left(-\int_{t_0}^{s} a(\tau)\,d\tau\right)$$

for each $s \in \mathbb{I}_0$. Integrating this equality both sides between t_0 and $t \in \mathbb{I}_0$, multiplying the equality thus obtained by

$$\exp\left(\int_{t_0}^{t} a(\tau)\,d\tau\right),$$

we deduce (1.3.4), and this completes the proof. □

Remark 1.3.1 From (1.3.4) it follows that every solution of (1.3.3) may be continued as a solution of the same equation to the whole interval $\mathbb{I}$.

1.3.3 *Homogeneous Equations*

A *homogeneous equation* is an equation of the form

$$x' = h\left(\frac{x}{t}\right), \tag{1.3.5}$$

where $h : \mathbb{I} \to \mathbb{R}$ is continuous and $h(r) \neq r$ for each $r \in \mathbb{I}$.

Theorem 1.3.3 *If $h : \mathbb{I} \to \mathbb{R}$ is continuous and $h(r) \neq r$ for each $r \in \mathbb{I}$, then the general solution of* (1.3.5) *is given by*

$$x(t) = tu(t)$$

for $t \neq 0$, where u is the general solution of the equation with separable variables

$$u' = \frac{1}{t}\left(h(u) - u\right).$$

Proof. We have merely to express x' by means of u and to impose the condition that x be a solution of the equation (1.3.5). □

An important class of differential equations which can be reduced to homogeneous equations is

$$x' = \frac{a_{11}x + a_{12}t + b_1}{a_{21}x + a_{22}t + b_2}, \tag{1.3.6}$$

where a_{ij} and b_i, $i, j = 1, 2$ are constants and

$$\begin{cases} a_{11}^2 + a_{12}^2 + b_1^2 > 0 \\ a_{21}^2 + a_{22}^2 + b_2^2 > 0. \end{cases}$$

According to the compatibility of the linear algebraic system

$$\begin{cases} a_{11}x + a_{12}t + b_1 = 0 \\ a_{21}x + a_{22}t + b_2 = 0, \end{cases} \tag{$\mathcal{AS}$}$$

we distinguish between three different cases. More precisely we have:
Case I. If the system $(\mathcal{AS})$ has a unique solution (ξ, η) then, by means of the change of variables

$$\begin{cases} x = y + \xi \\ t = s + \eta, \end{cases}$$

(1.3.6) can be equivalently rewritten under the form of the homogeneous equation below

$$y' = \frac{a_{11}\dfrac{y}{s} + a_{12}}{a_{21}\dfrac{y}{s} + a_{22}};$$

Case II. If the system $(\mathcal{AS})$ has infinitely many solutions, then there exists $\lambda \neq 0$ such that

$$(a_{11}, a_{12}, b_1) = \lambda\,(a_{21}, a_{22}, b_2)$$

and therefore (1.3.6) reduces to $x' = \lambda$;
Case III. If the system $(\mathcal{AS})$ is incompatible then there exists $\lambda \neq 0$ such that

$$\begin{cases} (a_{11}, a_{12}) = \lambda\,(a_{21}, a_{22}) \\ (a_{11}, a_{12}, b_1) \neq \lambda\,(a_{21}, a_{22}, b_2) \end{cases}$$

and, by means of the substitution $y = a_{21}x + a_{22}t$ the equation reduces to an equation with separable variables.

1.3.4 *Bernoulli Equations*

An equation of the form

$$x' = a(t)x + b(t)x^{\alpha}, \tag{1.3.7}$$

where $a, b : \mathbb{I} \to \mathbb{R}$ are non-identically zero continuous functions which are not proportional on $\mathbb{I}$, and $\alpha \in \mathbb{R} \setminus \{0, 1\}$, is called *Bernoulli equation.*

Remark 1.3.2 The restrictions imposed on the data a, b and α can be explained by the simple observations that: if $a \equiv 0$ then (1.3.7) is with separable variables; if there exists $\lambda \in \mathbb{R}$ such that $a(t) = \lambda b(t)$ for each $t \in \mathbb{I}$, (1.3.7) is with separable variables too; if $b \equiv 0$ then (1.3.7) is linear and homogeneous; if $\alpha = 0$ then (1.3.7) is linear; if $\alpha = 1$ then (1.3.7) is linear and homogeneous.

Theorem 1.3.4 *If $a, b : \mathbb{I} \to \mathbb{R}$ are continuous and non-identically zero on $\mathbb{I}$ and $\alpha \in \mathbb{R} \setminus \{0, 1\}$ then x is a positive solution of the equation* (1.3.7) *if and only if the function y, defined by*

$$y(t) = x^{1-\alpha}(t) \tag{1.3.8}$$

for each $t \in \mathrm{Dom}(x)$, is a positive solution of the linear non-homogeneous equation

$$y' = (1 - \alpha)a(t)y + (1 - \alpha)b(t). \tag{1.3.9}$$

Proof. Let x be a positive solution of the equation (1.3.7). Expressing x' as a function of y and y' and using the fact that x is a solution of (1.3.7) we deduce that y is a positive solution of (1.3.9). A similar argument shows that if y is a positive solution of the equation (1.3.9), then x given by (1.3.8) is, in its turn, a positive solution of (1.3.7), and the proof is complete. $\square$

1.3.5 *Riccati Equations*

An equation of the form

$$x' = a(t)x + b(t)x^2 + c(t), \tag{1.3.10}$$

where $a, b, c : \mathbb{I} \to \mathbb{R}$ are continuous, with b and c non-identically zero on $\mathbb{I}$ is called *Riccati Equation.*

By definition we have excluded the cases $b \equiv 0$ when (1.3.10) is a linear equation and $c \equiv 0$ when (1.3.10) is a Bernoulli equation with $\alpha = 2$.

Remark 1.3.3 In general, there are no effective methods of solving a given Riccati equation, excepting the fortunate case when we dispose of an *a priori* given particular solution. The next theorem refers exactly to this particular but important case.

Theorem 1.3.5 *Let $a, b, c : \mathbb{I} \to \mathbb{R}$ be continuous with b and c non-identically zero on $\mathbb{I}$. If $\varphi : \mathbb{J} \to \mathbb{R}$ is a solution of* (1.3.10)*, then the general solution of* (1.3.10) *on $\mathbb{J}$ is given by*

$$x(t) = y(t) + \varphi(t),$$

where y is the general solution of the Bernoulli equation

$$y' = (a(t) + 2b(t)\varphi(t))y + b(t)y^2.$$

Proof. One verifies by direct computation that $x = y + \varphi$ is a solution of the equation (1.3.10) if and only if $y = x - \varphi$ is a solution of the Bernoulli equation above. $\square$

1.3.6 *Exact Differential Equations*

Let D be a nonempty and open subset in $\mathbb{R}^2$ and let $g, h : D \to \mathbb{R}$ be two functions of class C^1 on D, with $h(t, x) \neq 0$ on D. An equation of the form

$$x' = \frac{g(t, x)}{h(t, x)} \tag{1.3.11}$$

is called *exact* if there exists a function of class C^2, $F : D \to \mathbb{R}$, such that

$$\begin{cases} \dfrac{\partial F}{\partial t}(t, x) = -g(t, x) \\[2ex] \dfrac{\partial F}{\partial x}(t, x) = h(t, x). \end{cases} \tag{1.3.12}$$

The condition above shows that $-g(t,x)\,dt + h(t,x)\,dx$ is the differential dF of the function F calculated at $(t,x) \in D$.

Theorem 1.3.6 *If* (1.3.11) *is an exact equation, then its general solution is implicitly given by*

$$F(t,x) = c, \tag{1.3.13}$$

where $F : D \to \mathbb{R}$ *satisfies* (1.3.12), *and* c *ranges over* $F(D)$.

Proof. If (1.3.11) is an exact differential equation then x is one of its solutions if and only if

$$-g(t,x(t))\,dt + h(t,x(t))\,dx(t) = 0$$

for $t \in \mathrm{Dom}(x)$, equality which, by virtue of the fact that F satisfies (1.3.12), is equivalent to

$$dF(t,x(t)) = 0$$

for each $t \in \mathrm{Dom}(x)$. Since this last equality is, in its turn, equivalent to (1.3.13), the proof is complete. □

Theorem 1.3.7 *If D is a simply connected domain, then a necessary and sufficient condition in order that* (1.3.11) *be exact is*

$$\frac{\partial h}{\partial t}(t,x) = -\frac{\partial g}{\partial x}(t,x),$$

for each $(t,x) \in D$.

For the proof see Theorem 5 in [Nicolescu *et al.* (1971b)], p. 187.

1.3.7 *Equations Reducible to Exact Differential Equations*

In general if the system (1.3.12) has no solutions the method of finding the general solution of (1.3.11) described above is no longer applicable. There are however some specific cases in which, even though (1.3.12) has no solutions, (1.3.11) can be reduced to an exact equation. We describe in what follows such a method of reduction known under the name of *the integrant factor method*. More precisely, if (1.3.11) is not exact, one looks for a function $\rho : D \to \mathbb{R}$ of class C^1 with $\rho(t,x) \neq 0$ for each $(t,x) \in D$ such that

$$-\rho(t,x)g(t,x)\,dt + \rho(t,x)h(t,x)\,dx$$

be the differential of a function $F : D \to \mathbb{R}$. Assuming that D is simply connected, from Theorem 1.3.7, we know that a necessary and sufficient condition in order that this happen is that

$$h(t,x)\frac{\partial \rho}{\partial t}(t,x) + g(t,x)\frac{\partial \rho}{\partial x}(t,x) + \left(\frac{\partial g}{\partial x}(t,x) + \frac{\partial h}{\partial t}(t,x)\right)\rho(t,x) = 0$$

for each $(t,x) \in D$. This is a first-order partial differential equation with the unknown function ρ. We shall study the possibility of solving such kind of equations later on in Chapter 6. By then, let us observe that, if

$$\frac{1}{h(t,x)}\left(\frac{\partial g}{\partial x}(t,x) + \frac{\partial h}{\partial t}(t,x)\right) = f(t)$$

does not depend on x, we can look for a solution ρ of the equation above which does not depend on x too. This function ρ is a solution of the linear homogeneous equation

$$\rho'(t) = -f(t)\rho(t).$$

Analogously, if $g(t,x) \neq 0$ for $(t,x) \in D$ and

$$\frac{1}{g(t,x)}\left(\frac{\partial g}{\partial x}(t,x) + \frac{\partial h}{\partial t}(t,x)\right) = k(x),$$

does not depend on t, we can look for a solution ρ of the equation above which does not depend on t too.

1.3.8 *Lagrange Equations*

A differential equation of the non-normal form

$$x = t\varphi(x') + \psi(x')$$

in which φ and ψ are functions of class C^1 from $\mathbb{R}$ in $\mathbb{R}$ and $\varphi(r) \neq r$ for each $r \in \mathbb{R}$, is called *Lagrange Equation*. This kind of differential equation can be integrated by using the so-called *parameter method*. By this method we can find only the solutions of class C^2 under the parametric form

$$\begin{cases} t = t(p) \\ x = x(p), \quad p \in \mathbb{R}. \end{cases}$$

More precisely, let x be a solution of class C^2 of the Lagrange equation. Differentiating both sides of the equation, we get

$$x' = \varphi(x') + t\varphi'(x')\,x'' + \psi'(x')\,x''.$$

Denoting by $x' = p$, we have $x'' = p'$ and consequently

$$\frac{dp}{dt} = -\frac{\varphi(p) - p}{t\varphi'(p) + \psi'(p)}.$$

Assuming now that p is invertible and denoting its inverse by $t = t(p)$, the above equation may be equivalently rewritten as

$$\frac{dt}{dp} = -\frac{\varphi'(p)}{\varphi(p) - p}\,t - \frac{\psi'(p)}{\varphi(p) - p}.$$

But this is a linear differential equation which can be solved by the variation of constants method. We will find then $t = \theta(p, c)$ for $p \in \mathbb{R}$, with c constant, from where, using the initial equation, we deduce *the parametric equations* of the general C^2 solution of the Lagrange Equation, i.e.

$$\begin{cases} t = \theta(p, c) \\ x = \theta(p, c)\varphi(p) + \psi(p), \quad p \in \mathbb{R}. \end{cases}$$

1.3.9 *Clairaut Equations*

An equation of the form

$$x = tx' + \psi(x'),$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is of class C^1 is called *Clairaut equation.* This can be solved also by the parameter method. More precisely, let x be a solution of class C^2 of the equation. Differentiating both sides the equation, we get

$$x''(t + \psi'(x')) = 0.$$

Denoting by $x' = p$, the equation above is equivalent to $p'(t + \psi'(p)) = 0$. If $p' = 0$ it follows that $x(t) = ct + d$, with $c, d \in \mathbb{R}$, from where, imposing the condition on x to satisfy the equation, we deduce the so-called *general solution of the Clairaut equation*

$$x(t) = ct + \psi(c)$$

for $t \in \mathbb{R}$, where $c \in \mathbb{R}$. Obviously, these equations represent a family of straight lines. If $t + \psi'(p) = 0$ we deduce

$$\begin{cases} t = -\psi'(p) \\ x = -p\psi'(p) + \psi(p), \quad p \in \mathbb{R}, \end{cases}$$

system that defines a plane curve called the *singular solution of the Clairaut equation* and which, is nothing but the envelope of the family of straight lines in the general solution. We recall that the *envelope* of a family of straight lines is a curve with the property that the family of straight lines coincides with the family of all tangents to the curve.

Remark 1.3.4 In general, Clairaut equation admits certain solutions which are merely of class C^1. Such a solution can be obtained by continuing a particular arc of curve of the singular solution with those half-tangents at the endpoints of the arc in such a way to get a C^1 curve. See the solutions to Problems 1.11 and 1.12.

1.3.10 *Higher-Order Differential Equations*

In what follows we shall present two classes of n^{th}-order scalar differential equations which, even though they can not be solved by quadratures, they can be reduced to equations of order strictly less than n. Let us consider for the beginning the *incomplete* n^{th}*-order scalar differential equation*

$$F(t, x^{(k)}, x^{(k+1)}, \ldots, x^{(n)}) = 0, \tag{1.3.14}$$

where $0 < k < n$ and $F : D(F) \subset \mathbb{R}^{n-k+2} \to \mathbb{R}$. By means of the substitution $y = x^{(k)}$ this equation reduces to an $(n-k)^{\text{th}}$-order scalar differential equation with the unknown function y

$$F(t, y, y', \ldots, y^{(n-k)}) = 0.$$

Let us assume for the moment that we are able to obtain the general solution $y = y(t, c_1, c_2, \ldots, c_{n-k})$ of the latter equation. In these circumstances, we can obtain the general solution $x(t, c_1, c_2, \ldots, c_n)$ of the equation (1.3.14) by integrating k-times the identity $x^{(k)} = y$. Namely, for $a \in \mathbb{R}$ suitably chosen, we have

$$x(t, c_1, c_2, \ldots, c_n) = \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} y(s, c_1, c_2, \ldots, c_{n-k})\, ds$$

$$+ \sum_{i=1}^{k} c_{n-k+i} t^{i-1},$$

where $c_{n-k+1}, c_{n-k+2}, \ldots, c_n \in \mathbb{R}$ are constants appeared in the iterated integration process.

Example 1.3.1 Find the general solution of the third-order scalar differential equation

$$x''' = -\frac{1}{t}x'' + 3t, \quad t > 0.$$

The substitution $x'' = y$ leads to the non-homogeneous linear equation

$$y' = -\frac{1}{t}y + 3t, \quad t > 0$$

whose general solution is $y(t, c_1) = t^2 + c_1/t$ for $t > 0$. Integrating two times the identity $x'' = y$ we get $x(t, c_1, c_2, c_3) = t^4/12 + c_1(t\,\ln t - t) + c_2 t + c_3$.

A second class of higher-order differential equations which can be reduced to equations whose order is strictly less than the initial one is the class of autonomous higher-order differential equations. So, let us consider the autonomous n^{th}-order differential equation

$$F(x, x', \dots, x^{(n)}) = 0, \tag{1.3.15}$$

where $F : D(F) \subset \mathbb{R}^{n+1} \to \mathbb{R}$. Let us denote by $p = x'$, and let us express p as a function of x. To this aim let us observe that

$$\begin{cases} x'' = \dfrac{dp}{dt} = \dfrac{dp}{dx}\dfrac{dx}{dt} = \dfrac{dp}{dx}p, \\[2ex] x''' = \dfrac{d}{dt}\left(\dfrac{dp}{dx}p\right) = \dfrac{d}{dx}\left(\dfrac{dp}{dx}p\right)p, \\ \vdots \\ x^{(n)} = \dots \qquad . \end{cases}$$

In this way, for each $k = 1, 2, \dots, n$, $x^{(k)}$ can be expressed as a function of p, $\frac{dp}{dx}, \dots, \frac{dp^{k-1}}{dx^{k-1}}$. Substituting in (1.3.15) the derivatives of x as functions of p, $\frac{dp}{dx}, \dots, \frac{dp^{n-1}}{dx^{n-1}}$ we get an $(n-1)^{\text{th}}$-order differential equation.

Example 1.3.2 The second-order differential equation $x'' + \frac{g}{\ell}\sin x = 0$, i.e. *the pendulum equation*, reduces by the method described above to the first-order differential equation (with separable variables) $p\frac{dp}{dx} = -\frac{g}{\ell}\sin x$ whose unknown function is $p = p(x)$.

1.4 Some Mathematical Models

In this section we shall present several phenomena in Physics, Biology, Chemistry, Demography whose evolutions can be described highly accurate by means of some differential equations, or even systems of differential equations. We begin with an example from Physics, became well-known due to its use in archeology as a tool of dating old objects. We emphasize that, in this example, as in many others that will follow, we shall substitute the discrete mathematical model, which is the most realistic by a continuously differentiable one, and this for pure mathematical reasons. More precisely, in order to take advantage of the concepts and results of Mathematical Analysis, we shall assume that every function which describes the evolution in time of the state of the system: the number of individuals in a given species, the number of molecules in a given substance, *etc.*, is of class C^1 on its interval of definition, even though, in reality, this takes values in a very large but finite set. From a mathematical point of view this reduces to the substitution of the discontinuous function x_r, whose graph is illustrated in Figure 1.4.1 as a union of segments which are parallel to the Ot axis, by the function x whose graph is a curve of class C^1. See Figure 1.4.1.

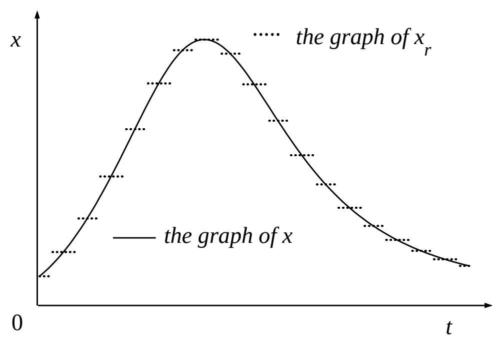

Figure 1.4.1

1.4.1 *Radioactive Disintegration*

In 1902 Ernest Rutherford Lord of Nelson[19] and Sir Frederick Soddy[20] have formulated *the law of radioactive disintegration* saying that *the instantaneous rate of disintegration of a given radioactive element is proportional to the number of radioactive atoms existing at the time considered, and does not depend on any other external factors.* Therefore, denoting by $x(t)$ the number of non disintegrated atoms at the time t and assuming that x is a function of class C^1 on $[0, +\infty)$, by virtue of the above mentioned law, we deduce that

$$-x' = ax$$

for every $t \geq 0$, where $a > 0$ is a constant, specific to the radioactive element, called *disintegration constant* and which can be determined experimentally with a sufficient degree of accuracy. This is a first-order linear homogeneous differential equation, whose general solution is given by

$$x(t) = ce^{-at} = x(0)e^{-at}$$

for $t \geq 0$, with $c \in \mathbb{R}_+$.

1.4.2 *The Carbon Dating Method*

This method[21] is essentially based on similar considerations. So, following [Hubbard and West (1995)], Example 2.5.4, p. 85, we recall that living organisms, besides the stable isotope C^{12}, contain a small amount of radioactive isotope C^{14} arising from cosmic ray bombardment. We notice that C^{14} enters the living bodies during, and due to, some specific exchange processes, such that the ration C^{14}/C^{12} is kept constant. If an organism dies, these exchange processes stop, and the radioactive C^{14} begins to decrease at a constant rate, whose approximate value (determined experimentally) is $1/8000$, i.e. one part in 8000 per year. Consequently, if $x(t)$ represents this ratio C^{14}/C^{12}, after t years from the death, we conclude that the function

[19] British chemist and physicist born in New Zealand (1871-1937). Laureate of the Nobel Prize for Chemistry in 1908, he has succeeded the first provoked transmutation of one element into another: the Nitrogen into Oxygen by means of the alpha radiations (1919). He has proposed the atomic model which inherited his name.

[20] British chemist (1877–1956). Laureate of the Nobel Prize for Chemistry in 1921.

[21] The carbon-14 method has been proposed around 1949 by Willard Libby.

$t \mapsto x(t)$ satisfies

$$x' = -\frac{1}{8000}x.$$

Consequently, if we know $x(T)$, we can find the number T, of years after death, by means of

$$T = 8000 \ln \frac{r_0}{x(T)},$$

where r_0 is the constant ratio C^{14}/C^{12} in the living matter. For more details on similar methods of dating see [Braun (1983)].

1.4.3 *Equations of Motion*

The equations of motion of n-point particles in the three-dimensional Euclidean space are described by means of *Newton second law* saying that "*Force equals mass times acceleration*". Indeed, in this case, this fundamental law takes the following mathematical expression

$$m_i x_i''(t) = F_i(x_i(t)), \quad i = 1, 2, \ldots, n,$$

where x_i is the Cartesian coordinate of the i^{th}-particle of mass m_i and F_i is the force acting on that particle. According to what kind of forces are involved: strong, weak, gravitational, or electromagnetic, we get various equations of motion. The last two forces, i.e. occurring in gravitation and electromagnetism, can be expressed in a rather simple manner in the case when the velocities of the particles are considerably less than the speed of light. In these cases, the F_i's are the gradients of newtonian and coulombic potentials, i.e.

$$F_i(x_i) = \sum_{j \neq i} \frac{x_j - x_i}{\|x_j - x_i\|^3}(k m_i m_j - e_i e_j),$$

where k is the gravitational constant and e_i is the charge of the i^{th}-particle. For a more detailed discussion on this subject see [Thirring (1978)].

As concerns the case of only one particle moving in the one-dimensional space, i.e. in a straight line, we mention:

1.4.4 *The Harmonic Oscillator*

Let us consider a particle of mass m that moves on a straight line under the action of an elastic force. We denote by $x(t)$ the abscissa of the particle at the time t and by $F(x)$ the force exercised upon the particle in motion situated at the point of abscissa x. Since the force is elastic, $F(x) = -kx$ for each $x \in \mathbb{R}$, where $k > 0$. On the other hand, the motion of the particle should obey Newton's Second Law which, in this specific case, takes the form $F(x(t)) = ma(t)$, where $a(t)$ is the acceleration of the particle at the time t. But $a(t) = x''(t)$ and denoting by $\omega^2 = k/m$, from the considerations above, it follows that x has to verify the second-order scalar linear differential equation:

$$x'' + \omega^2 x = 0,$$

called the *equation of the harmonic oscillator*. As we have already seen in Example 1.2.3, the general solution of this equation is

$$x(t, c_1, c_2) = c_1 \sin \omega t + c_2 \cos \omega t$$

for $t \in \mathbb{R}$.

1.4.5 *The Mathematical Pendulum*

Let us consider a pendulum of length ℓ and let us denote by $s(t)$ the length of the arc curve described by the free extremity of the pendulum by the time t. We have $s(t) = \ell x(t)$, where $x(t)$ is the measure expressed in radian units of the angle between the pendulum at the time t and the vertical axis Oy. See Figure 1.4.2.

The force which acts upon the pendulum is $F = mg$, where g is the acceleration of gravitation. This force can be decomposed along two components, one having the direction of the thread, and another one having the direction of the tangent at the arc of circle described by the free end of the pendulum. See Figure 1.4.2. The component having the direction of the thread is counterbalanced by the resistance of the latter, so that the motion takes place only under the action of the component $-mg \sin x(t)$.

But x should obey Newton' Second Law, which in this case takes the form of the second-order scalar differential equation $m\ell x'' = -mg \sin x$, or equivalently

$$x'' + \frac{g}{\ell} \sin x = 0,$$

nonlinear equation called the *equation of the mathematical pendulum* , or *the equation of the gravitational pendulum.*

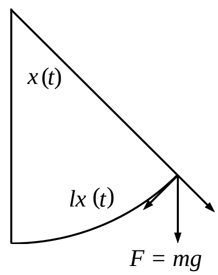

Figure 1.4.2

If we intend to study only the small oscillations, we can approximate $\sin x$ by x and we obtain the *equation of the small oscillations of the pendulum*

$$x'' + \frac{g}{\ell}x = 0,$$

a second-order scalar linear differential equation. For this equation, which is formally the same with that of the harmonic oscillator, we know the general solution, i.e.

$$x(t, c_1, c_2) = c_1 \sin\sqrt{\frac{g}{\ell}}t + c_2 \cos\sqrt{\frac{g}{\ell}}t$$

for $t \in \mathbb{R}$, where $c_1, c_2 \in \mathbb{R}$.

1.4.6 *Two Demographic Models*

A first demographic model describing the growth of the human population was proposed in 1798 by Thomas Robert Malthus.[22] We shall present here a continuous variant of the model proposed by Malthus. More precisely, if we denote by $x(t)$ the population, i.e. the number of individuals of a given species at the time t, and by $y(t)$ the subsistence, i.e. the resources of living, according to *Malthus' Law*: *the instantaneous rate of change of x at the time t is proportional with $x(t)$, while the instantaneous rate of*

[22] British economist (1766–1834). In his *An essay on the principle of population as it affects the future improvement of society* (1798) he has enunciated the principle stipulating that *a population, which evolves freely increases in a geometric ratio, while subsistence follows an arithmetic ratio growth.* This principle, expressed as a discrete mathematical model, has had a deep influence on the economical thinking even up to the middle of the XX century.

change of the subsistence is constant at any time. Then we have the following mathematical model expressed by means of a system of first-order differential equations of the form

$$\begin{cases} x' = cx \\ y' = k, \end{cases}$$

where c and k are strictly positive constants. This system of uncoupled equations (in the sense that each equation contains only one unknown function) can be solved explicitly. Its general solution is given by

$$\begin{cases} x(t, \xi) = \xi e^{ct} \\ y(t, \eta) = \eta + kt \end{cases}$$

for $t \geq 0$, where ξ and η represent the population and respectively the subsistence, at the time $t = 0$. One may see that this model describes rather well the real phenomenon only on very short intervals of time. For this reason, some more refined and more realistic models have been proposed. The aim was to take into consideration that, at any time, the number of individuals of a given species can not exceed a certain critical value which depends on the subsistence at that time. So, if we denote by $h > 0$ the quantity of resources necessary to one individual to remain alive after the time t, we may assume that x and y satisfy a system of the form

$$\begin{cases} x' = cx\left(\dfrac{y}{h} - x\right) \\ y' = k. \end{cases}$$

This system describes a more natural relationship between the subsistence and the growth, or decay, of a given population. In certain models, as for instance in that one proposed in 1835 by Verhulst, for simplicity, one considers $k = 0$, which means that the subsistence is constant ($y(t) = \eta$ for each $t \in \mathbb{R}$). Thus, one obtains a first-order nonlinear differential equation of the form

$$x' = cx(b - x),$$

for $t \geq 0$, where $b = \eta/h > 0$. This equation, i.e. the *Verhulst model*, known under the name of *logistic equation*, is with separable variables and can be integrated. More precisely, the general solution is

$$x = \frac{b\mu e^{cbt}}{1 + \mu e^{cbt}}$$

for $t \geq 0$, where $\mu \geq 0$ is a constant. To this solution we have to add the singular solution $x = b$, eliminated during the integration process. In order to individualize a certain solution x from the general one we have to determine the corresponding constant μ. Usually this is done by imposing the initial condition

$$x(0) = \frac{b\mu}{1+\mu} = \xi,$$

where ξ represents the number of the individuals at the time $t = 0$, number which is assumed to be known. We deduce that the solution $x(\cdot, \xi)$ of the logistic equation that satisfies the initial condition $x(0, \xi) = \xi$ is given by

$$x(t, \xi) = \frac{b\xi e^{cbt}}{b + \xi(e^{cbt} - 1)}$$

for each $t \geq 0$.

All the models described above can be put under the general form

$$x' = d(t, x),$$

where $d(t, x)$ represents the difference between the rate of birth and the rate of mortality corresponding to the time t and to a population x.

1.4.7 *A Spatial Model in Ecology*

Following [Neuhauser (2001)], we consider an infinite number of sites which are linked by migration and we assume that all sites are equally accessible and no explicit spatial distances between sites are taken into consideration. We denote by $x(t)$ the number of occupied sites and we assume that the time is scaled so that the rate at which the sites become vacant equals 1. Then, assuming that the colonization rate x' is proportional to the product of the number of occupied sites and the vacant sites, we get the so-called *Levins Model*

$$x' = \lambda x(1 - x) - x$$

which is formally equivalent to the logistic equation.

1.4.8 *The Prey-Predator Model*

Immediately after the First World War, in the Adriatic Sea area, a significant decay of the fish population has been observed. This decay, at the first glance in contradiction with the fact that almost all fishermen in the area, enrolled in the army, were in the impossibility to practice their usual job, was a big surprise. Under these circumstances, it seems to be quite natural to expect rather a growth instead of a decay of the fish population. In his attempt to explain this strange phenomenon, Vito Volterra[23] has proposed a mathematical model describing the evolution of two species both living within the same area, but which compete for surviving. Namely, in [Volterra (1926)], he considered two species of animals living in the same region, the first one having at disposal unlimited subsistence, species called *prey*, and the second one, called *predator*, having as unique source of subsistence the members of the first species. Think of the case of herbivores versus carnivores. Denoting by $x(t)$ and respectively by $y(t)$ the population of the prey species, and respectively of the predator one at the time t, and assuming that both x and y are function of class C^1, we deduce that x and y have to satisfy the system of first-order nonlinear differential equations

$$\begin{cases} x' = (a - ky)x \\ y' = -(b - hx)y, \end{cases} \tag{1.4.1}$$

where a, b, k, h are positive constants. The first equation is nothing else than the mathematical expression of the fact that the instantaneous rate of growth of x at the time t is proportional with the population of the prey species at the time considered ($x' = ax - \dots$) while the instantaneous rate of decay of x at the same time t is proportional with the number of all possible contacts between prey and predators at the same time t ($x' = \cdots - kyx$). Analogously, the second equation expresses the fact that the instantaneous rate of decay of y at the time t is proportional with the population of the predator species at that time t ($y' = -by \dots$) while the instantaneous rate of growth of y at the same time t is proportional with the number of all possible contacts between prey and predators. It should be noticed that the very same model was been proposed earlier by [Lotka (1925)] and therefore the system (1.4.1) is known under the name of *Lotka–Volterra System*

As we shall see later on[24], each solution of the Lotka–Volterra System (1.4.1) with nonnegative initial data has nonnegative components as long as it exists, while each solution with positive initial data is periodic (with the principal period depending on the initial data). The trajectory of such solution is illustrated in Figure 1.4.3 (a), while its graph in Figure 1.4.3 (b).

[23]Italian mathematician (1860–1940) with notable contributions in Functional Analysis and in Applied Mathematics (especially in Physics and in Biology).

[24]See Problems 6.1, 6.3, 6.4 .

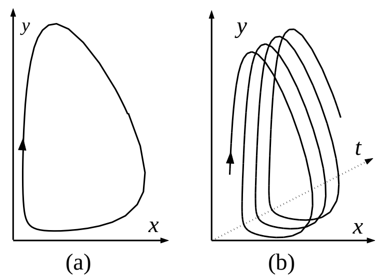

Figure 1.4.3

For this reason the function $t \mapsto x(t) + y(t)$, which represents the total number of animals in both species at the time t, is periodic too, and thus it has infinitely many local minima. Under these circumstances, it is not difficult to realize that, the seemingly non-understandable decay of the fish population in the Adriatic Sea was nothing else but a simple consequence of the fact that the moment in question (the end of the First World War) was quite close to a local minimum of the function above.

Finally, let us observe that the system above has two constant solutions called (for obvious reasons) *stationary solutions, or equilibria*: $(0,0)$ and $(b/h, a/k)$. The first one has the property that, there exist solutions of the system, which start from initial points as close as we wish to $(0,0)$, but which do not remain close to $(0,0)$ as t tends to infinity. Indeed, if at a certain moment the predator population is absent it remains absent for all t, while the prey population evolves obeying the Malthus' law. More precisely, the solution starting from the initial point $(\xi, 0)$, with $\xi > 0$, is $(x(t), y(t)) = (\xi e^{at}, 0)$ for $t \geq 0$, and this obviously, moves off $(0,0)$ as t tends to infinity. For this reason we say that $(0,0)$ is *unstable* with respect to small perturbations in the initial data. We shall see later on that the second stationary solution is stable with respect to small perturbations in the initial data. Roughly speaking, this means that, all solutions having the initial data close enough to $(b/h, a/k)$ are defined on the whole half-axis and remain close to the solution $(b/h, a/k)$ on the whole domain of definition. The precise definition of this concept will be formulated in Section 5.1. See Definition 5.1.1.

1.4.9 *The Spreading of a Disease*

In 1976 A. Lajmanovich and J. Yorke have proposed a model of the spread of a disease which confers no immunity. Following [Hirsch (1984)], we present a slight generalization of this model. We start with the description of a very specific variant and then we shall approach the model in its whole generality. More precisely, let us consider a disease who could affect a given population and who confers no immunity. This means that anyone who does not have the disease at a given time is susceptible to infect, even though he or she has already been infected, but meanwhile recovered. Let us denote by p the population which is assumed to be constant (assumption which is plausible if, for instance, during the spreading of the disease there are neither births, nor deaths) and by x the number of infected people in the considered population. As we have already mentioned at the beginning of this section, we may assume that x is a positive continuously differentiable function of the time variable t. Consequently, $p - x$ is a nonnegative continuously differentiable function too. Obviously, for each $t \geq 0$, $p - x(t)$ represents the number of those susceptible to be infected at the time t. Then, if we assume that, at any time t, the instantaneous rate of change of the number of infected members is proportional to the number of all possible contacts between infected and non-infected members, number which obviously equals $x(t)(p - x(t))$, we deduce that x must obey the following nonlinear differential equation

$$x' = ax(p - x),$$

where $a > 0$ is constant. This is an equation with separable variables, of the very same form as that described in the Verhulst's model, and whose general solution is given by

$$x = \frac{p\mu e^{apt}}{1 + \mu e^{apt}},$$

where η is a positive constant. To this general solution we have to add the singular stationary solution $x = p$, eliminated in the integration process. As in the case of the logistic equation, the solution $x(\cdot, \xi)$ of the equation above, which satisfies the initial condition $x(0, \xi) = \xi$, is

$$x(t, \xi) = \frac{p\xi e^{apt}}{p + \xi(e^{apt} - 1)}$$

for each $t \geq 0$. It is of interest to note that, for each $\xi > 0$, we have

$$\lim_{t \to +\infty} x(t, \xi) = p,$$

relation which shows that, *in the absence of any external intervention* (*cure*), *a population which has at the initial moment a positive number* $\xi > 0$ *of infected, tends to be entirely infected.* The graph of $x(\cdot, \xi)$ is illustrated in Figure 1.4.4.

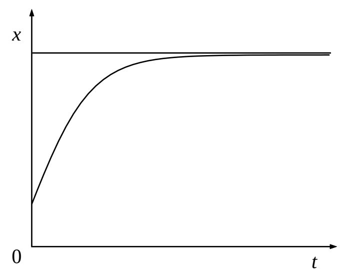

Figure 1.4.4

We may now proceed to a more general case. More precisely, let us consider that the population in question is divided into n disjoint classes (on social criteria, for instance) each one having a constant number of members. We denote by p_i the cardinal of the class i and by x_i the number of infected in the class i, $i = 1, 2, \ldots, n$. Then, the number of susceptible in the class i is $p_i - x_i$. As above, from pure mathematical reasons, we shall consider that x_i is a positive continuously differentiable function of the time variable t. We denote by R_i the rate of infection corresponding to the class i and by C_i the rate of recovering corresponding to the same class i. For the sake of simplicity, we shall assume that R_i depends only on $x = (x_1, x_2, \ldots, x_n)$, while C_i depends only on x_i, $i = 1, 2, \ldots, n$. Finally, it is fairly realistic to consider that $\frac{\partial R_i}{\partial x_j} \geq 0$ for $i, j = 1, 2, \ldots, n$, relations which express the fact that the rate of infection R_i is increasing with respect to each of its arguments x_j, that represents the number of infected in the class j.

Let us observe that all these assumptions lead to the mathematical model described by the system of first-order nonlinear differential equations

$$x_i' = R_i(x) - C_i(x_i) \quad (i = 1, 2, \ldots, n).$$

We mention that the model proposed by A. Lajmanovich and J. Yorke has the specific form

$$x_i' = \sum_{j=0}^{n} a_{ij} x_j (p_i - x_i) - k_i x_i \quad (i = 1, 2, \ldots, n),$$

where $a_{ij} \geq 0$ and $k_i \geq 0$, for $i, j = 1, 2, \ldots, n$ and was obtained via analogous considerations as those used for the simplified model, i.e. to that one corresponding to a single class.

For more details on models in both population dynamics and ecology see [Neuhauser (2001)].

1.4.10 *Lotka Model*

In 1920 A. J. Lotka considered a chemical reaction mechanism described by

$$\begin{cases} A + X \xrightarrow{k_1} 2X \\ X + Y \xrightarrow{k_2} 2Y \\ Y \xrightarrow{k_3} B, \end{cases} \tag{1.4.2}$$

where X and Y are intermediaries, k_1, k_2 and k_3 are the reaction rate constants, and the concentrations of both the reactant A and the product B are kept constant. See [Lotka (1920a)] and [Lotka (1920b)]. Noticing that the signification of the first relation is that one molecule of A combines with one molecule of X giving two molecules of X, the signification of the next two relations becomes obvious.

Before obtaining the corresponding mathematical model of these reactions, we recall for easy reference a fundamental law which governs chemical reactions, i.e., *the law of mass action.* Namely this asserts that: *the rate of a chemical reaction is proportional to the active concentrations of the reactants, i.e. only to that amounts of reactants that are taking part in the reaction.* For instance, for the irreversible reaction $X + Y \longrightarrow A$, if x and y denote the active concentrations of X and Y respectively, the law of mass action says that $x' = -kxy$, where $k > 0$ is the *rate constant of the reaction.* If one assumes that the reaction is reversible with rate constants of reaction k_1 and k_{-1}, i.e. $X + Y \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} A$, then the active concentrations x and y must satisfy $x' = -k_1 xy + k_{-1} a$. Finally, for the simplest irreversible reaction $X \longrightarrow C$, the law of mass action implies that $x' = -kx$, while for the reversible one $X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} C$, says that $x' = -k_1 x + k_{-1} c$.

Now, coming back to (1.4.2), let us denote the concentrations of A, B, X and Y by a, b, x and y respectively, and let us observe that, by virtue of the law of mass action just mentioned, x and y must obey the kinetic equations

$$\begin{cases} x' = k_1 a x - k_2 x y \\ y' = -k_3 y + k_2 x y. \end{cases} \tag{1.4.3}$$

We emphasize that the system (1.4.3) is formally equivalent to the Lotka–Volterra system (1.4.1), and thus all the considerations made for the latter applies here as well. For this reason, in all what follows, we will refer to either system (1.4.1) or (1.4.3) to as the Lotka–Volterra system, or to the *pray-predator system.* For more details on this subject see [Murray (1977)], pp. 136–141.

1.4.11 *An Autocatalytic Generation Model*

Following [Nicolis (1995)], let us consider a tank containing a substance X whose concentration at the time t is denoted by $x(t)$, and another one A whose concentration $a > 0$ is kept constant, and let us assume that in the tank take place the following reversible chemical reactions:

$$\begin{cases} A + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 2X \\ X \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} B, \end{cases}$$

in which B is a residual product whose concentration at the time t is $b(t)$.[25]

Here $k_i \geq 0$, $i = \pm 1, \pm 2$ are the reaction rate constants of the four reactions in question. The mathematical model describing the evolution of this chemical system, obtained by means of the law of mass action is

$$\begin{cases} x' = k_1 a x - k_{-1} x^2 - k_2 x + k_{-2} b \\ b' = k_2 x - k_{-2} b. \end{cases}$$

[25] This model has been proposed in 1971 by Schlögl in order to describe some isothermal autocatalytic chemical reactions. For more details on such kind of models the interested reader is referred to [Nicolis (1995)].

If the second reaction does not take place, situation which is described mathematically by $k_2 = k_{-2} = 0$, then the system above reduces to

$$x' = k_1 a x - k_{-1} x^2.$$

Let us notice the remarkable similarity of the equation above with the logistic equation in the Verhulst's model as well with the equation describing the spread of a disease.

1.4.12 *An RLC Circuit Model*

Following [Hirsch and Smale (1974)], pp. 211–214, let us consider an electric circuit consisting of a resistance R, a coil L, and a capacitor C in which the sense of currents on each of the three portions of the circuit are illustrated in Figure 1.4.5.

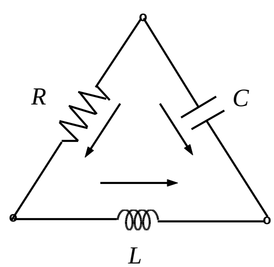

Figure 1.4.5

Let us denote by $i(t) = (i_R(t), i_L(t), i_C(t))$ the state of the current in the circuit at the time t. Here i_R, i_L, i_C represent the currents on the portions of the circuit containing the resistance R, the coil L and respectively the capacitor C. Analogously, let $v(t) = (v_R(t), v_L(t), v_C(t))$ be the state of the voltages in the circuit at the time t. Following Kirchhoff' Laws, we deduce

$$\begin{cases} i_R(t) = i_L(t) = -i_C(t) \\ v_R(t) + v_L(t) - v_C(t) = 0, \end{cases}$$

while from the generalized Ohm's Law $g(i_R(t)) = v_R(t)$ for each $t \geq 0$.

Finally, from Faraday's Law, we obtain

$$\begin{cases} \mathcal{L}\dfrac{di_L}{dt} = v_L \\ \\ \mathcal{C}\dfrac{dv_C}{dt} = i_C \end{cases}$$

for each $t \geq 0$, where $\mathcal{L} > 0$ and $\mathcal{C} > 0$ are the *inductance* of L and respectively the *capacity* of C. From these relations we observe that i_L and v_C satisfy the system of first-order nonlinear differential equations

$$\begin{cases} \mathcal{L}\dfrac{di_L}{dt} = v_C - g(i_L) \\ \\ \mathcal{C}\dfrac{dv_C}{dt} = -i_L \end{cases}$$

for $t \geq 0$.

For simplicity, let us assume now that $\mathcal{L} = 1$ and $\mathcal{C} = 1$, and let us denote by $x = i_L$ and $y = v_C$. Then the previously considered system can be rewritten under the form

$$\begin{cases} \dfrac{dx}{dt} = y - g(x) \\ \\ \dfrac{dy}{dt} = -x \end{cases}$$

for $t \geq 0$. Assuming in addition that g is of class C^1, differentiating both sides the first equation and using the second one in order to eliminate y, we finally get

$$x'' + g'(x)x' + x = 0$$

for $t \geq 0$. This is the *Liénard Equation.* In the case in which $g(x) = x^3 - x$ for each $x \in \mathbb{R}$, the equation above takes the form

$$x'' + (3x^2 - 1)x' + x = 0$$

for $t \geq 0$ and it is known as the *Van der Pol Equation.* For a detailed study of mathematical models describing the evolution of both current and voltage in electrical circuits see also [Hirsch and Smale (1974)], Chapter 10. For many other interesting mathematical models see [Braun (1983)].

1.5 Integral Inequalities

In this section we include several inequalities very useful in proving the boundedness of solutions of certain differential equations or systems. We start with the following nonlinear integral inequality.

Lemma 1.5.1 (Bihari) *Let $x : [\,a,b\,] \to \mathbb{R}_+$, $k : [\,a,b\,] \to \mathbb{R}_+$ and $\omega : \mathbb{R}_+ \to \mathbb{R}_+^*$ be three continuous functions with ω nondecreasing on $\mathbb{R}_+$ and let $m \geq 0$. If*

$$x(t) \leq m + \int_a^t k(s)\,\omega(x(s))\,ds$$

for each $t \in [\,a,b\,]$, then

$$x(t) \leq \Phi^{-1}\left(\int_a^t k(s)\,ds\right)$$

for each $t \in [\,a,b\,]$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\Phi(u) = \int_m^u \frac{d\eta}{\omega(\eta)}$$

for each $u \in \mathbb{R}_+$.

Proof. Let us observe that it suffices to prove the lemma in the case in which $m > 0$ because the case $m = 0$ can be obtained from the preceding one by passing to the limit for m tending to 0. So, let $m > 0$, and let us consider the function $y : [\,a,b\,] \to \mathbb{R}_+^*$ defined by

$$y(t) = m + \int_a^t k(s)\,\omega(x(s))\,ds$$

for each $t \in [\,a,b\,]$. Obviously y is of class C^1 on $[\,a,b\,]$. In addition, since, $x(t) \leq y(t)$ for $t \in [\,a,b\,]$ and ω is nondecreasing, it follows that

$$y'(t) = k(t)\,\omega(x(t)) \leq k(t)\,\omega(y(t))$$

for each $t \in [\,a,b\,]$. This relation can be rewritten under the form

$$\frac{y'(s)}{\omega(y(s))} \leq k(s)$$

for each $s \in [a,b]$. Integrating both sides of the last inequality from a to t, we obtain

$$\Phi(y(t)) \leq \int_a^t k(s)\,ds$$

for each $t \in [a,b]$. As Φ is strictly increasing, it is invertible on its range, which includes $[0,+\infty)$, and has strictly increasing inverse. From the last inequality we get

$$y(t) \leq \Phi^{-1}\left(\int_a^t k(s)\,ds\right),$$

relation which, along with $x(t) \leq y(t)$ for $t \in [a,b]$, completes the proof.$\square$

The next two consequences of Lemma 1.5.1 are useful in applications.

Lemma 1.5.2 (Gronwall) *Let* $x : [a,b] \to \mathbb{R}_+$ *and* $k : [a,b] \to \mathbb{R}_+$ *be two continuous functions and let* $m \geq 0$. *If*

$$x(t) \leq m + \int_a^t k(s)\,x(s)\,ds$$

for each $t \in [a,b]$, *then*

$$x(t) \leq m \exp\left(\int_a^t k(s)\,ds\right)$$

for each $t \in [a,b]$.

Proof. Let us remark that, for each $\varepsilon > 0$, we have

$$x(t) \leq m + \int_a^t k(s)(x(s)+\varepsilon)\,ds$$

for each $t \in [a,b]$. Taking $\omega : \mathbb{R}_+ \to \mathbb{R}_+^*$, defined by $\omega(r) = r + \varepsilon$ for each $r \in \mathbb{R}_+$, from Lemma 1.5.1, we obtain

$$x(t) \leq (m+\varepsilon)\exp\left(\int_a^t k(s)\,ds\right) - \varepsilon$$

for each $\varepsilon > 0$ and $t \in [a,b]$. Passing to the limit for ε tending to 0 in this inequality, we get the conclusion of the lemma. The proof is complete. $\square$

Some generalizations of Gronwall's Lemma 1.5.2 are stated in Section 6. See Problems 1.16 and 1.17.

Lemma 1.5.3 (Brezis) *Let* $x : [a, b] \to \mathbb{R}_+$ *and* $k : [a, b] \to \mathbb{R}_+$ *be two continuous functions and let* $m \geq 0$*. If*

$$x^2(t) \leq m^2 + 2\int_a^t k(s)\, x(s)\, ds$$

for each $t \in [a, b]$*, then*

$$x(t) \leq m + \int_a^t k(s)\, ds$$

for each $t \in [a, b]$*.*

Proof. As in the proof of Lemma 1.5.2, let us observe that, for each $\varepsilon > 0$, we have $x^2(t) \leq m^2 + 2\int_a^t k(s)\sqrt{x^2(s) + \varepsilon}\, ds$ for each $t \in [a, b]$. This inequality and Lemma 1.5.1 with $\omega : \mathbb{R}_+ \to \mathbb{R}_+^*$, defined by $\omega(r) = 2\sqrt{r + \varepsilon}$ for each $r \in \mathbb{R}_+$, yield $x^2 \leq \left(\sqrt{m^2 + \varepsilon} + \int_a^t k(s)\, ds\right)^2 - \varepsilon$ for each $\varepsilon > 0$ and $t \in [a, b]$. We complete the proof by passing to the limit for ε tending to 0 in this inequality and by extracting the square root both sides in the inequality thus obtained. $\square$

For a generalization of Lemma 1.5.3, see Problem 1.18.

1.6 Exercises and Problems

Problem 1.1 *Find a plane curve for which the ratio of the ordinate by the subtangent*[26] *equals the ratio of a given positive number* k *by the difference of the ordinate by the abscissa.*[27] ([Halanay (1972)], p. 7)

Problem 1.2 *Find a plane curve passing through the point* $(3, 2)$ *for which the segment of any tangent line contained between the coordinate axes is divided in half at the point of tangency.* ([Demidovich (1973)], p. 329)

[26] We recall that the subtangent to a given curve of equation $x = x(t)$, $t \in [a, b]$ at a point $(t, x(t))$ equals $x(t)/x'(t)$.

[27] This problem, considered to be the first in the domain of Differential Equations, has been formulated by Debeaune and conveyed, in 1638, by Mersenne to Descartes. The latter has realized not only the importance of the problem but also the impossibility to solve it by known (at that time) methods.

Exercise 1.1 *Solve the following differential equations.*

(1) $x' \cos^2 t \cot x + \tan t \sin^2 x = 0$. (2) $tx' = x + x^2$.
(3) $tx'x = 1 - t^2$. (4) $x' = (t + x)^2$.
(5) $x' = (8t + 2x + 1)^2$. (6) $x'(4t + 6x - 5) = -(2t + 3x + 1)$.
(7) $x'(4t - 2x + 3) = -(2t - x)$. (8) $x'(t^2 x - x) + tx^2 + t = 0$.

Problem 1.3 *Find a plane curve passing through the point* $(1, 2)$ *whose segment of the normal at any point of the curve lying between the coordinate axes is divided in half by the current point.* ([Demidovich (1973)], 2758, p. 330)

Problem 1.4 *Find a plane curve whose subtangent is of constant length* a. ([Demidovich (1973)], 2759, p. 330)

Problem 1.5 *Find a plane curve in the first quadrant whose subtangent is twice the abscissa of the point of tangency.* ([Demidovich (1973)], 2760, p. 330)

Exercise 1.2 *Solve the following differential equations.*

(1) $tx' = x - t$. (2) $tx' = -(t + x)$.
(3) $t^2 x' = x(t - x)$ (4) $2txx' = t^2 + x^2$.
(5) $(2\sqrt{tx} - t)x' = -x$. (6) $tx' = x + \sqrt{t^2 + x^2}$.
(7) $(4x^2 + 3tx + t^2)x' = -(x^2 + 3tx + 4t^2)$. (8) $2txx' = 3x^2 - t^2$.

Problem 1.6 *Find the equation of a curve that passes through the point* $(1, 0)$ *and having the property that the segment cut off by the tangent line at any current point* P *on the* t*-axis equals the length of the segment* OP. ([Demidovich (1973)], 2779, p. 331)

Problem 1.7 *Let* $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a continuous function for which there exists a real number* m *such that* $f(\lambda t, \lambda^m x) = \lambda^{m-1} f(t, x)$ *for each* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$ *and each* $\lambda \in \mathbb{R}_+$. *Show that, by the substitution* $x(t) = t^m y(t)$, *the differential equation* $x' = f(t, x)$, *called quasi-homogeneous, reduces to an equation with separable variables. Prove that the differential equation* $x' = x^2 - \frac{2}{t^2}$ *is quasi-homogeneous and then solve it.* ([Glăvan *et al.* (1993)], p. 34)

Exercise 1.3 *Solve the following differential equations.*

(1) $tx' = x + tx$. (2) $tx' = -2x + t^4$.
(3) $tx' = -x + e^t$. (4) $(x^2 - 3t^2)x' + 2tx = 0$.
(5) $tx' = -x - tx^2$. (6) $2txx' = x^2 - t$.
(7) $(2t - t^2 x)x' = -x$. (8) $tx' = -2x(1 - tx)$.

Problem 1.8 *Let* x, x_1, x_2 *be solutions of the linear equation* $x' = a(t)x + b(t)$, *where* a, b *has continuous functions on* $\mathbb{I}$. *Prove that the ratio*

$$R(t) = \frac{x_2(t) - x(t)}{x(t) - x_1(t)}$$

is constant on $\mathbb{I}$. What is the geometrical meaning of this result?

Problem 1.9 *Let x_1, x_2 be solutions of the Bernoulli equation $x' = a(t)x + b(t)x^2$, where a, b are continuous functions on $\mathbb{I}$. Prove that, if $x_1(t) \neq 0$ and $x_2(t) \neq 0$ on $\mathbb{J} \subset \mathbb{I}$, then the function y, defined by $y(t) = \frac{x_1(t)}{x_2(t)}$ for each $t \in \mathbb{J}$, satisfies the linear equation $y' = b(t)[x_1(t) - x_2(t)]y$.*

Problem 1.10 *Let x, x_1, x_2, x_3 be solutions of the Riccati equation*

$$x' = a(t)x + b(t)x^2 + c(t),$$

where a, b, c are continuous functions on $\mathbb{I}$. Prove that the ratio

$$B(t) = \frac{x_2(t) - x(t)}{x_2(t) - x_1(t)} : \frac{x_3(t) - x(t)}{x_3(t) - x_1(t)}$$

is constant on $\mathbb{I}$.

Exercise 1.4 *Solve the following differential equations.*

(1) $(t + 2x)x' + t + x = 0.$ (2) $2tx' + t^2 + 2x + 2t = 0.$
(3) $(3t^2x - x^2)x' - t^2 + 3tx^2 - 2 = 0.$ (4) $(t^2x + x^3 + t)x' - t^3 + tx^2 + x = 0.$
(5) $(x^2 - 3t^2)x' + 2tx = 0.$ (6) $2txx' - (t + x^2) = 0.$
(7) $tx' - x(1 + tx) = 0.$ (8) $t(x^3 + \ln t)x' + x = 0.$

Exercise 1.5 *Solve the following differential equations.*

(1) $x = \dfrac{1}{2}tx' + x'^3.$ (2) $x = x' + \sqrt{1 - x'^2}.$
(3) $x = (1 + x')t + x'^2.$ (4) $x = -\dfrac{1}{2}x'(2t + x').$
(5) $x = tx' + x'^2.$ (6) $x = tx' + x'.$
(7) $x = tx' + \sqrt{1 + x'^2}.$ (8) $x = tx' + \dfrac{1}{x'}.$

Problem 1.11 *Find a plane curve for which the distance of a given point to any line tangent to this curve is constant.* ([Demidovich (1973)], 2831, p. 340)

Problem 1.12 *Find the curve for which the area of the triangle formed by a tangent line at any point and by the coordinate axes is constant.* ([Demidovich (1973)], 2830, p. 340)

Problem 1.13 *Prove that, for a heavy liquid rotating about the vertical symmetry axis in a cylindric tank, the free surface is situated on a paraboloid of revolution.* ([Demidovich (1973)], 2898, p. 344)

Problem 1.14 *Find the relationship between the air pressure and the altitude if it is known that the pressure is of $1kgf$ per $1cm^2$ at the sea level and of $0.92kgf$ per $1cm^2$ at an altitude of $500m$.* ([Demidovich (1973)], 2899, p. 344)

Problem 1.15 *According to Hooke's law an elastic band of length l increases in length klF (k=constant) due to a tensile force F. By how much will the band increase in length due to its weight W if it is suspended at one end? (The initial length of the band is l).* ([Demidovich (1973)], 2900, p. 344)

Problem 1.16 (Bellman's Inequality) *Let $x : [a, b] \to \mathbb{R}_+$, $h : [a, b] \to \mathbb{R}$ and $k : [a, b] \to \mathbb{R}_+$ be three continuous functions. If*

$$x(t) \leq h(t) + \int_a^t k(s)\, x(s)\, ds$$

for each $t \in [a, b]$, then

$$x(t) \leq h(t) + \int_a^t k(s)\, h(s) \exp\left(\int_s^t k(\tau)\, d\tau\right) ds$$

for each $t \in [a, b]$.

Problem 1.17 *Let $x : [a, b] \to \mathbb{R}_+$, $v : [a, b] \to \mathbb{R}$ and $k : [a, b] \to \mathbb{R}_+$ be three continuous functions and $\xi \in \mathbb{R}$. If*

$$x(t) \leq \xi + \int_a^t [k(s)\, x(s) + v(s)]\, ds$$

for each $t \in [a, b]$, then

$$x(t) \leq \xi \exp\left(\int_a^t k(s)\, ds\right) + \int_a^t v(s) \exp\left(\int_s^\tau k(\tau)\, d\tau\right) ds$$

for each $t \in [a, b]$. ([Halanay (1972)], p. 196)

Problem 1.18 *If $x : [a, b] \to \mathbb{R}_+$ and $k : [a, b] \to \mathbb{R}_+$ are continuous and*

$$x^p(t) \leq m^p + p \int_a^t k(s)\, x^{p-1}(s)\, ds$$

for each $t \in [a, b]$, where $m \geq 0$ and $p > 1$, then

$$x(t) \leq m + \int_a^t k(s)\, ds$$

for each $t \in [a, b]$.

Problem 1.19 *Let $f : \mathbb{R} \to \mathbb{R}$ be non-increasing and let $x, y : [0, T] \to \mathbb{R}$ be two functions of class C^1. If $x'(t) + f(x(t)) \leq y'(t) + f(y(t))$ for each $t \in [0, T]$ and $x(0) \leq y(0)$ then $x(t) \leq y(t)$ for each $t \in [0, T]$.*

Chapter 2

The Cauchy Problem

This chapter is exclusively dedicated to the introduction and study of the fundamental concepts and results concerning the main topic of this book: the so-called Cauchy problem, or the initial-value problem. In the first section we define the Cauchy problem for a given differential equation and the basic concepts referring to: local solution, saturated solution, global solution, *etc*. In the second section we prove that a sufficient condition in order that a Cauchy problem have at least one local solution is the continuity of the function f. In the third one we present several specific situations in which every two solutions of a certain Cauchy problem coincide on the common part of their domains. The existence of saturated solutions as well as of global solutions is studied in the fourth section. In the fifth section we prove several results concerning the continuous dependence of the saturated solutions on the initial data and on the parameters, while in the sixth one we discuss the differentiability of saturated solutions with respect to the data and to the parameters. The seventh section reconsiders all the problems previously studied in the case of the n^{th}-order scalar differential equation. The last section contains several exercises and problems illustrating the most delicate aspects of the abstract theory.

2.1 General Presentation

Let $\mathbb{I}$ be a nontrivial interval in $\mathbb{R}$, Ω a nonempty and open subset in $\mathbb{R}^n$, $f : \mathbb{I} \times \Omega \to \mathbb{R}^n$ a given function, $a \in \mathbb{I}$ and $\xi \in \Omega$.

The Cauchy problem, or *the initial-value problem* for a first-order differential system with data $\mathcal{D} = (\mathbb{I}, \Omega, f, a, \xi)$ consists in finding a C^1-function