6

NONCONSTANT COEFFICIENT SECOND ORDER LINEAR EQUATIONS AND SERIES SOLUTIONS

6.1 INTRODUCTION

In Chapter 2, the general theory for the initial value problem

$$a(t)y'' + b(t)y' + c(t)y = 0,$$

$$y(t_0) = r, \qquad y'(t_0) = s,$$

was discussed. There it was assumed that $a(t)$, $b(t)$, and $c(t)$ were continuous functions on some open interval containing t_0 and that $a(t)$ did not vanish at t_0. The theory stated that the solution $y(t)$ could be expressed uniquely as

$$y(t) = c_1y_1(t) + c_2y_2(t),$$

where $y_1(t)$ and $y_2(t)$ were a linearly independent pair of solutions of the differential equation, and the constants c_1 and c_2 depended on the initial values r and s. Furthermore, the existence of such a pair of linearly independent solutions $y_1(t)$ and $y_2(t)$ was also guaranteed by the theory.

Much of the remainder of Chapter 2 was devoted to a discussion of the constant coefficient case

$$ay'' + by' + cy = 0.$$

In this case finding a pair of linearly independent solutions depended solely on solving the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

The type of solutions depended on the nature of the roots of the characteristic equation (real and unequal, real and equal, or complex) but the solutions could be *explicitly found.*

The reader may well ask whether this simplicity of the solution procedure carries over in some analogous fashion to the nonconstant coefficient case. The answer is a definite NO! With the exception of the special case of the Euler differential equation,

$$t^2y'' + bty' + cy = 0$$

(see Example 2 and Exercise 14 of Section 2.6), there are no general techniques to reduce the solving of nonconstant coefficient linear differential equations to an algebraic process.

However, if we are willing to extend our notion of the solution of a differential equation to allow for the solution to be expressed as an infinite series, then there is a solution procedure that is applicable to a large class of nonconstant coefficient linear differential equations. The infinite series will be a power series in the independent variable, possibly multiplied by a known function, and the coefficients of the power series can be determined recursively. Given the generality of the problem, one could hardly wish for a more satisfactory outcome.

For instance, a solution of

$$y'' + \frac{1}{t}y' + y = 0$$

is given by the power series

$$y(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^4(2!)^2} - \frac{t^6}{2^6(3!)^2} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^{2n}(n!)^2},$$

as we shall see later in this chapter. The function given by the power series, denoted by $J_0(t)$, is an oscillatory function with an infinite number of zeros on the positive axis, and there are tables and numerical procedures to estimate it for any value of its argument. This is not such a marked contrast from the familiar function $\cos t$, which is a solution of $y'' + y = 0$ and which can also be represented by a power series.

The function $J_0(t)$, called the *Bessel function of the first kind of order zero,* occurs in a number of problems in heat conduction in solids and vibration of membranes. It is an example of what are called *special functions,* many of which arise as series solutions of second order linear differential equations. A few of these special functions will be discussed at the end of this chapter—they are useful tools for the kitbag of any applied scientist.

6.2 SERIES SOLUTIONS—PART 1

In the remainder of this chapter we will denote the independent variable by x rather than t. Therefore the second order linear equation will be written as

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

with solutions $y = y(x)$. The use of x rather than t is somewhat traditional, since many of the problems associated with series solutions or special functions arise from physical problems where x is a spatial rather than a temporal variable.

To motivate the use of infinite series, we start with a simple example that can be solved explicitly.

EXAMPLE 1 Consider the first order differential equation

$$y' = 2xy,$$

whose general solution is $y(x) = Ae^{x^2}$, where A is an arbitrary constant. Assume that the solution can be represented by a convergent power series; hence

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and the task is to find the a_n.

SOLUTION Since a convergent power series can be differentiated term by term and also multiplied by $2x$, the series can be substituted into the differential equation to obtain

$$y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} = 2xy(x) = \sum_{n=0}^{\infty} 2a_n x^{n+1}.$$

Expanding both sides gives

$$[0 \cdot a_0 + 1 \cdot a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots]$$
$$= [2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + 2a_5 x^6 + \cdots],$$

and by comparing the coefficients of like powers of x we have

$$0 \cdot a_0 = 0, \quad 1 \cdot a_1 = 0, \quad 2a_2 = 2a_0, \quad 3a_3 = 2a_1,$$
$$4a_4 = 2a_2, \quad 5a_5 = 2a_3, \quad 6a_6 = 2a_4, \quad \ldots .$$

This implies that a_0 is arbitrary and

$$a_1 = 0, \quad a_2 = a_0, \quad a_3 = \frac{2}{3} a_1 = 0, \quad a_4 = \frac{a_2}{2} = \frac{a_0}{2},$$
$$a_5 = \frac{2}{5} a_3 = 0, \quad a_6 = \frac{a_4}{3} = \frac{a_0}{3 \cdot 2}, \quad \ldots ;$$

hence

$$y(x) = a_0\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right],$$

which is the first four terms of the Taylor series for $a_0 e^{x^2}$.

The reader can easily see that the general recursion relation for the a_n is $na_n = 2a_{n-2}$, $n = 2, 3, \ldots$. Since a_0 is arbitrary and a_1 is zero, this implies that

$$a_{2n+1} = 0, \qquad n = 0, 1, 2, \ldots,$$

$$a_{2n} = \frac{2a_{2n-2}}{2n} = \frac{1}{n}\cdot\frac{2a_{2n-4}}{2n-2} = \frac{1}{n(n-1)}\cdot\frac{2a_{2n-6}}{2n-4} = \cdots = \frac{1}{n!}a_0.$$

Therefore

$$y(x) = \sum_{n=0}^{\infty}\frac{a_0}{n!}x^{2n} = a_0\sum_{n=0}^{\infty}\frac{x^{2n}}{n!} = a_0 e^{x^2},$$

and the power series solution is exactly the Taylor series of the general solution.■

We now return to the second order equation

$$y'' + p(x)y' + q(x)y = 0 \tag{6.2.1}$$

and consider first the simplest case, in which the coefficient functions $p(x)$ and $q(x)$ are smooth, well-behaved functions in some neighborhood of a given point, which is assumed to be $x = 0$ for simplicity of notation. Consequently, we will assume that $p(x)$ and $q(x)$ have infinite power series representations

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

where these series converge in some neighborhood of $x = 0$. This leads to the following definition.

Definition

If the coefficients $p(x)$ and $q(x)$ of equation (6.2.1) can be represented by power series that are convergent in some neighborhood of $x = 0$, then $x = 0$ is said to be an *ordinary point* of the differential equation.

With the above assumptions it is natural to suppose that the solution $y(x)$ will also have a power series representation convergent in some neighborhood of $x = 0$, and therefore

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{6.2.2}$$

Since a power series can be differentiated term by term, the solution procedure is to substitute the series for $y(x)$ into (6.2.1), with $p(x)$ and $q(x)$ replaced by their power series representation, and compare coefficients of like powers of x. This will lead to a recursion relation by which the a_n's can be determined step by step, and possibly a general expression can be derived that will give all the a_n's. (In the case where the ordinary point is $x_0 \neq 0$, one would use instead $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, and the algebraic procedure is the same.) We summarize the above discussion.

Solution procedure for ordinary point case

1. Replace the coefficients $p(x)$ and $q(x)$ in the differential equation (6.2.1) by their power series representations if necessary.
2. Differentiate the series (6.2.2) successively term by term and substitute the series for $y(x)$, $y'(x)$ and $y''(x)$ into the differential equation.
3. Combine terms and compare coefficients of like powers of x. This will lead to a recursion relation by which the coefficients a_n can be determined step by step. Possibly a general recursion relation can be obtained by which all the a_n's can be determined.

This procedure is best illustrated by an example.

EXAMPLE 2 Find the series solution of *Airy's equation*

$$y'' - xy = 0. \tag{6.2.3}$$

SOLUTION Since $p(x) = 0$ and $q(x) = -x$ the point $x = 0$ is an ordinary point. From (6.2.2) we obtain the relations

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

through term-by-term differentiation. After multiplying the series for $y(x)$ by x, substitution into (6.2.3) gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{6.2.4}$$

If just the first few terms of the solution are desired, one can expand the above sums to get

$$(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots) - (a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \cdots) = 0.$$

Now, combining the coefficients of like powers of x and setting them equal to zero gives the relations

$$2a_2 = 0, \quad 6a_3 - a_0 = 0, \quad 12a_4 - a_1 = 0, \quad 20a_5 - a_2 = 0,$$
$$30a_6 - a_3 = 0, \quad 42a_7 - a_4 = 0, \quad \ldots,$$

and consequently

$$a_2 = 0, \quad a_3 = \frac{a_0}{6}, \quad a_4 = \frac{a_1}{12}, \quad a_5 = \frac{a_2}{20} = 0,$$
$$a_6 = \frac{a_3}{30} = \frac{a_0}{180}, \quad a_7 = \frac{a_4}{42} = \frac{a_1}{504}, \ldots .$$

Therefore

$$y(x) = a_0 + a_1 x + 0 + \frac{a_0}{6}x^3 + \frac{a_1}{12}x^4 + 0 + \frac{a_0}{180}x^6 + \frac{a_1}{504}x^7 + \cdots$$
$$= a_0\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right] + a_1\left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right],$$

where a_0 and a_1 are arbitrary constants. ■

Examining the last expression we see that it can be written in the form

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where $y_1(x)$ will have a power series representation in $3n$th powers of x, where $n = 0, 1, 2, \ldots$, and $y_2(x)$ in $(3n + 1)$st powers of x, where $n = 0, 1, 2, \ldots$. Clearly, the two expressions are linearly independent and they represent two linearly independent solutions of Airy's equation. The standard notation for them is

$$\text{Ai}(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots,$$
$$\text{Bi}(x) = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots;$$

Ai(x) and Bi(x) are called the Airy functions, and the reader is referred to [10], where they are extensively tabulated and many of their properties are given. The two arbitrary constants a_0 and a_1 can be determined only if initial or boundary conditions are given.

If a general expression for the series representations of Ai(x) and Bi(x) is desired, we must return to relation (6.2.4) and try to derive a general recursion relation from it. The procedure used is to shift indices in the two series until they can be combined. Note that the first series leads off with x^0, whereas the second leads off with x, so write

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Now both series lead off with x, but the first starts with $n = 3$, whereas the second starts with $n = 0$, so reindex the second as

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0.$$

Both sums can now be combined to obtain

$$2a_2 + \sum_{n=3}^{\infty} [n(n-1)a_n - a_{n-3}]x^{n-2} = 0.$$

From the above follow the relations

$$a_2 = 0, \qquad a_n = \frac{a_{n-3}}{n(n-1)}, \qquad n = 3, 4, 5, \ldots, \tag{6.2.5}$$

which clearly imply that

$a_2, a_5, a_8, \ldots, a_{3n+2}, \ldots$ are all zero,

$a_3, a_6, a_9, \ldots, a_{3n}, \ldots$ all depend on a_0,

$a_4, a_7, a_{10}, \ldots, a_{3n+1}, \ldots$ all depend on a_1.

The relations (6.2.5) are the desired general recursion relations.

The matter of reindexing the series, so as to be able to combine terms, is somewhat arbitrary. For instance, in the expression

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

one could reindex the first series instead, so that it starts with $n = 0$. This would give

$$2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2)a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0,$$

and the recursion relation would be

$$a_2 = 0, \qquad a_{n+3} = \frac{a_n}{(n+3)(n+2)}, \qquad n = 0, 1, 2, \ldots.$$

It is easily seen that this would give exactly the same result.

The reader should now study the analysis below to see how to use (6.2.5) to get the general series expressions for Ai(x) and Bi(x). First,

$$a_n = \frac{a_{n-3}}{n(n-1)}$$

implies that

$$a_{3n} = \frac{a_{3n-3}}{3n(3n-1)} = \frac{a_{3(n-1)}}{3^2 n(n-\frac{1}{3})},$$

and now apply the recursion relation to a_{3n-3} :

$$a_{3n-3} = \frac{a_{3n-6}}{(3n-3)(3n-4)}$$

or

$$a_{3(n-1)} = \frac{a_{3(n-2)}}{3(n-1)(3n-4)} = \frac{a_{3(n-2)}}{3^2(n-1)(n-\frac{4}{3})}.$$

Substitute the last expression for $a_{3(n-1)}$ in the expression for a_{3n} to get

$$a_{3n} = \frac{a_{3(n-2)}}{3^4 n(n-1)(n-\frac{1}{3})(n-\frac{4}{3})}.$$

Proceeding backwards in this fashion, one finally obtains

$$a_{3n} = \frac{a_0}{3^{2n}\, n!(n-\frac{1}{3})(n-\frac{4}{3})\cdots(\frac{2}{3})}, \qquad n = 1, 2, \ldots,$$

and since a_0 is arbitrary, the choice of $a_0 = 1$ gives

$$\mathrm{Ai}(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{3^{2n}\, n!(n-\frac{1}{3})(n-\frac{4}{3})\cdots(\frac{2}{3})} x^{3n}.$$

The reader may wish to work out the second case starting with a_{3n+1} and letting $a_1 = 1$ to get

$$\mathrm{Bi}(x) = x + \sum_{n=1}^{\infty} \frac{1}{3^{2n}\, n!(n+\frac{1}{3})(n-\frac{2}{3})\cdots(\frac{4}{3})} x^{3n+1},$$

and to show, by using the ratio test, that both series converge for $-\infty < x < \infty$.

A brief word should be said about convergence of the power series obtained by the method just described. Suppose one is given the linear differential equation

$$y'' + p(x)y' + q(x)y = f(x),$$

where $p(x)$, $q(x)$, and $f(x)$ have power series representations convergent for $|x - x_0| < r$, $r > 0$, so that x_0 is an *ordinary point*. Then the power series for the solution, obtained formally by substituting the power series

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

and recursively solving for the coefficients, will also converge for $|x - x_0| < r$. This useful result, which is proved in more advanced texts, means that there is no need to test for convergence of the power series obtained for the solution. It will have the same radius of convergence as the smallest radius of convergence of the power series for the coefficients.

EXERCISES 6.2

1. Use the power series method to find the general solutions of
   a) $y'' + y = 0$;
   b) $y'' - 4y = 0$.

   Verify that you obtain the series for $\sin x$ and $\cos x$ in (a), and for e^{2x} and e^{-2x} in (b).

In Exercises 2–5, find the recurrence relation for the coefficients and the first six nonzero terms of the series solutions with the ordinary point, $x_0 = 0$.

2. $y'' - xy' - 2y = 0$
3. $y'' + x^2y = 0$
4. $y'' + y' + xy = 0, \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' - (1 + x^2)y = 0, \quad y(0) = -2, \quad y'(0) = 2$

By expressing the coefficients in a power series, substituting $y(x) = \Sigma_{n=0}^{\infty} a_n x^n$, and equating like powers of x, find the first five nonzero terms of the series solutions of Exercises 6–8.

6. $y'' + 2y' + (\sin x)y = 0$
7. $y'' + e^x y = 0, \quad y(0) = 1, \quad y'(0) = -1$
8. $y'' + (\alpha + \beta \cos 2x)y = 0$ (Mathieu's equation: α, β are parameters.)

In Exercises 9–13, the ordinary point $x_0 \neq 0$ so the series solution will be of the form $y(x) = \Sigma_{n=0}^{\infty} a_n(x - x_0)^n$. Find its first five nonzero terms.

9. $y'' + (x - 1)y' + 2y = 0$ with the ordinary point $x_0 = 1$.
10. $y'' + (x^2 - 1)y = 0$ with the ordinary point $x_0 = 1$.
    (*Hint:* $x^2 - 1 = 2(x - 1) + (x - 1)^2$.)
11. $y'' + xy' + (\ln x)y = 0$ with the ordinary point $x_0 = 1$.
    (*Hint:* $x = 1 + (x - 1)$.)

12. $y'' + (\ln x)y = 0$ with the ordinary point $x_0 = 1$.

13. $y'' + (\sin x)y = 0$ with the ordinary point $x_0 = \pi/2$.

14. Series solutions can be used occasionally to find approximations to solutions of nonlinear equations. Find the first three nonzero terms of the series solutions of the following:

a) $y' = y^2 + (1 + x^2)$, $y(0) = 0$. (*Hint:* The initial conditions imply that $y(x) = a_1 x + a_2 x^2 + \cdots$, and a useful fact is

$$\left(\sum_{k=1}^{n} c_k\right)^2 = \sum_{k=1}^{n} c_k^2 + 2\sum_{j \neq k} c_j c_k.)$$

b) $y' = 1/y + x^2$, $y(0) = 1$. (*Hint:* $y(x) = 1 + a_1 x + a_2 x^2 + \cdots$; recall that

$$(1 + u)^{-1} = 1 - u + u^2 - \cdots + (-1)^n u^n + \cdots, \qquad |u| < 1.)$$

Method of Taylor Series. Given the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = a, \qquad y'(0) = b,$$

where $p(x)$ and $q(x)$ are smooth functions near $x = 0$, one can find $y''(0)$ by direct substitution:

$$\begin{aligned} y''(0) &= -p(0)y'(0) - q(0)y(0) \\ &= -bp(0) - aq(0). \end{aligned}$$

Differentiating the equation and then substituting again will give $y'''(0)$. For example,

$$y'''(0) + p'(0)y'(0) + p(0)y''(0) + q'(0)y(0) + q(0)y'(0) = 0$$

implies that

$$y'''(0) = -bp'(0) - p(0)y''(0) - aq'(0) - bq(0).$$

Proceeding in this manner, one can use the derivatives obtained to develop the Taylor series of the solution:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots + \frac{y^{(n)}(0)}{n!}x^n + \cdots.$$

In Exercises 15–20, use this method to find the first four nonzero terms of the series solutions.

15. $y'' + x^2 y = 0$, $y(0) = 0$, $y'(0) = -1$

16. Exercise 4 above

17. $y'' + (e^{2x})y' + y = 0$, $y(0) = 0$, $y'(0) = 1$

18. Exercise 5 above

19. $y'' + xy = e^{2x}, \quad y(0) = 0, \quad y'(0) = 1$
20. $y'' - xy' + y = 4 \sin x, \quad y(0) = 1, \quad y'(0) = -1$
21. Obtain the general series expression for the Airy function $\text{Bi}(x)$.

6.3 SERIES SOLUTIONS—PART 2

In the previous section we considered series solutions for

$$y'' + p(x)y' + q(x)y = 0,$$

where it was assumed that $p(x)$ and $q(x)$ were smooth functions and had convergent power series expansions in some neighborhood of a given point x_0. In this case the desired series solutions were obtained by substitution and comparison of coefficients of like powers of $x - x_0$. Does this procedure work when $p(x)$ or $q(x)$ are singular at $x = x_0$? The answer is NO, except when the singularity is of a special kind, but fortunately this class of equations contains many of the equations that arise in applied mathematics and mathematical physics.

Suppose as before that the given point in question is $x_0 = 0$, and suppose further that the differential equation can be written in the form

$$y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0, \tag{6.3.1}$$

where $p(x)$ and $q(x)$ are smooth functions in a neighborhood of $x = 0$; hence

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

In this case $x = 0$ is said to be a *regular singular point,* and it is this type of singularity for which the method of power series, suitably modified, also works.

Definition

If the functions $p(x)$ and $q(x)$ of equation (6.3.1) can be represented by power series convergent in some neighborhood of $x = 0$, then $x = 0$ is said to be a *regular singular point* of the differential equation.

The following are examples of differential equations with a regular singular point at $x = 0$. Note that one must sometimes modify the coefficients to get the equation into the form of (6.3.1).

1. $x^2 y'' + y = 0$ or $y'' + \dfrac{1}{x^2} y = 0$ with $p(x) = 0$, $q(x) = 1$;

2. $4x^2y'' - 3xy' + 2y = 0$ or $y'' + \dfrac{-3/4}{x}y' + \dfrac{1/2}{x^2}y = 0$ with
$p(x) = -\dfrac{3}{4}, q(x) = \dfrac{1}{2}$;

3. $xy'' + (x + 2)y' + e^x y = 0$ or $y'' + \dfrac{x + 2}{x}y' + \dfrac{xe^x}{x^2}y = 0$
with $p(x) = x + 2, q(x) = xe^x$;

4. $y'' + 2y' + \dfrac{4}{x}y = 0$ or $y'' + \dfrac{2x}{x}y' + \dfrac{4x}{x^2}y = 0$ with $p(x) = 2x$, $q(x) = 4x$.

THE METHOD OF FROBENIUS

The *method of Frobenius* provides a technique for finding a series solution, but the series will be of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \tag{6.3.2}$$

where r is some number *(not necessarily an integer!)* to be determined from the differential equation. This is the modification incurred by the fact that $x = 0$ is a regular singular point. Term-by-term differentiation of the expression for $y(x)$ gives

$$y'(x) = \sum_{n=0}^{\infty} (n + r)a_n x^{n+r-1}$$

and

$$y''(x) = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-2}.$$

Now write the differential equation (6.3.1) in the form

$$x^2y'' + xp(x)y' + q(x)y = 0$$

and substitute the expansions for $p(x)$, $q(x)$, $y(x)$, $y'(x)$, and $y''(x)$ to obtain

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n\right) \sum_{n=0}^{\infty} (n + r)a_n x^{n+r}$$
$$+ \left(\sum_{n=0}^{\infty} q_n x^n\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

The term x^r can be factored out of each term, and combining the coefficients of like powers of x we obtain the expression

$$x^r[r(r-1)a_0 + p_0 r a_0 + q_0 a_0]$$
$$+ x^r[(1+r)ra_1 + p_0(1+r)a_1 + p_1 r a_0 + q_0 a_1 + q_1 a_0]x$$
$$+ x^r[\quad]x^2 + \cdots + x^r[\quad]x^n + \cdots = 0,$$

where we have explicitly written down the coefficients of $x^0 = 1$ and x. Equating to zero the first bracketed term and assuming $a_0 \neq 0$ gives

$$r(r-1) + p_0 r + q_0 = r^2 + (p_0 - 1)r + q_0 = 0. \tag{6.3.3}$$

This quadratic polynomial in r is called the *indicial equation,* and its two roots, r_1 and r_2, will be the admissible values of the exponent r in the expression (6.3.2) for the solution. Note that p_0 and q_0 are merely the respective values of $p(0)$ and $q(0)$ and can be easily obtained from the differential equation.

The theory of the regular singular point case, which the reader may wish to examine in more detail in some of the references, leads to the following solution procedure.

Solution procedure for the regular singular point case

1. Find $p_0 = p(0)$; and $q_0 = q(0)$, then find the roots r_1 and r_2 of the indicial equation (6.3.3).
2. Choose as r_1 that root of the indicial equation for which the real part of $r_1 - r_2$ is nonnegative; if r_1 and r_2 are real and unequal then r_1 would be the largest root, for instance. Then for $r = r_1$, let the solution be represented by the series

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n.$$

3. Replace $p(x)$ and $q(x)$ by their power series representations, substitute $y_1(x)$ into the differential equation and proceed as in the ordinary point case. One can always obtain recursively all the coefficients a_n and the series obtained will converge in some deleted neighborhood of $x = 0$. By a deleted neighborhood we mean the set of all x satisfying $0 < |x| < \alpha$ for some $\alpha > 0$. In some cases the series may also converge at $x = 0$.
4. If r_2 is the second root and $r_1 - r_2$ is not zero or a positive integer, then one can proceed as in (1). The series

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

with the b_n's determined recursively converges in some deleted neighborhood of $x = 0$ and is a solution. The two solutions, $y_1(x)$ and $y_2(x)$, are linearly independent.

5. If $r_1 - r_2$ equals zero or a positive integer, then there are two possibilities: Either a solution of the form $y_2(x)$ above exists or the second solution is of the form

$$y_2(x) = y_1(x)\,\beta \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n,$$

with β a constant dependent on $y_1(x)$ and $p(x)$.

In the case where the regular singular point $x_0 \neq 0$ one would use $p_0 = p(x_0)$ and $q_0 = q(x_0)$ in the indicial equation. The solution corresponding to the root r_1 would then be of the form

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)$$

and the solution procedure above applies. (See Exercise 8 of Section 6.3.)

The possibility (5) above is the so-called *logarithmic case,* and if $r_1 - r_2$ equals zero, it always occurs since the indicial equation has only one root. If $r_1 - r_2$ is a positive integer, there is a technique using $p(x)$ and $y_1(x)$ to determine whether the logarithmic case occurs; it uses the reduction of order formula to be discussed shortly, and the interested reader is referred to [2] or [3] for details.

The problem with the case where the real part of $r_1 - r_2$ is a positive integer is that if one substitutes the series $y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$ into the differential equation, then one cannot find a recursion relation for all the b_n. Specifically, for some positive integer m one can no longer solve for b_m in terms of the previous b_j and the process comes to a halt. One must resort to other techniques to find a second linearly independent solution.

A simple example of the logarithmic case, which the reader has seen before, is the Euler differential equation

$$x^2 y'' + axy' + by = 0, \quad \text{where } a, b = \text{const.}$$

Clearly, $x = 0$ is a regular singular point and $p(x) = a$, $q(x) = b$, so the indicial equation is

$$r^2 + (a - 1)r + b = 0.$$

If the roots of the indicial equation are r_1 and r_2 and both are real and equal, then two linearly independent solutions are

$$y_1(x) = x^{r_1} \quad \text{and} \quad y_2(x) = x^{r_1} \ln x = y_1(x) \ln x.$$

This is an example of case (5) above with $\beta = 1$ and $\sum_{n=0}^{\infty} b_n x^n = 0$.

The Euler differential equation can be used to illustrate two observations about regular singular points. The first is that solutions need not be undefined at a regular singular point. For instance, a linearly independent pair of solutions of

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$$

are $y_1(x) = x$ and $y_2(x) = x^2$ and both are smooth, well-defined functions at the regular singular point $x_0 = 0$. The example also illustrates the second observation, namely that the solutions at a regular singular point need not be full infinite series, but may merely be polynomials. Some examples of equations with a regular singular point and their corresponding indicial equations are given below.

EXAMPLE 1 Find the indicial equation, its roots, and the general form of the solution for

a) $y'' - [1/(2x)]y' + [(1 + x^2)/(2x^2)]y = 0$;
b) $y'' + (2/x)y' + xy = 0$;
c) $y'' + (3/x)y' + [(1 + x)/x^2]y = 0$.

SOLUTION

a) Because $p(x) = -1/2$ and $q(x) = 1/2 + (1/2)x^2$; we have $p_0 = -1/2$, $q_0 = 1/2$. The indicial equation is

$$r^2 + \left(-\frac{1}{2} - 1\right)r + \frac{1}{2} = (r - 1)\left(r - \frac{1}{2}\right) = 0$$

with roots $r_1 = 1$ and $r_2 = 1/2$, and $r_1 - r_2 = 1/2$, which is not zero or a positive integer. We conclude that two linearly independent series solutions,

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n,$$

can be found.

b) If the last term is written as $x = x^3/x^2$, it is seen that $p(x) = 2$, $q(x) = x^3$, hence $p_0 = 2$, $q_0 = 0$. The indicial equation is

$$r^2 + (2 - 1)r = r(r + 1) = 0$$

with roots $r_1 = 0$ and $r_2 = -1$, so $r_1 - r_2 = 1$ a positive integer. Corresponding to the root $r_1 = 0$ there is a solution $y_1(x) = \Sigma_{n=0}^{\infty} a_n x^n$, and the second solution is either a logarithmic case or has the form $y_2(x) = x^{-1}\Sigma_{n=0}^{\infty} b_n x^n$, corresponding to $r_2 = -1$, which turns out to be the case.

c) We have $p(x) = 3$, $q(x) = 1 + x$, and $p_0 = 3$, $q_0 = 1$. The indicial equation is

$$r^2 + (3 - 1)r + 1 = (r + 1)^2 = 0$$

with roots $r_1 = r_2 = -1$ and $r_1 - r_2 = 0$. This is a logarithmic case, and the solutions are of the form

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n$$

corresponding to $r_1 = -1$, and

$$y_2(x) = y_1(x)\,\beta \ln x + x^{-1}\sum_{n=0}^{\infty} b_n x^n.$$ ■

Once the value of r_1 is determined from the indicial equation, finding the series solution for $y_1(x)$ proceeds as in the case of an ordinary point. The series is substituted and like powers of x are compared after shifting indices if necessary.

EXAMPLE 2 Find $y_1(x)$ for the differential equation of Example 1(c) above:

$$x^2y'' + 3xy' + (1 + x)y = 0.$$

SOLUTION Set $y_1(x) = \Sigma_{n=0}^{\infty} a_n x^{n-1}$, then differentiate the series twice, term by term, and substitute it into the differential equation to get

$$x^2\sum_{n=0}^{\infty}(n-1)(n-2)a_n x^{n-3} + 3x\sum_{n=0}^{\infty}(n-1)a_n x^{n-2} + (1+x)\sum_{n=0}^{\infty} a_n x^{n-1} = 0.$$

Multiply each series by its coefficient and combine terms to obtain

$$\sum_{n=0}^{\infty}[(n-1)(n-2) + 3(n-1) + 1]a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

When $n = 0$ in the first series, the coefficient of x^{-1} is zero, so we can shift its index by one to get

$$\sum_{n=0}^{\infty}(n+1)^2 a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}[(n+1)^2 a_{n+1} + a_n]x^n = 0.$$

Hence $a_{n+1} = -a_n/(n+1)^2$, $n = 0, 1, 2, \ldots$, which implies

$$a_1 = -a_0, \qquad a_2 = \frac{-a_1}{2^2} = \frac{a_0}{2^2}, \qquad a_3 = \frac{-a_2}{3^2} = \frac{-a_0}{3^2 \cdot 2^2}, \text{ etc.,}$$

and in general,

$$a_n = \frac{(-1)^n a_0}{n^2 \cdots 3^2 \cdot 2^2} = \frac{(-1)^n a_0}{(n!)^2}.$$

Since a_0 is arbitrary, it may be set equal to unity, and the first solution is

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n.$$

Usually writing down the first few terms of the series expansion, after combining those series that lead off with the same power of x, will give a clue as to how to shift the indices. Careful bookkeeping does the rest! ■

The following exercises deal, for the most part, with the nonlogarithmic case. To obtain the second solution in the logarithmic case, a technique called *the method of reduction of order* is often used. That is the subject of the next section.

EXERCISES 6.3

The equations of Exercises 1–7 have a regular singular point at $x = 0$. Find the indicial equation, determine its roots, and state what general form the series solutions will have.

1. $2x^2y'' + (3x - 2x^2)y' - (x + 1)y = 0$
2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
3. $4xy'' + 2y' - y = 0$
4. $xy'' + y' + x^2y = 0$
5. $x^2y'' + 5xy' + 4(\cos x)y = 0$
6. $x^2y'' + (2x + x^2)y' - 2y = 0$
7. $16x^2y'' + 24xy' - 3y = 0$
8. If a differential equation has a regular singular point at $x = a$, then it can be written in the form

$$y'' + \frac{p(x)}{x - a}y' + \frac{q(x)}{(x - a)^2}y = 0,$$

where $p(x)$ and $q(x)$ are smooth functions in a neighborhood of $x = a$. By assuming a solution of the form

$$y(x) = (x - a)^r \sum_{n=0}^{\infty} a_n(x - a)^n,$$

show that r must be a root of the indicial equation

$$r^2 + (p_0 - 1)r + q_0 = 0,$$

where $p_0 = p(a)$, $q_0 = q(a)$.

In Exercises 9–12 find the regular singular points, the corresponding indicial equation, and its roots, and state what general form the series solution will have.

9. $(x + 1)y'' + \frac{3}{2}y' + y = 0$

10. $(x - 1)y'' + xy' + y = 0$

11. $2(x - 2)^2y'' - \dfrac{6(x - 2)}{x}y' + 3y = 0$

12. $(1 - x^2)y'' - 2xy' + 6y = 0$

The equations in Exercises 13–17 have a regular singular point at $x = 0$ and are the case $r_1 - r_2 \neq 0$ or a positive integer. Find the recursion relation and the first four nonzero terms of the series expansion for each of the two linearly independent solutions. Find a general expression for each solution if possible.

13. $2xy'' + y' - y = 0$

14. $2x^2y'' - xy' + (1 - x^2)y = 0$

15. $4xy'' + 2y' + y = 0$

16. $3x^2y'' + 4xy' - 2y = 0$

17. $2x^2y'' - 5(\sin x)\, y' + 3y = 0$; find only the first three nonzero terms of each solution.

The equations in Exercises 18–22 have a regular singular point at $x = 0$ and $r_1 - r_2$ is a positive integer (the nonlogarithmic case). Both series solutions can often be found by substituting $y(x) = x^{r_2} \sum_{n=0}^{\infty} a_n x^n$, where r_2 is the *smaller* root of the indicial equation. Find the recursion relation and the first four nonzero terms of the series expansion for each of the two linearly independent solutions. Find a general expression for each solution if possible.

18. $xy'' + (3 + x^3)y' + 3x^2y = 0$

19. $xy'' + 2y' + x^2y = 0$

20. $xy'' + (4 + x)y' + 2y = 0$

21. $x(1 - x)y'' - (4 + x)y' + 4y = 0$

22. $x(1 - x)y'' - (4 + x)y' + 4y = 0$, solve for the regular singular point $x = 1$.

6.4 THE METHOD OF REDUCTION OF ORDER AND THE LOGARITHMIC CASE OF A REGULAR SINGULAR POINT

Suppose we are given the second order linear equation

$$y'' + p(x)y' + q(x)y = 0, \tag{6.4.1}$$

where $p(x)$ and $q(x)$ are continuous for all x in some neighborhood of $x = x_0$ but may be discontinuous or singular at x_0 itself. For instance, this will be the case if x_0 is a regular singular point. Suppose further the happy circumstance that a nontrivial solution $y_1(x)$ of (6.4.1) is known. This solution could be the result of a fortuitous guess or of diligent labor, as in the case of a regular singular point, where $y_1(x)$ is the series solution corresponding to r_1, the largest root of the indicial equation.

The method of reduction of order deals with the following question. Given a nontrivial solution $y_1(x)$ of (6.4.1), can one find a second linearly independent solution $y_2(x)$? The answer is YES, and the method is especially useful when $y_1(x)$ is given by an infinite series and one wishes to obtain the first few terms of the series representation of $y_2(x)$. For the logarithmic case of the regular singular point it is an especially effective tool.

Suppose $y_1(x)$ is a solution of (6.4.1). The method of reduction of order, like the method of variation of parameters, assumes that the second solution can be expressed as $y_2(x) = y_1(x)\, u(x)$, where $u(x)$ is to be determined. Therefore

$$y_2 = y_1 u, \qquad y_2' = y_1 u' + y_1' u, \qquad y_2'' = y_1 u'' + 2y_1' u' + y_1'' u,$$

and since $y_2(x)$ is assumed to be a solution, this implies that

$$y_2'' + p(x)y_2' + q(x)y_2 = [y_1'' + p(x)y_1' + q(x)y_1]u + y_1 u'' + (2y_1' + p(x)y_1)u' = 0.$$

Since $y_1(x)$ is a solution, the coefficient of u is zero, and so $u = u(x)$ must satisfy the relation

$$y_1 u'' + [2y_1' + p(x)y_1]u' = 0$$

or

$$u'' + \left[2\frac{y_1'}{y_1} + p(x)\right] u' = 0.$$

The last relation is a first order linear homogeneous equation in $w = u'$ and can be solved by using the methods of Chapter 1. The solution is

$$u'(x) = \exp\left[-2\int^x \frac{y_1'(s)}{y_1(s)}\, ds - \int^x p(s)\, ds\right] = \frac{\exp\left[-\int^x p(s)\, ds\right]}{y_1(x)^2}.$$

(We have ignored the immaterial constant of integration.) Integrating once more gives $u(x)$ and therefore $y_2(x)$. Following is a summary of the previous analysis.

Method of reduction of order. Given a solution $y_1(x)$ of equation (6.4.1), to find a second linearly independent solution $y_2(x)$, let $y_2(x) = y_1(x)u(x)$. Substitute and solve the differential equation obtained for $u(x)$ to get

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^r p(s)\, ds\right]}{y_1(r)^2}\, dr, \qquad \textbf{(6.4.2)}$$

which is the *reduction of order* formula.

To show that $y_1(x)$ and $y_2(x)$ are a linearly independent pair of solutions, we compute their Wronskian. The formulas above for u' and y_2' imply that

$$y_2'(x) = y_1'(x) \int^x \frac{\exp\left[-\int^r p(s)\,ds\right]}{y_1(r)^2}\,dr + \frac{\exp\left[-\int^x p(s)\,ds\right]}{y_1(x)},$$

and therefore

$$y_1(x)y_2'(x) = y_1'(x)y_1(x) \int^x \frac{\exp\left[-\int^r x\, p(s)\,ds\right]}{y_1(r)^2}\,dr + \exp\left[-\int^x p(s)\,ds\right]$$

$$= y_1'(x)y_2(x) + \exp\left[-\int^x p(s)\,ds\right].$$

Consequently

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \exp\left[-\int^x p(s)\,ds\right] \neq 0.$$

Therefore $y_1(x)$ and $y_2(x)$ are a linearly independent pair. Note that if $p(x) = 0$, so that the differential equation is

$$y'' + q(x)y = 0,$$

then the reduction of order formula is simply

$$y_2(x) = y_1(x) \int^x \frac{1}{y_1(r)^2}\,dr.$$

EXAMPLE 1 Use the reduction of order formula to construct a second solution to each of the following differential equations.

a) The constant coefficient equation

$$y'' + 2by' + b^2y = 0;$$

b) the Euler equation

$$x^2y'' - 7xy' + 16y = 0, \qquad x \neq 0;$$

c) the nonconstant coefficient equation

$$y'' - \frac{2}{x^2+1}y = 0.$$

SOLUTION

a) This equation has a solution $y_1(x) = e^{-bx}$ and is the case where the characteristic polynomial has a double root $\lambda = b$. Since $p(x) = 2b$, the reduction of order formula gives

$$y_2(x) = e^{-bx} \int^x \frac{\exp\left[-\int^r 2b\, ds\right]}{(e^{-br})^2}\, dr = e^{-bx} \int^x \frac{e^{-2br}}{e^{-2br}}\, dr = xe^{-bx}.$$

b) A trial solution of the type $y_1(x) = x^k$ yields $k = 4$. Rewriting the equation as

$$y'' - \frac{7}{x} y' + \frac{16}{x^2} y = 0, \quad x \neq 0,$$

we see that $p(x) = -7/x$ and

$$\exp\left[-\int^r \left(-\frac{7}{s}\right) ds\right] = \exp\,[7 \ln r] = r^7.$$

Hence

$$y_2(x) = x^4 \int^x \frac{r^7}{(r^4)^2}\, dr = x^4 \ln x.$$

c) Careful examination reveals that $y_1(x) = x^2 + 1$ is a solution. Since $p(x) = 0$, the second linearly independent solution is

$$y_2(x) = (x^2 + 1) \int^x \frac{1}{(r^2 + 1)^2}\, dr = (x^2 + 1)\left[\frac{x}{2(x^2 + 1)} + \frac{1}{2} \tan^{-1} x\right]$$

$$= \frac{x}{2} + \frac{1}{2}(x^2 + 1) \tan^{-1} x. \qquad \blacksquare$$

It is often the case that the integral in the reduction of order formula cannot be evaluated. Nevertheless the expression for the second solution may still be useful, for instance to determine the asymptotic behavior of the general solution.

Returning to the topic of series solutions of linear differential equations, it was mentioned above that the first solution $y_1(x)$ could always be expressed as an infinite series. The use of the reduction of order formula to find the second linearly independent solution $y_2(x)$ will then involve the squaring and inverting of an infinite series, followed by a term-by-term integration. There is a systematic way to accomplish this, and the procedure is a direct generalization of arithmetical operations on polynomials.

The Cauchy product formula for series. Given the two power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n, \qquad \sum_{n=0}^{\infty} b_n(x - x_0)^n,$$

their product is given by the power series

$$\begin{aligned}\left[\sum_{n=0}^{\infty} a_n(x - x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x - x_0)^n\right] &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j b_{n-j}\right)(x - x_0)^n \\ &= a_0b_0 + (a_0b_1 + a_1b_0)(x - x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x - x_0)^2 \\ &\quad + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)(x - x_0)^n + \cdots.\end{aligned}$$

The formula is merely a systematic way of combining in the product all those terms that have the same power of $x - x_0$. If the two series are the same, this gives the formula for squaring a series:

The Cauchy product formula for squaring a series

$$\begin{aligned}\left[\sum_{n=0}^{\infty} a_n(x - x_0)^n\right]^2 = {} & a_0^2 + 2a_0a_1(x - x_0) + (a_1^2 + 2a_0a_2)(x - x_0)^2 \\ & + (2a_0a_3 + 2a_1a_2)(x - x_0)^3 \\ & + (a_2^2 + 2a_0a_4 + 2a_1a_3)(x - x_0)^4 + \cdots \\ & + (a_n^2 + 2a_0a_{2n} + \cdots + 2a_{n-1}a_{n+1})(x - x_0)^{2n} \\ & + (2a_0a_{2n+1} + \cdots + 2a_na_{n+1})(x - x_0)^{2n+1} + \cdots.\end{aligned}$$

EXAMPLE 2 Find the first four nonzero terms of the series for $\cos^2 x$.

SOLUTION Since

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

then $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_0 = 1, \qquad a_2 = -\frac{1}{2!} = -\frac{1}{2}, \qquad a_4 = \frac{1}{4!} = \frac{1}{24}$$

$$a_0 = -\frac{1}{6!} = -\frac{1}{720}, \ldots.$$

Therefore the series for $\cos^2 x$ will have only even terms and the Cauchy product formula can be used to compute its first four nonzero terms:

$$\cos^2 x = (1)^2 + \left[(0)^2 + 2(1)\left(-\frac{1}{2}\right)\right]x^2 + \left[\left(-\frac{1}{2}\right)^2 + 2(1)\left(\frac{1}{24}\right) + 2(0)(0)\right]x^4$$
$$+ \left[(0)^2 + 2(1)\left(-\frac{1}{720}\right) + 2(0)(0) + 2\left(-\frac{1}{2}\right)\left(\frac{1}{24}\right)\right]x^6 + \cdots$$
$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots . \qquad \blacksquare$$

In the reduction of order formula there appears an expression of the form $1/y_1(x)^2$, so if $y_1(x)$ is represented by a power series, it can be squared by using the above formula. But then it must be inverted or, equivalently, the series representation of an expression of the form

$$\frac{1}{\sum_{n=0}^{\infty} b_n(x - x_0)^n}$$

must be computed. To accomplish this, let the last expression equal the series $\Sigma_{n=0}^{\infty} q_n(x - x_0)^n$, where the q_n are to be determined. Therefore we have the relation

$$\left[\sum_{n=0}^{\infty} b_n(x - x_0)^n\right]\left[\sum_{n=0}^{\infty} q_n(x - x_0)^n\right] = 1,$$

and now use the Cauchy product formula on the left side and then equate the coefficients of like powers of $x - x_0$ to obtain

$$b_0q_0 = 1, \quad b_0q_1 + b_1q_0 = 0, \quad b_0q_2 + b_1q_1 + b_2q_0 = 0, \ldots,$$
$$b_0q_n + b_1q_{n-1} + \cdots + b_nq_0 = 0, \ldots .$$

Now solve for q_0 in the first equation, substitute its value in the second equation, and solve for q_1, use q_0 and q_1 in the third equation to find q_2, etc. The above formulas are a systematic recursive approach to what amounts to "long division."

It is assumed in the above division process that $b_0 \neq 0$. If $b_0, b_1, \ldots, b_{k-1}$ are all zero, but $b_k \neq 0$, then the above quotient may be written as

$$\frac{1}{\sum_{n=0}^{\infty} b_n(x - x_0)^n} = \frac{1}{(x - x_0)^k} \frac{1}{\sum_{n=0}^{\infty} b_{k+n}(x - x_0)^n},$$

and the division algorithm applied to the second term.

EXAMPLE 3 Find the first few terms of $1/g(x)^2$, where

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n+2} x^n.$$

SOLUTION We have

$$a_0 = 1, \quad a_1 = \frac{1}{3}, \quad a_2 = \frac{1}{4}, \quad a_3 = \frac{1}{5}, \ldots, \quad a_n = \frac{1}{n+2},$$

and by the Cauchy product formula

$$\begin{aligned} g(x)^2 &= (1 + \tfrac{1}{3}x + \tfrac{1}{4}x^2 + \cdots)(1 + \tfrac{1}{3}x + \tfrac{1}{4}x^2 + \cdots \\ &= 1^2 + \left[2 \cdot 1 \cdot \frac{1}{3}\right]x + \left[\left(\frac{1}{3}\right)^2 + 2 \cdot 1 \cdot \frac{1}{4}\right]x^2 \\ &\quad + \left[2 \cdot 1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{3} \cdot \frac{1}{4}\right]x^3 + \cdots \\ &= 1 + \frac{2}{3}x + \frac{11}{18}x^2 + \frac{17}{30}x^3 + \cdots. \end{aligned}$$

To find the first few terms of $1/g(x)^2 = q_0 + q_1x + q_2x^2 + \cdots$, one must solve the equation

$$(1 + \tfrac{2}{3}x + \tfrac{11}{18}x^2 + \cdots)(q_0 + q_1x + q_2x^2 + \cdots) = 1.$$

Therefore

$$1 \cdot q_0 = 1, \quad 1 \cdot q_1 + \frac{2}{3}q_0 = 0, \quad 1 \cdot q_2 + \frac{2}{3}q_1 + \frac{11}{18}q_0 = 0,$$

$$1 \cdot q_3 + \frac{2}{3}q_2 + \frac{11}{18}q_1 + \frac{17}{30}q_0 = 0, \ldots,$$

and we successively solve these to obtain

$$q_0 = 1, \quad q_1 = -\frac{2}{3}, \quad q_2 = -\frac{1}{6}, \quad q_3 = -\frac{13}{270}, \quad \ldots.$$

Therefore

$$\frac{1}{g(x)^2} = 1 - \frac{2}{3}x - \frac{1}{6}x^2 - \frac{13}{270}x^3 + \cdots. \qquad \blacksquare$$

After the above detour into the multiplication and inversion of power series, it is time to return to our principal topic—series solutions of linear differential equations around a regular singular point. The method of reduction of order will be used to obtain the first few terms of the series representation of the second solution $y_2(x)$, when the roots r_1 and r_2 of the indicial equation satisfy $r_1 - r_2 = 0$, the logarithmic case. This will be accomplished via some examples.

EXAMPLE 4 Find the first few terms of the logarithmic solution to

$$y'' + \frac{1}{x}y' + y = 0.$$

SOLUTION In Section 6.1 it was stated that a solution is

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \cdots.$$

Since $p(x) = 1/x$, the reduction of order formula gives

$$y_2(x) = J_0(x) \int^x \frac{\exp\left[-\int^r \frac{1}{s}\,ds\right]}{J_0(r)^2}\,dr = J_0(x) \int^x \frac{1}{rJ_0(r)^2}\,dr.$$

By rewriting the equation in the form

$$y'' + \frac{1}{x}y' + \frac{x^2}{x^2}y = 0,$$

we see that $x = 0$ is a regular singular point, and since $p_0 = 1$, $q_0 = 0$, the indicial equation is $r^2 + (1 - 1)r + 0 = r^2 = 0$ with roots $r_1 = r_2 = 0$. This is the logarithmic case.

To find the first few terms of $y_2(x)$, first calculate $J_0(x)^2$. Since

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\frac{1}{4}, \quad a_3 = 0, \quad a_4 = \frac{1}{64}, \quad \ldots,$$

we have

$$\begin{aligned} J_0(x)^2 &= 1^2 + (2 \cdot 1 \cdot 0)x + \left[0^2 + 2 \cdot 1 \cdot \left(-\frac{1}{4}\right)\right]x^2 \\ &\quad + \left[2 \cdot 1 \cdot 0 + 2 \cdot 0 \cdot \left(-\frac{1}{4}\right)\right]x^3 \\ &\quad + \left[\left(-\frac{1}{4}\right)^2 + 2 \cdot 1 \cdot \frac{1}{64} + 2 \cdot 0 \cdot 0\right]x^4 + \cdots \\ &= 1 - \frac{1}{2}x^2 + \frac{3}{32}x^4 + \cdots. \end{aligned}$$

To find $1/J_0(x)^2$, the following relations have to be solved:

$$1 \cdot q_0 = 1, \quad 1 \cdot q_1 + 0 \cdot q_0 = 0, \quad 1 \cdot q_2 + 0 \cdot q_1 + \left(-\frac{1}{2}\right)q_0 = 0,$$

$$1 \cdot q_3 + 0 \cdot q_2 + \left(-\frac{1}{2}\right)q_1 + 0 \cdot q_0 = 0,$$

$$1 \cdot q_4 + 0 \cdot q_3 + \left(-\frac{1}{2}\right)q_2 + 0 \cdot q_1 + \frac{3}{32}q_0 = 0, \ldots$$

to obtain

$$q_0 = 1, \quad q_1 = 0, \quad q_2 = \frac{1}{2}, \quad q_3 = 0, \quad q_4 = \frac{5}{32}, \ldots.$$

Therefore

$$\begin{aligned} y_2(x) &= J_0(x) \int^x \frac{1}{r}\left(1 + \frac{1}{2}r^2 + \frac{5}{32}r^4 + \cdots\right) dr \\ &= J_0(x)\left[\ln x + \frac{1}{4}x^2 + \frac{5}{128}x^4 + \cdots\right]. \end{aligned}$$

■

EXAMPLE 5 Find the first few terms of the logarithmic solution to

$$y'' + \frac{3}{x}y' + \frac{1 + x}{x^2}y = 0.$$

SOLUTION In the final example of the previous section, a solution was found to be

$$y_1(x) = x^{-1}\left[1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \cdots\right].$$

The point $x = 0$ is a regular singular point and the roots of the indicial equation are $r_1 = r_2 = -1$, so again this is the logarithmic case. Since $p(x) = 3/x$, then

$$\exp\left[-\int^r \frac{3}{s}\, ds\right] = \frac{1}{r^3},$$

and the second solution is

$$y_2(x) = y_1(x) \int^x \frac{1}{r^3 y_1(r)^2}\, dr = y_1(x) \int^x \frac{1}{r} \frac{1}{\left[1 - r + \dfrac{r^2}{4} - \dfrac{r^3}{36} + \cdots\right]^2}\, dr.$$

Letting $g(r)$ be the expression in the brackets, we use the Cauchy product formula to get

$$g(r)^2 = 1 - 2r + \tfrac{3}{2}r^2 - \tfrac{5}{9}r^3 + \cdots.$$

To find $1/g(r)^2$ we must solve the relations

$$1 \cdot q_0 = 1, \quad 1 \cdot q_1 + (-2)q_0 = 0,$$

$$1 \cdot q_2 + (-2)q_1 + \left(\frac{3}{2}\right)q_0 = 0,$$

$$1 \cdot q_3 + (-2)q_2 + \left(\frac{3}{2}\right)q_1 + \left(-\frac{5}{9}\right)q_0 = 0,$$

to obtain

$$\frac{1}{g(r)^2} = 1 + 2r + \frac{5}{2}r^2 + \frac{23}{9}r^3 + \cdots.$$

Therefore

$$\begin{aligned} y_2(x) &= y_1(x)\int^x \frac{1}{r}\left[1 + 2r + \frac{5}{2}r^2 + \frac{23}{9}r^3 + \cdots\right] dr \\ &= y_1(x)\ln x + y_1(x)\left[2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \cdots\right]. \end{aligned}$$

A further multiplication of series would give

$$y_2(x) = y_1(x)\ln x + [2 - \frac{3}{4}x + \frac{11}{108}x^2 + \cdots]. \qquad \blacksquare$$

EXAMPLE 6 Find a solution to $y'' + (2/x)y' + xy = 0$ that is singular at $x = 0$.

SOLUTION The indicial equation is $r^2 + r = 0$ with roots $r_1 = 0$, $r_2 = -1$. Since $r_1 - r_2 = 1$ (a positive integer), this may or may not be a logarithmic case. The solution $y_1(x)$ corresponding to $r_1 = 0$ has the series representation (see Exercise 19 of the previous section)

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Since $p(x) = 2/x$ then

$$\exp\left[\int^r \frac{2}{s}\,ds\right] = \frac{1}{r^2},$$

and by the reduction of order formula gives

$$y_2(x) = y_1(x)\int^x \frac{1}{r^2 y_1(r)^2}\,dr = y_1(x)\int^x \frac{1}{r^2\left(\sum_{n=0}^{\infty} a_n r^n\right)^2}\,dr$$

$$= y_1(x)\int^x \frac{1}{r^2}[b_0 + b_1 r + b_2 r^2 + \cdots]\,dr$$

$$= y_1(x)[-b_0 x^{-1} + b_1 \ln x + b_2 x + \cdots],$$

where

$$\frac{1}{\left(\sum_{n=0}^{\infty} a_n x^n\right)^2} = \sum_{n=0}^{\infty} b_n x^n.$$

There will be a logarithmic term if $b_1 \neq 0$; otherwise the solution will be of the form

$$y_2(x) = x^{-1}\sum_{n=0}^{\infty} c_n x^n$$

corresponding to the root $r_2 = -1$. The reader may wish to find the first few a_n's and consequently show that $b_1 = 0$. Therefore the singular solution is of the form above, and not the logarithmic case. ■

EXERCISES 6.4

Given the linear differential equations of Exercises 1–7 and one solution $y_1(x)$, use the reduction of order formula to find a second linearly independent solution $y_2(x)$.

1. $y'' - 4y' + 13y = 0, \quad y_1(x) = e^{2x}\cos 3x$
2. $y'' - 9y = 0, \quad y_1(x) = e^{3x}$
3. $y'' - xy' + y = 0, \quad y_1(x) = x$
4. $x^2y'' + 4xy' - 10y = 0, \quad y_1(x) = x^2$
5. $y'' - \dfrac{6}{x^2}y = 0, \quad y_1(x) = \dfrac{1}{x^2}$
6. $(1 - x^2)y'' - 2xy' + 2y = 0, \quad y_1(x) = x$
7. $y'' - 2(1 + \tan^2 x)y = 0, \quad y_1(x) = \tan x$

Given the following series

$$P(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} x^n,$$

$$Q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n,$$

compute the first four nonzero terms of the expressions in Exercises 8–13.

8. $P(x)^2$ **9.** $\dfrac{1}{P(x)^2}$ **10.** $\dfrac{1}{Q(x)^2}$ **11.** $P(x)\,Q(x)$

12. $\dfrac{Q(x)}{P(x)}$ **13.** $P(x)\,Q^2(x)$

Given the functions $p(x)$ and the series $y_1(x)$, compute the first three nonzero terms of the expression

$$\int^x \frac{\exp\left[-\int^r p(s)\,ds\right]}{y_1(r)^2}\,dr$$

in Exercises 14–18.

14. $p(x) = \dfrac{3}{x}$,

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

15. $p(x) = -\dfrac{2}{x}$,

$$y_1(x) = 1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{16}x^3 + \cdots$$

16. $p(x) = \dfrac{1}{2x}$, $y_1(x) = 1 + x + x^2 + x^3 + \cdots$

17. $p(x) = \dfrac{1}{x}$, $y_1(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{9} - \cdots$

18. $p(x) = -\dfrac{1}{3x}$, $y_1(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$

The equations of Exercises 19–24 have a regular singular point at $x = 0$ and are examples of the logarithmic case. Find the recursion relation and the first four nonzero terms of the series expansion of the solution corresponding to the root r_1. If possible, find a general expression for this solution. Then use the method of reduction of order to find the first few terms of the second linearly independent solution.

19. $xy'' + y' - xy = 0$

20. $x^2y'' + xy' + xy = 0$

21. $x^2y'' + xy' + (x - 1)y = 0$

22. $xy'' + y = 0$

23. $x^2y'' - 3xy' + (4x + 4)y = 0$

24. $x^2y'' + 5xy' + (3 - x^2)y = 0$

6.5 SOME SPECIAL FUNCTIONS AND TOOLS OF THE TRADE

THE GAMMA FUNCTION

This function arises so frequently in series representations of solutions that it is worth discussing briefly. The *Gamma function,* $\Gamma(x)$, is defined by the definite integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt,$$

which converges for all *positive* x. One sees immediately that $\Gamma(1) = 1$, and a simple integration by parts shows that

$$\Gamma(x + 1) = \int_0^\infty e^{-t}t^x\,dt = -e^{-t}t^x\Big|_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\,dt = 0 + x\Gamma(x).$$

The *fundamental identity* $\Gamma(x + 1) = x\,\Gamma(x)$ implies, in particular, that for $x = n$, a positive integer, the following holds:

$$\begin{aligned}\Gamma(n + 1) &= n\Gamma(n) = n(n - 1)\Gamma(n - 1)\\ &= \cdots = n(n - 1)\cdots 2 \cdot 1 = n!\end{aligned}$$

since $\Gamma(1) = 1$. Therefore the Gamma function extends the definition of the factorial $x!$, previously defined only for $x = 1, 2, \ldots$, to a function $\Gamma(x)$ that is defined for all values of $x > 0$ and that agrees with $n!$ when $x = n + 1$. Its importance will become clearer in the next section when we discuss the Bessel function.

The fundamental identity also implies that one needs to know only the values of $\Gamma(x + 1)$ for $0 \le x \le 1$ to be able to compute $\Gamma(x)$ for any $x > 0$. For if $x > 2$, one can always find a positive integer n and real number α, $0 \le \alpha \le 1$, so that $x = n + \alpha + 1$. Then by successively using the fundamental identity we obtain

$$\begin{aligned}\Gamma(x) &= \Gamma(n + \alpha + 1) = (n + \alpha)\Gamma(n + \alpha)\\ &= (n + \alpha)(n + \alpha - 1)\Gamma(n + \alpha - 1) = \cdots \qquad \textbf{(6.5.1)}\\ &= (n + \alpha)(n + \alpha - 1)\cdots(1 + \alpha)\Gamma(1 + \alpha);\end{aligned}$$

hence computing $\Gamma(x)$ becomes a matter of knowing $\Gamma(1 + \alpha)$ from a table, then performing a series of multiplications. Note that $\Gamma(x) = \Gamma(x + 1)/x$ and $\Gamma(1) = 1$ implies that $\Gamma(x) \to \infty$ as $x \to 0^+$.

EXAMPLE 1 Find $\Gamma(3.77)$.

SOLUTION

$$\Gamma(3.77) = 2.77\,\Gamma(2.77) = (2.77)(1.77)\,\Gamma(1.77)$$
$$\cong (2.77)(1.77)(0.92376) = 4.52910,$$

where the approximate value of $\Gamma(1.77)$ was obtained from a table. Extensive tables of $\Gamma(x)$ can be found, for instance, in [11]. ■

Observe that by using the fundamental identity in reverse, $\Gamma(x)$ can be defined for $x < 0$, where x is not a negative integer. For instance, to find $\Gamma(-\frac{4}{3})$, we write

$$\left(-\frac{4}{3}\right)\Gamma\left(-\frac{4}{3}\right) = \Gamma\left(-\frac{4}{3} + 1\right) = \Gamma\left(-\frac{1}{3}\right),$$
$$\left(-\frac{1}{3}\right)\Gamma\left(-\frac{1}{3}\right) = \Gamma\left(-\frac{1}{3} + 1\right) = \Gamma\left(\frac{2}{3}\right),$$

and

$$\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{5}{3}\right).$$

All of this implies that

$$\Gamma\left(-\frac{4}{3}\right) = \left(-\frac{3}{4}\right)\left(-3\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{3}\right) = \frac{27}{8}\Gamma\left(\frac{5}{3}\right),$$

and the last value can be obtained from tables.

Finally we note the useful and important fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This result is often proved in advanced calculus texts.

THE BESSEL FUNCTION

A differential equation that arises frequently in boundary value problems and problems in mathematical physics is the *Bessel equation:*

$$y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0, \qquad \textbf{(6.5.2)}$$

where α is a given parameter. For this discussion, α will be assumed to be a real number. The solutions of (6.5.2) occur in many types of potential problems involving cylindrical boundaries, as well as in such areas as elasticity theory, fluid mechanics, and electromagnetic field theory.

The Bessel equation is so important that literally hundreds of volumes of tables of its solutions, their derivatives, and their zeros for both integer and fractional values of α have been published. For instance, in the late 1940s the Harvard Computation Laboratory published 12 volumes, each of about 650 pages, giving values of solutions to (6.5.2) for $\alpha = 0, 1, 2, \ldots, 135$. This was done by means of computers that by today's standards would be called prehistoric; today many of those tables have been replaced by stored computational packages and subroutines.

One sees immediately that $x = 0$ is a regular singular point of the Bessel equation and that $p(x) = 1$, $q(x) = -\alpha^2 + x^2$. Hence $p_0 = 1$, $q_0 = -\alpha^2$, and its indicial equation is

$$r^2 + (1 - 1)r - \alpha^2 = r^2 - \alpha^2 = 0.$$

Therefore its roots are $r_1 = \alpha$ and $r_2 = -\alpha$ and $r_1 - r_2 = 2\alpha$, which means there are two distinct cases:

1. $\alpha \neq 0$ and not a positive integer. Then 2α is not zero and not a positive integer, and there are two linearly independent solutions of the form

$$J_\alpha(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n, \qquad J_{-\alpha}(x) = x^{-\alpha} \sum_{n=0}^{\infty} b_n x^n$$

   called *Bessel functions of fractional or nonintegral order* α *and* $-\alpha$.
2. $\alpha = 0$ or a positive integer, say $\alpha = m$. In this case one solution is of the form

$$J_m(x) = x^m \sum_{n=0}^{\infty} a_n x^n;$$

   it is called the *Bessel function of order m*. The second linearly independent solution is of the logarithmic type and is denoted by $Y_m(x)$, the *Bessel function of the second kind of order m*.

In the earlier sections of this chapter we introduced $J_0(x)$. In what follows we will develop the series representation of $J_\alpha(x)$, $\mathrm{Re}(\alpha) \geq 0$, and list some of its important properties.

Write the Bessel equations as

$$x^2y'' + xy' + (-\alpha^2 + x^2)y = 0,$$

let $y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$, and substitute the series to obtain

$$x^2 \sum_{n=0}^{\infty} [(n+\alpha)(n+\alpha-1)]a_n x^{n+\alpha-2} + x\sum_{n=0}^{\infty} (n+\alpha)a_n x^{n+\alpha-1} - \alpha^2 \sum_{n=0}^{\infty} a_n x^{n+\alpha} + x^2 \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0.$$

The first three series can be combined to get

$$\sum_{n=0}^{\infty} [(n+\alpha)^2 - \alpha^2]a_n x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0.$$

Note that in the first series the term corresponding to $n = 0$ is zero. Therefore write down the term for $n = 1$ and reindex to obtain

$$[(1+\alpha)^2 - \alpha^2]a_1 x^{1+\alpha} + \sum_{n=0}^{\infty} \{[(n+2+\alpha)^2 - \alpha^2]a_{n+2} + a_n\}x^{n+\alpha+2} = 0.$$

Since $\alpha \geq 0$, this implies that $a_1 = 0$ and that

$$a_{n+2} = -\frac{a_n}{(n+2+\alpha)^2 - \alpha^2} = -\frac{a_n}{(n+2)(n+2+2\alpha)}, \quad n = 0, 1, \ldots.$$

Hence $a_1 = a_3 = \cdots = a_{2n+1} = 0$ and

$$a_2 = -\frac{a_0}{2(2+2\alpha)} = -\frac{a_0}{2^2(1+\alpha)},$$

$$a_4 = -\frac{a_2}{4(4+2\alpha)} = \frac{a_0}{2^4 \cdot 2(2+\alpha)(1+\alpha)},$$

$$a_6 = -\frac{a_4}{6(6+2\alpha)} = -\frac{a_0}{2^6 \cdot 3 \cdot 2(3+\alpha)(2+\alpha)(1+\alpha)}, \ldots,$$

$$a_{2n} = (-1)^n \frac{a_0}{2^{2n} n!(n+\alpha)(n-1+\alpha)\cdots(1+\alpha)}, \quad n = 1, 2, \ldots.$$

We conclude that

$$J_\alpha(x) = a_0 x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(n+\alpha)(n-1+\alpha)\cdots(1+\alpha)} x^{2n}.$$

We note that the series can be written in powers of $x/2$, and if we examine the denominator of the general term and compare it to the expression (6.5.1) for the

Gamma function, we see that considerable simplification is obtained by letting $a_0 = [2^\alpha\Gamma(\alpha + 1)]^{-1}$. We obtain the compact form

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n}.$$

In comparing this expression with the previous one we can see the notational convenience of using the Gamma function. We now examine the various possibilities for different values of α.

Case 1. Since $\alpha = m$, where $m = 0$ or m is a positive integer, we have

$$\Gamma(n + m + 1) = (n + m)!,$$

and we obtain the *Bessel function of order m:*

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n + m)!} \left(\frac{x}{2}\right)^n.$$

From the series it follows

$$J_0(0) = 1, \quad J_m(0) = 0, \quad m = 1, 2, \ldots.$$

A more extensive analysis shows that $J_m(x)$ is an oscillatory function that approaches zero as $x \to \infty$, very similar to a damped cosine or sine function. For instance,

$$J_0(x_j) = 0 \quad \text{for} \quad x_j \cong 2.405, 5.520, 8.654, 11.97, \ldots$$

$$J_1(x_j) = 0 \quad \text{for} \quad x_j \cong 3.832, 7.016, 10.17, 13.32, \ldots$$

$$J_2(x_j) = 0 \quad \text{for} \quad x_j \cong 5.136, 8.417, 11.62, 14.80, \ldots.$$

A graph of $J_m(x)$ for $m = 0, 1, 2$ is shown in Fig. 6.1; these three Bessel functions occur often in applications. Tables of the Bessel functions and their zeros can be found, for instance, in [11].

As mentioned above, when $\alpha = m$, $m = 0$ or a positive integer, the second linearly independent solution of the Bessel equation is $Y_m(x)$, the *Bessel function of the second kind of order m,* and it is of logarithmic type. Its series representation is very complicated, but it is also an oscillatory function which, however, becomes unbounded as $x \to 0$. A graph of $Y_0(x)$ and $Y_1(x)$ is shown in Fig. 6.2.

There are many interesting properties and identities for the Bessel functions that we do not have space to discuss; the reader could browse through [5] or [8]

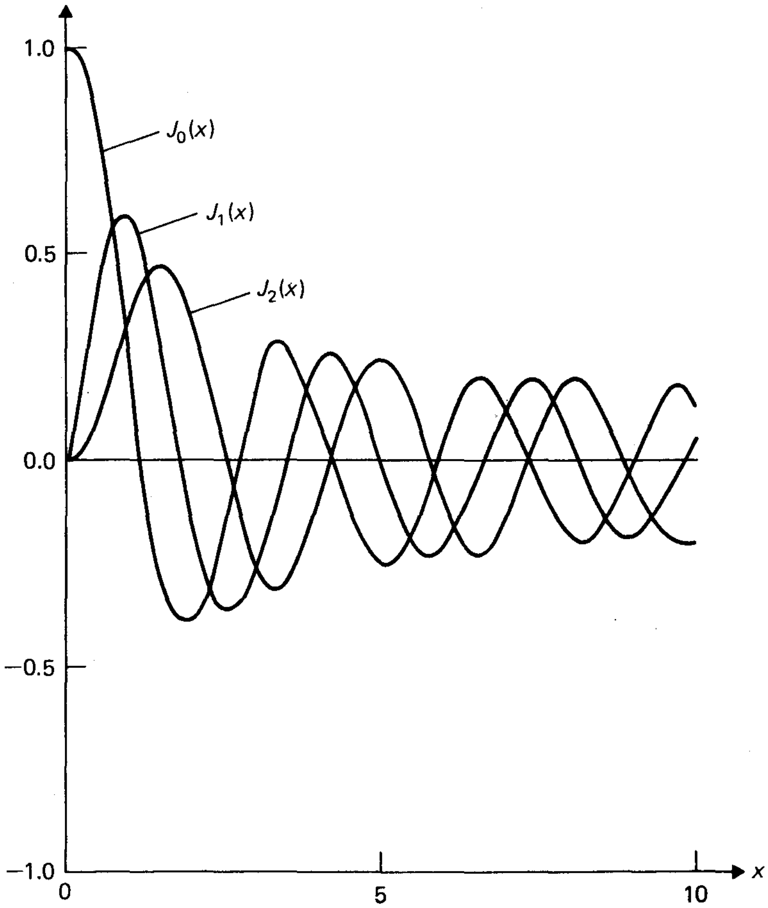

Figure 6.1 Graphs of $J_0(x)$, $J_1(x)$, and $J_2(x)$.

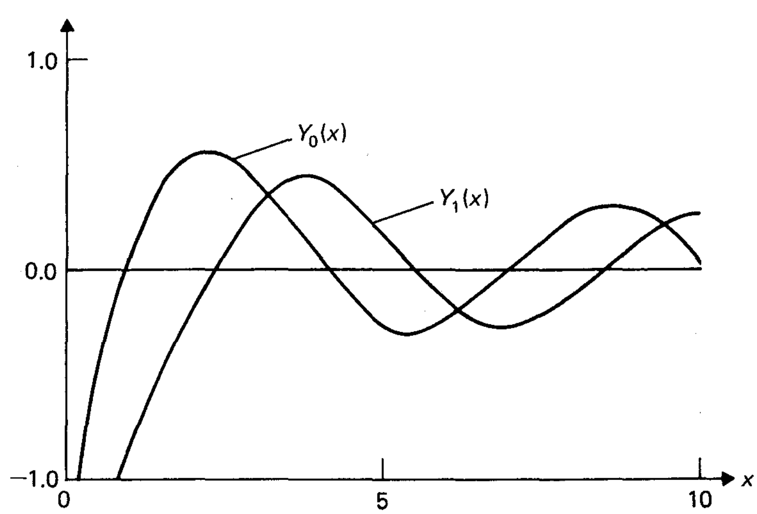

Figure 6.2 Graphs of $Y_0(x)$ and $Y_1(x)$.

to see the vast literature devoted to Bessel functions $J_m(x)$ and $Y_m(x)$. For this discussion we only need to state that:

When $\alpha = m$, $m = 0$ or a positive integer, a general solution of the Bessel equation (6.5.2) is

$$y(x) = aJ_m(x) + bY_m(x),$$

where a and b are arbitrary constants.

Case 2. When $\alpha \neq 0$ or a positive integer, the theory of regular singular points tells us that the two *Bessel functions of fractional order* α *and* $-\alpha$,

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n}$$

and

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - \alpha + 1)} \left(\frac{x}{2}\right)^{2n},$$

are a fundamental pair of solutions of the Bessel equation (6.5.2). Hence:

When $\alpha \neq 0$ or a positive integer, the general solution of the Bessel equation (6.5.2) is

$$y(x) = aJ_\alpha(x) + bJ_{-\alpha}(x),$$

where a and b are arbitrary constants.

The function $J_\alpha(x)$ is oscillatory and approaches zero as $x \to \infty$, and for large x, $J_\alpha(x)$ looks very much like a damped sine wave.

Finally, a frequently useful and time-saving fact, which allows us to express many solutions of equations with regular singular points at $x = 0$ in terms of Bessel functions, is the following.

Given an equation with a regular singular point at $x = 0$ in the form

$$x^2y'' + (1 - 2s)xy' + [(s^2 - r^2\alpha^2) + a^2r^2x^{2r}]y = 0, \qquad \textbf{(6.5.3)}$$

where s, r, a, and α are given constants, every solution can be written in the form

$$y(x) = c_1 x^s J_\alpha(ax^r) + c_2 x^s J_{-\alpha}(ax^r),$$

where c_1 and c_2 are arbitrary constants. If $\alpha = 0$ or a positive integer, then we replace $J_{-\alpha}(ax^r)$ with $Y_\alpha(ax^r)$.

This result is obtained by using a change of variable, and the verification is left to the reader in Exercise 9 (p. 465).

EXAMPLE 2

a) In the previous section on regular singular points, the series expression for one solution of

$$x^2y'' + 3xy' + (1 + x)y = 0$$

and the first few terms of the second solution (of logarithmic type) were developed. By comparison with the differential equation (6.5.3), we find:

$$1 - 2s = 3, \quad s^2 - r^2\alpha^2 = 1, \quad a^2r^2 = 1, \quad 2r = 1,$$

which, when solved, gives $r = \frac{1}{2}$, $a = 2$, $s = -1$, $\alpha = 0$. Therefore a fundamental pair of solutions is

$$y_1(x) = x^{-1}J_0(2x^{1/2}), \quad y_2(x) = x^{-1}Y_0(2x^{1/2}).$$

b) Let $s = \frac{1}{2}$, $r = a = 1$ in (6.5.3) to obtain

$$x^2y'' + [(\tfrac{1}{4} - \alpha^2) + x^2]y = 0$$

and the above fact tells us that a fundamental pair of solutions are $x^{1/2}J_\alpha(x)$ and either $x^{1/2}J_{-\alpha}(x)$ or $x^{1/2}Y_\alpha(x)$. However, if $\alpha = \frac{1}{2}$, the above equation, after canceling x^2, becomes $y'' + y = 0$, whose fundamental pair of solutions are $\sin x$ and $\cos x$. This implies that $J_{1/2}(x)$ and $J_{-1/2}(x)$ can be expressed in terms of $x^{-1/2}\sin x$ and $x^{-1/2}\cos x$. In fact it can be shown that (see Exercise 7, p. 464)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

so $J_{1/2}(x)$ and $J_{-1/2}(x)$ can be computed without the use of special tables. ■

The above examples show the advantage of the general form (6.5.3), namely, that the solutions of a large class of equations with a regular singular point at $x = 0$ can be expressed in terms of well-tabulated functions. Hence, the computation of complicated series solutions is avoided, which is a real benefit!

THE AIRY FUNCTIONS

The *Airy functions* Ai(x), Bi(x) are the fundamental pair of solutions of *Airy's equation*

$$y'' - xy = 0, \qquad \textbf{(6.5.4)}$$

as was shown before. Similarly, Ai($-x$), Bi($-x$) denote the fundamental pair of solutions of

$$y'' + xy = 0, \qquad \textbf{(6.5.5)}$$

and both pairs of functions are tabulated, for example, in [10]. By writing (6.5.5) in the form

$$x^2y'' + x^3y = 0$$

we see that it is in the general form (6.5.3) with $1 - 2s = 0$, $s^2 - r^2\alpha^2 = 0$, $a^2r^2 = 1$, and $2r = 3$. The solution of these equations is $r = \frac{3}{2}$, $s = \frac{1}{2}$, $a = \frac{2}{3}$, and $\alpha = \frac{1}{3}$; therefore Ai($-x$) and Bi($-x$) can be written as linear combinations of

$$\sqrt{x}\,J_{1/3}\left(\frac{2}{3}\,x^{3/2}\right) \quad \text{and} \quad \sqrt{x}\,J_{-1/3}\left(\frac{2}{3}\,x^{3/2}\right).$$

In fact, for $x > 0$,

$$\text{Ai}(-x) = \frac{1}{3}\sqrt{x}\left[J_{1/3}\left(\frac{2}{3}\,x^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\,x^{3/2}\right)\right],$$

$$\text{Bi}(-x) = -\sqrt{\frac{x}{3}}\left[J_{1/3}\left(\frac{2}{3}\,x^{3/2}\right) - J_{-1/3}\left(\frac{2}{3}\,x^{3/2}\right)\right],$$

and again we see how ubiquitous are the Bessel functions!

THE LEGENDRE POLYNOMIALS

The differential equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \qquad \textbf{(6.5.6)}$$

with λ a given parameter, is called *Legendre's equation;* it occurs frequently in the analysis of potential problems on spherical domains. One sees that $x = 0$ is an ordinary point since

$$p(x) = \frac{-2x}{1 - x^2}, \qquad q(x) = \frac{\lambda}{1 - x^2}$$

clearly are smooth functions for $|x| < 1$. From the equation in the form

$$y'' + \frac{1}{x-1}\frac{2x}{1+x}y' + \frac{1}{(x-1)^2}\frac{1-x}{1+x}\lambda y = 0, \qquad \textbf{(6.5.7)}$$

it follows that $x = 1$ is a regular singular point (and so is $x = -1$ by a similar rewriting).

For the regular singular point at $x = 1$ it is seen from (6.5.7) that

$$p(x) = \frac{2x}{1+x} \quad \text{and} \quad q(x) = \left(\frac{1-x}{1+x}\right)\lambda.$$

Therefore $p(1) = 1$, $q(1) = 0$, and the indicial equation is

$$r^2 + (1-1)r + 0 = r^2 = 0$$

with roots $r_1 = r_2 = 0$. This is the logarithmic case, and two linearly independent solutions valid in a deleted neighborhood of $x = 1$ are of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x-1)^n, \qquad y_2(x) = y_1(x)\beta \ln|x-1| + \sum_{n=0}^{\infty} b_n(x-1)^n.$$

A similar analysis can be made for $x = -1$, which is also a logarithmic case. The interesting case occurs when we examine the ordinary point $x = 0$.

Since $x = 0$ is an ordinary point, both linearly independent solutions have a series representation $y(x) = \Sigma_{n=0}^{\infty} a_n x^n$. Differentiating the series term by term and substituting it in (6.5.6) gives

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} [-n(n-1) - 2n + \lambda]a_n x^n = 0.$$

Now shift the index by two in the first sum and combine terms to obtain

$$\sum_{n=0}^{\infty} \{(n+2)(n+1)a_{n+2} + [-n(n+1) + \lambda]a_n\}x^n = 0,$$

which leads to the recurrence relation

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n, \quad n = 0, 1, 2, \ldots. \qquad \textbf{(6.5.8)}$$

It is seen that $a_2, a_4, \ldots, a_{2n} \ldots$ depend on a_0, while $a_3, a_5, \ldots, a_{2n+1}, \ldots$ depend on a_1, with a_0, a_1 arbitrary. This leads to two series, one in even powers of x and the other in odd powers of x, which are a linearly independent pair of solutions of Legendre's equation in a neighborhood of $x = 0$.

However, the interesting case in many applications is when the parameter $\lambda = m(m + 1)$ for some nonnegative integer m, in which case the relation (6.5.8) tells us that:

1. If m is even, then $a_2, a_4, \ldots, a_m$ will not be zero, but $a_{m+2}, a_{m+4}, \ldots$ will all be zero. Therefore one solution will be an even polynomial $P_m(x)$ of degree m and the other solution will be a power series in odd powers of x.
2. If m is odd, then $a_3, a_5, \ldots, a_m$ will not be zero, but $a_{m+2}, a_{m+4}, \ldots$ will all be zero. Therefore one solution will be an odd polynomial $P_m(x)$ of degree m and the other solution will be a power series in even powers of x.

The polynomial solutions $P_m(x)$ described above are called the *Legendre polynomials* corresponding to $\lambda = m(m + 1)$. They are extremely important because they are the *only* solutions of Legendre's equation that are defined for $x = 0$ and are bounded at $x = \pm 1$ (the power series solutions diverge at $x = \pm 1$ like $\ln(1 \mp x)$).

The recursion relation (6.5.8) may be used to construct some Legendre polynomials, for instance,

1. If $\lambda = 20 = 4 \cdot 5$, then $m = 4$ and

$$a_{0+2} = a_2 = \frac{(0)(1) - 20}{(0 + 2)(0 + 1)} a_0 = -10a_0,$$

$$a_{2+2} = a_4 = \frac{(2)(3) - 20}{(2 + 2)(2 + 1)} a_2 = \frac{-14}{12}(-10a_0) = \frac{35}{3} a_0,$$

hence $P_4(x) = a_0(1 - 10x^2 + \frac{35}{3}x^4)$. It is conventional to choose a_0 so $P_4(1) = 1$; therefore

$$P_4(x) = \frac{3}{8}(1 - 10x^2 + \frac{35}{3}x^4).$$

2. If $\lambda = 30 = 5 \cdot 6$, then $m = 5$ and

$$a_{1+2} = a_3 = \frac{(1)(2) - 30}{(1 + 2)(1 + 1)} a_1 = -\frac{14}{3} a_1,$$

$$a_{3+2} = a_5 = \frac{(3)(4) - 30}{(3 + 2)(3 + 1)} a_3 = \frac{-18}{20}\frac{-14}{3} a_1 = \frac{21}{5} a_1,$$

and choosing a_1 so that $P_5(1) = 1$, we have

$$P_5(x) = \frac{15}{8}(x - \frac{14}{3}x^3 + \frac{21}{5}x^5).$$

The reader may check that

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{3}{2}\left(\frac{5}{3}x^3 - x\right),$$

and for a further check, may wish to use the *Rodrigues' formula,*

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2 - 1)^m.$$

This formula generates all the Legendre polynomials by successive differentiation.

Finally, we remark that the Legendre polynomials are an example of a *family of orthogonal polynomials on* $-1 \leq x \leq 1$. By this we mean that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \quad \text{if} \quad n \neq m.$$

This fact is extremely useful in boundary value problems and in approximation theory.

EXERCISES 6.5

1. Given
   a) $\Gamma(1.185) = 0.92229$; evaluate $\Gamma(5.185)$ and $\Gamma(-3.815)$.
   b) $\Gamma(1.910) = 0.96523$; evaluate $\Gamma(4.1910)$ and $\Gamma(-2.090)$.
2. Stirling's formula for an asymptotic approximation of $n! = \Gamma(n + 1)$ is

$$\Gamma(n+1) \approx n^n e^{-n}\sqrt{2\pi n},$$

meaning that

$$\lim_{n\to\infty} \frac{\Gamma(n+1)}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

This means the approximation is good as n gets large, in the sense of a small relative error *not* a small absolute error. Use Stirling's formula to find an approximation of $100! \cong 9.3326 \times 10^{157}$ and compute the relative and absolute errors. (*Hint:* For computing, use the logarithmic form

$$\ln \Gamma(n+1) \approx \left(n + \frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln 2\pi.)$$

3. The Maclaurin series for e^{-x} is

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n;$$

and since it is an alternating series, the error in stopping at N terms is less than the magnitude of the $(N + 1)$st term.
   a) Use Stirling's formula to approximate the error in estimating e^{-10} with the first 20, 25, and 30 terms of the series.

b) How many terms would be needed to compute e^{-20} with an error of less than 10^{-10}?

One can infer from the above that for computational purposes, power series can be very inefficient!

4. By differentiating the series for the Bessel function term by term, show that

$$xJ_p'(x) = pJ_p(x) - xJ_{p+1}(x),$$
$$xJ_p'(x) = -pJ_p(x) + xJ_{p-1}(x).$$

5. The two relations in Exercise 4 imply the important recursion relation

$$J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x),$$

which allows one to compute higher order Bessel functions in terms of lower order ones. Use it and the expressions given on p. 459 for $J_{1/2}(x)$ and $J_{-1/2}(x)$ to obtain the values of

a) $J_{3/2}(1.76)$,
b) $J_{-3/2}(0.587)$,
c) $J_{5/2}(6.78)$.

6. Use the substitution $y = u/\sqrt{x}$ directly in Bessel's equation to obtain the differential equation

$$u'' + \left(1 - \frac{\alpha^2 - \frac{1}{4}}{x^2}\right)u = 0$$

with solutions $x^{1/2}J_\alpha(x)$.

7. Using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, show directly from the series representation for $J_{1/2}(x)$ and $J_{-1/2}(x)$ that

$$\sin x = \sqrt{\frac{\pi x}{2}}J_{1/2}(x), \qquad \cos x = \sqrt{\frac{\pi x}{2}}J_{-1/2}(x).$$

(You will need the identity 6.5.1.)

8. The integral representation of $J_n(x)$,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta, \qquad n = 0, 1, 2, \ldots,$$

was obtained by F. W. Bessel (1784–1846) in his study of astronomical orbits.

a) Use it to show that $J_{-n}(x) = (-1)^n J_n(x)$ and $J_0'(x) = -J_1(x)$.
b) Use it and a quadrature formula (e.g., Simpson's rule) to show that 2.405 is an approximate zero of $J_0(x)$.
c) Similarly, show that 3.833 is an approximate zero of $J_1(x)$.

9. Given the Bessel equation

$$z^2w'' + zw' + (z^2 - \alpha^2)w = 0,$$

let $z = ax^r$, $y = x^s w$ and obtain the differential equation (6.5.3).

Express the solutions of the differential equations in Exercises 10–14 in terms of Bessel functions by using (6.5.3).

10. $x^2y'' + 5xy' + (3 + 4x^2)y = 0$
11. $y'' + 4xy = 0$
12. $x^2y'' + 3xy' + (1 + x)y = 0$
13. $2x^2y'' - xy' + (1 + x^2)y = 0$
14. $xy'' + y = 0$
15. Using Rodrigues' formula directly,
    a) find $P_2(x)$, $P_3(x)$, and $P_4(x)$;
    b) show that $\int_{-1}^{1} x^r P_m(x)\, dx = 0$, $r = 0, 1, \ldots, m - 1$. (*Hint:* Use successive integration by parts.)
16. Show that the Legendre equation can be written in the form

$$\frac{d}{dx}[(1 - x^2)y'] = -\lambda y$$

and use this to show the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0, \quad n \neq m.$$

(*Hint:* Let $\lambda = m(m + 1)$ and $y(x) = P_m(x)$ in the differential equation, then multiply both sides by $P_n(x)$ and integrate on $-1 \leq x \leq 1$ by parts. Repeat the process with $\lambda = n(n + 1)$, etc.)

17. The function $y(\theta) = P_n(\cos\theta)$, $0 < \theta < \pi$, where $P_n(x)$ is a Legendre polynomial, arises in potential theory for a spherical body. Show that $y(\theta)$ satisfies

$$y'' + (\cot\theta)y' + n(n + 1)y = 0.$$

18. The Chebyshev equation is

$$(1 - x^2)y'' - xy' + \lambda^2 y = 0.$$

    a) Show that $x = \pm 1$ are regular singular points, and determine the nature of the two linearly independent solutions valid near them.
    b) Show that $x = 0$ is an ordinary point and that if $\lambda^2 = m^2$, m an integer, one of the linearly independent solutions valid near $x = 0$ is a polynomial $T_m(x)$. Find $T_m(x)$, $m = 0, 1, 2, 3, 4$.

19. The Hermite equation is

$$y'' - 2xy' + 2\lambda y = 0.$$

Show that $x = 0$ is an ordinary point and that if $\lambda = m$, m a nonnegative integer, then one of the linearly independent solutions is a polynomial $H_m(x)$. Find $H_m(x)$, $m = 0, 1, 2, 3, 4$.

SUMMARY

The problem of finding two linearly independent solutions or a general solution of the second order linear differential equation

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = 0$$

when the coefficients $a(t)$, $b(t)$, and $c(t)$ are constants is a simple algebraic task. However, when they are not, there is no general technique for finding solutions. But for a large class of equations, a method wherein the solution is represented by a power series can be employed.

The first case discussed is where $a(t)$, $b(t)$, and $c(t)$ are smooth, well-behaved functions in the neighborhood of some point t_0, and $a(t_0) \neq 0$. Then any solution $y(t)$ can be represented by a power series $y(t) = \Sigma_0^\infty a_n(t - t_0)^n$ that is substituted into the differential equation, and the coefficients a_n are found term by term via a recursion relation.

The second case discussed is where $a(t) = (t - t_0)^2$, $b(t) = (t - t_0)p(t)$, and $c(t) = q(t)$ where $p(t)$ and $q(t)$ are smooth well-behaved functions in a neighborhood of $t = t_0$. In this case a solution can be represented by an infinite series

$$y(t) = (t - t_0)^r \sum_0^\infty a_n (t - t_0)^n$$

where the value of r is determined by a quadratic equation. This series can be substituted into the differential equation and the coefficients a_n determined recursively. A second linearly independent solution can be found, but its series form depends on the nature of the roots of the quadratic equation.

Many of the special functions of mathematical physics, such as the Bessel functions or the Legendre polynomials, arise as series solutions of second order differential equations of the type described above. The chapter concludes with a brief description of some of these functions.

MISCELLANEOUS EXERCISES

6.1. A solution $y(x)$ of an ordinary differential equation is said to be *oscillatory* if it vanishes infinitely often on a half line $x_0 \leq x < \infty$. The following comparison theorem is useful in determining whether solutions are oscillatory.

Given the two second order linear equations

$$y'' + p(x)y = 0 \qquad (1)$$

$$y'' + q(x)y = 0 \qquad (2)$$

with $q(x) \geq p(x)$ for $x \geq x_0$. If the solutions of (1) are oscillatory, then so are those of (2), and the zeros of the solutions of (2) are closer together than those of the solutions of (1).

Use the theorem to study the Airy equation

$$y'' + xy = 0, \quad x \geq 1,$$

and by comparison with equations of the form $y'' + k^2y = 0$ show that its solutions are oscillatory. Furthermore show that:

a) The zeros of its solutions are less than π units apart.
b) The distance between successive zeros approaches zero as x approaches ∞.
c) There are either 1 or 2 zeros in the interval $1 \leq x \leq 4$. (A check with tables shows that $\mathrm{Ai}(-x)$ vanishes once and $\mathrm{Bi}(-x)$ vanishes twice in the interval.)

6.2. By transforming the Bessel equation to the form

$$y'' + \left[1 - \frac{\alpha^2 - \frac{1}{4}}{x^2}\right] y = 0$$

(whose solutions are $x^{1/2}J_\alpha(x)$ and $x^{1/2}J_{-\alpha}(x)$ or $x^{1/2}Y_\alpha(x)$), use the comparison theorem above to show that:

a) The Bessel functions are oscillatory.
b) The zeros of $J_\alpha(x)$ are separated by more than π units if $\alpha^2 < \frac{1}{4}$.

6.3. Show that $x = 0$ is a regular singular point of the differential equation

$$x^2y'' + (\sin x)y' - (\cos x)y = 0,$$

and that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. By expanding $\sin x$ and $\cos x$ in their Taylor series at $x = 0$ and by retaining enough terms, determine the first three nonzero terms of the series solution corresponding to r_1.

6.4. In the previous problem use the method of reduction of order to determine whether the second solution corresponding to $r_2 = -1$ is of logarithmic type or not.

6.5. Find the series solution of Laguerre's equation

$$xy'' + (1 - x)y' + \lambda y = 0$$

near the regular singular point $x_0 = 0$. Show that the solution reduces to a polynomial $L_m(x)$, called a Laguerre polynomial, if $\lambda = m$, a positive integer. Find $L_1(x)$, $L_2(x)$, and $L_3(x)$.

6.6. The Fourier–Legendre series expansion of a function $f(x)$, $-1 < x < 1$, is given by

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx.$$

It will converge to $f(x)$, for instance, for any function $f(x)$ that is continuous and has a piecewise continuous first derivative.

a) Using the orthogonality relation (see Exercise 16 of Section 6.5) and the fact that $\int_{-1}^{1} P_n(x)^2\, dx = 2/(2n + 1)$, obtain the above formula for a_n.

b) Show that if $f(x)$ is an even (odd) function, only even (odd) indexed terms will appear in the series.

c) Use the formula above to compute the first three nonzero terms of the Fourier–Legendre series for $f(x) = |x|$, $-1 < x < 1$. Graph your result.

d) Do the same as in (c) for $f(x) = e^x$, $-1 < x < 1$.

6.7. Find the solution of the boundary value problem

$$(1 - x)^2 y'' - 2xy' + 12y = 0, \qquad y(0) = 0, \qquad y(\tfrac{1}{2}) = 4.$$

6.8. Find the solution of the boundary value problem

$$y'' + \frac{1}{x} y' + \frac{x^2 - (1/4)}{x^2} y = 0, \qquad y(0) \text{ bounded}, \qquad y\left(\frac{\pi}{2}\right) = 1.$$

6.9. Using (6.5.3), determine for what values of α the boundary value problem

$$x^2 y'' - 3xy' + \left(\frac{15}{4} + x^2\right) y = 0, \qquad y\left(\frac{\pi}{2}\right) = 0, \qquad y(\alpha\pi) = 0,$$

will have nontrivial solutions or only the zero solution.

6.10. Derive the formula for the Laplace transform of t^α, where α is not necessarily an integer:

$$L\{t^\alpha\} = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}; \quad \alpha > -1, \quad \text{Re}(s) > 0.$$

REFERENCES

A majority of books on ordinary differential equations discuss series solutions and special functions. For a more detailed discussion than is given here we could suggest

1. W.E. Boyce and R.C. DiPrima, *Elementary Differential Equations,* 3rd ed., Wiley, New York, 1977.
2. E.A. Coddington, *An Introduction to Ordinary Differential Equations,* Prentice-Hall, Englewood Cliffs, N.J., 1961.

3. E.D. Rainville, *Intermediate Differential Equations,* 2nd ed., Macmillan, New York, 1964.
4. G.F. Simmons, *Differential Equations with Applications and Historical Notes,* McGraw-Hill, New York, 1972.

To get an idea of the depth of analysis devoted to special functions at the turn of this century, we recommend browsing through these two classics:

5. G.N. Watson, *A Treatise on the Theory of Bessel Functions,* 2nd ed., Cambridge University Press, Cambridge, 1944.
6. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis,* 4th ed., Cambridge University Press, Cambridge, 1965.

For a more accessible discussion of special functions we suggest

7. H. Hochstadt, *Special Functions of Mathematical Physics,* Holt, New York, 1961.
8. L.C. Andrews, *Special Functions for Engineers and Applied Mathematicians,* Macmillan, London, 1985.
9. H. Hochstadt, *The Functions of Mathematical Physics,* Wiley Interscience, New York, 1971.
10. F.W.J. Olver, *Introduction to Asymptotics and Special Functions,* Academic Press, New York, 1974.

For a book of tables and formulas of special functions the standby is

11. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions,* Applied Mathematics Series 55, National Bureau of Standards, Washington, D.C., 1964. (Also available from Dover Publications.)