Introduction to Differential Equations

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in collaboration with

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Some of the material in this text also appears in *Calculus of Several Variables*, by R. Creighton Buck and Alfred B. Wilcox. Houghton Mifflin, Boston, 1971.

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Contents

Preface

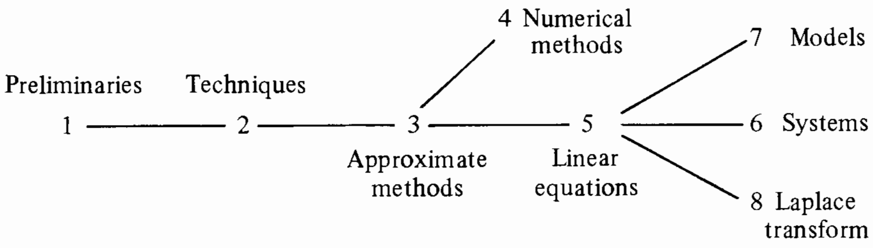
As the title indicates, this text is intended to be used in a first course in ordinary differential equations, following a standard course in elementary calculus. It is based on a one-semester three-hour-a-week course given at the University of Wisconsin; however, the book is adaptable also for a one-quarter course or a two-quarter course.

We have made a conscious effort to present two aspects of differential equations that we believe are of particular importance for today's students, namely modeling and computers. It is obvious that the advent of the programmable computer has had a major impact on trends in applied mathematics. We have therefore introduced computer awareness into our approach to the solution of differential equations without slighting the essential mathematical ideas of the subject and without turning the book into a text on numerical analysis or computing science.

A similar effort has been made to present "modeling" as an essential step in the application of mathematics to a real problem without requiring the reader to have a depth of experience in physics, economics, engineering, biology, and a host of other disciplines. We long ago learned that mathematics does not become easier if it is explained in terms of something else which is even more unfamiliar! Most students, however, find it extremely gratifying to discover that familiar mathematical techniques can be used to produce quantitative answers to practical questions dealing with the behavior of a rocket or to predict the future actions of two competing biological populations.

On the mathematical side, this text is not oriented toward abstract theory and proof, but rather toward concrete techniques and understandings. This does not mean that we have been afraid to bring in some modern concepts when we felt that they were useful or helped to tie ideas together into a more understandable package. We have also provided occasional glimpses of certain more advanced aspects of the subject, trying to convey a correct picture of the facts but omitting complete details and proofs.

The organization of the book is shown in the following chart:



Starred sections can be omitted safely without serious impairment of later material; these include the brief treatment of the existence and uniqueness theorems in Section 3.6, the convergence of the Euler method in Section 4.2, a discussion of more advanced numerical methods in Section 4.4, and Section 5.5 (our favorite!), which contains a comparison of the physical behavior and the mathematical behavior of the damped harmonic oscillator. We have also used the symbol in the margin to warn the reader when the water is getting deep.

If desired, the treatment of linear systems in Chapter 6 can be cut short after Section 6.3. The remainder of the chapter is devoted to the use of matrices and matrix-valued functions, particularly in the solution of systems with constant coefficients. Note that we do not assume previous knowledge of linear algebra; the treatment is self-contained and the small amount of matrix theory that is needed is developed here. (Of course, students who have already been exposed to a course in linear algebra will be able to move through this chapter more rapidly.)

We have included an appendix on Laplace transforms containing the usual applications of this method to the solution of single linear equations and linear systems.

We have included a section on infinite series (Section 3.2) which may be review for many. We also include a short section (Section 3.1) dealing with numerical sequences which is very elementary in nature but which we regard as very important; it contains much that is apt to be unfamiliar to students coming directly from calculus.

The work on numerical solution of differential equations is largely confined to Chapter 4. The emphasis is upon the description of certain standard algorithms and the rationale behind them, not upon computer programming or error analysis. In particular, we do not assume that readers have access to a computer or are able to write programs themselves. In order that students may appreciate better some of the problems that may arise, we have provided vicarious experience with computers by showing the results of certain experiments with specific algorithms and specific equations.

The book contains more than 500 exercises of varying difficulty, weighted toward the easier end rather than the honors level. Some of these carry the symbol PC to indicate that they involve arithmetic calculations that are more efficiently done with an inexpensive pocket calculator.

We also call especial attention to Chapter 7, which contains a variety of separate topics, including a discussion of Kepler's Laws, a treatment of the motion of a vibrating string and a vibrating circular drumhead, and a solution of the brachistochrone problem. Each topic illustrates some aspect of the way in which differential equations have been used to answer an interesting question. Each also serves to point the reader toward other areas of mathematics (for example, Fourier series, calculus of variations). This chapter, together with the list of miscellaneous word problems that follows it, can serve as a basis for open-ended class discussions, group projects, or individualized instruction.

If this text is to be used for a one-quarter course, we recommend that Chapters 1–5 be covered, omitting all starred sections. This can be covered in about 25 class meetings. If there is more time, the first three sections of Chapter 6 should be added, as well as Section 5.5.

In a full semester course, Chapters 1–6 should be covered, together with at least two of the following: Section 4.4, Section 7.1, Section 7.2, Section 7.3, and the appendix on Laplace transforms.

In a two-quarter course, the entire text should be covered.

If students have had some material on differential equations in the calculus course, it may be possible to cover Chapter 2 more rapidly, since portions of it will be review.

The approach we have used in this book was first tried by one of the authors some years ago at Wisconsin in an experimental course, funded in part by the National Science Foundation. However, the viewpoint was first explored while we were teaching ordinary differential equations at Brown University in 1948, and owes much to the frequent discussions with our colleagues George Carrier and J. B. Diaz on the philosophy of applied mathematics.

It is difficult to single out all the individuals whose influence is visible in the book. Much appreciation must certainly go to those with whom we worked for so long, starting in 1961, in connection with the CUPM panels on computing and on applied mathematics; it is evident that we have benefited much from conversations with Henry Pollak, Bob Walker, Bert Colvin, Ben Noble, Carl de Boor, Fred Brauer, John Nohel, and many, many others, both here at Wisconsin and elsewhere. We would also like to thank those who reviewed the manuscript: Philip Crooke, Vanderbilt University; Francis J. Murray, Duke University; Noboru Suzuki, University of California, Irvine; Thomas L. Sherman, Arizona State University; and James F. Hurley, University of Connecticut.

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Introduction to Differential Equations

1 Preliminaries

1.1 Models and Reality

Most people study mathematics because they are interested in using mathematics as a tool in some other discipline. Indeed, one of the rewards for many research mathematicians is seeing how a newly discovered mathematical theory makes it possible to better understand and deal with a portion of the world that was formerly a mystery—for example, the changing structure of the economy, or the strange nature of the electromagnetic winds that sweep across the solar system, or the mixture of human decisions that lies behind the play of a poker game.

In this respect, mathematics seems to have a unique versatility, and many scientists with a philosophical turn of mind have tried to understand why mathematics has proved to be so useful. Without attempting to answer this question, we can describe in very general terms the way a scientist uses mathematics. The key idea is given in the title of this section. One may think of the totality of mathematics thus far discovered or invented as contained in a large warehouse. Each mathematical concept or structure has a definite pattern and a definite collection of associated rules and techniques. A scientist who is interested in studying some aspect of reality comes to the warehouse to find a mathematical system which he thinks will be a model of that aspect of reality. He uses mathematical techniques to explore the behavior of the model and to answer questions about the model which reflect questions about nature.

Perhaps the diagram in Figure 1.1 will help to describe this process. Starting from reality (whatever that may be), we first pass to an idealized situation and then construct (or choose) a mathematical model. Both of these steps can be of great difficulty, and in many cases represent major scientific advances. The model itself can be as simple as a set of algebraic equations or so complex that an entire book is required to describe it. It should be possible to translate each question concerning reality into a parallel question about the model. The next step is to use specialized mathematical techniques to answer the questions about the model. Sometimes the questions are so difficult that no known techniques will answer them, and in this case, new techniques must be found or a new model chosen. When the questions can be answered, one is then ready to translate the answers back into the language of reality and if possible test them by experiment.

As a first illustration, let us reexamine briefly the way in which the basic ideas of calculus arise as models for aspects of the physical concept we call "motion." The first step was taken by Descartes, who realized that the idea of spatial position, or location, could be made concrete by the use of coordinate geometry; the second step was the realization that the subjective experience of time could be modeled by the real numbers. From these steps follows the basic model for the motion of a point along a line—that is, straight-line motion—as a function f from the time axis to one-dimensional space:

$$x = f(t) \tag{1.1}$$

Similarly, the model for the motion of a point in space becomes a function F

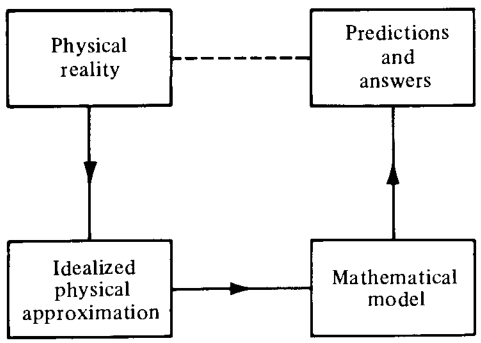

Figure 1.1

from the time axis to three-dimensional space

$$(x, y, z) = F(t) = (f(t), g(t), h(t)) \tag{1.2}$$

which specifies the position of the moving point at each instant of time by giving its coordinates. We note that the path, or track, of the moving point described by (1.2) is a curve in space; thus, as a dividend (1.2) becomes the equation in parametric form for a general space curve.

The simplest motions are the straight-line motions in which the point moves back and forth along a fixed line; the model for this is a single function of t, as in (1.1). Suppose we look for something associated with this model which can be calculated, and which can be connected with the intuitive concept of "speed" or "velocity." This is a familiar situation, and it leads to the standard definition

$$\text{Velocity at time } t = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$
$$= f'(t) = \frac{dx}{dt}$$

Historically, it happened that the same mathematical formula also arose in an entirely different context. In the study of curves, people wanted to know how to construct tangent lines to arbitrary curves, and this led to the study of the graphs of functions and then to the development of a formula for the slope of a curve at an arbitrary point. This turned out to be the same as the preceding formula for $f'(t)$.

It helps to think of three different contexts in which all this is going on: subjective reality (motion), geometry (curves in the plane), and analysis (functions). We can summarize the situation by means of the accompanying table.

Context	*Concept*	*Concept*
Reality	Straight-line motion	Velocity
Geometry	Curve	Slope
Analysis	Function	Derivative

All the concepts represented in the table can be displayed on a single diagram. (See Figure 1.2.) The graph of the function which describes a straight-line motion is a curve in the two-dimensional space-time plane, and the velocity of the moving point at a particular moment of time can be either read off the

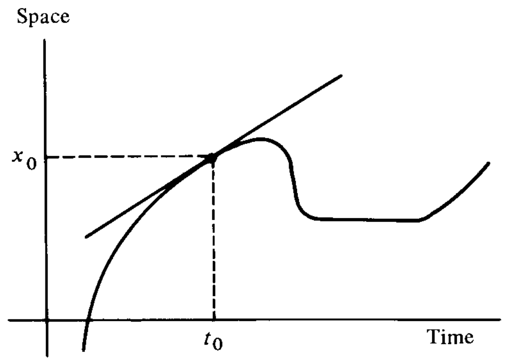

Figure 1.2

diagram as the slope of the curve at the corresponding point, or calculated as the value of the derivative function f' at t.

In the context of mathematical analysis divorced from all applications, differentiation becomes a process which is applied to one function f to yield a new function f'. It is then natural to examine the results of applying this process again, obtaining a new function f''. Returning to our table, it is logical to wonder if f'' has some obvious interpretation in either of the other two contexts—in geometry or in the real world. As it happens, in the geometry of curves, f'' does not correspond directly to any immediately relevant geometric concept (although it is related to the concept of curvature). However, in the study of motion, f'' fits at once into the experimental facts: it can be identified with the physical concept of acceleration, so that one may give precision to the intuitive notion of "force."

Things like this have happened often enough in the interplay between mathematics and science that some people believe that any mathematical concept that fits in the mathematical context must also correspond to some significant physical concept. Sample: Maxwell's equations were formulated partly for mathematical reasons; they predicted that certain experiments should produce "waves" that could be detected, and Hertz verified this in the laboratory.

Almost every branch of mathematics has served as a source for models of some aspect of reality. However, the most common models that have led to the most striking advances in science have been those associated with the calculus and, in particular, with differential equations. Without attempting to give an all-inclusive formal definition at this time, we may state simply that a *differential equation* is any equality relation involving a function and one or more of its derivatives.

Starting with the next section, we shall present a systematic treatment of some of the most useful techniques for analyzing a simple ordinary differential equation. Thus, in terms of Figure 1.1, we shall concentrate upon the right-hand side of the diagram, the passage from the mathematical model to the box called "predictions and answers." Before doing so, however, we shall examine a number of simple illustrations of the modeling process. We shall not be able to carry through to the final step until we have obtained the needed mathematical techniques, but we hope to be able to show how such models arise and how some of the translations are made.

The Pendulum Problem

Suppose that "physical reality" is a pendulum consisting of a spherical bob mounted at the end of a rod and swung from a pivot. We say that we "understand" the physics of this pendulum if we can predict the behavior of this pendulum. Can we answer such questions as the following: (1) How does the nature of the motion depend on the mass of the spherical weight? Or on the length of the rod, or the material of which it is made? (2) If I hold the rod out to form an angle θ_0 and release it, what will be the speed at the bottom of the swing and how long will it take the pendulum to reach the bottom? (3) Will the pendulum eventually stop swinging, and how long will that take?

Experience and experimentation suggest that we had better start with a simplified situation. (Example: Try swinging a weight on the end of a rubber band!) We restrict ourselves to the following physical approximation. We assume that the rod is completely rigid and of zero mass, that the bob is a point of mass M, that the pivot has no friction, that the force of gravity is constant, that there is no air resistance, and that Newton's laws are sufficient to describe the situation. (Note that the last assumption is equivalent to saying that we are willing to buy the Newtonian model for motion.) With

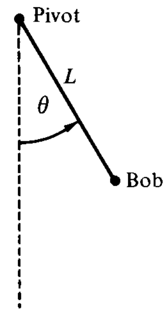

Figure 1.3

these assumptions, we can now move to the mathematical model. (See Figure 1.3.) The position of the pendulum swinging in a plane can be given at any moment of time by the angle θ. Thus, the motion of the pendulum can be translated into a function $\phi(t)$ defined for all $t \geq 0$ with $\theta = \phi(t)$ giving us the value of the angle at time t. We regard $t = 0$ as the instant at which we released the pendulum, so that we start with $\phi(0) = \theta_0$. Since we released the pendulum from rest, so that its velocity was zero at $t = 0$, we also have $\phi'(0) = 0$.

If we use Newton's laws to further restrict the nature of the function ϕ, we can derive a differential equation that must hold. In Figure 1.4, we show the force of gravity F acting on the bob of the pendulum in a vertical direction. The magnitude of F is Mg, where g is the acceleration of gravity and M is the mass of the bob. F can be resolved into two components, F_1 and F_2, as shown. Since the rod is absolutely rigid, F_1 has no effect. The motion of the pendulum bob is on a circle of radius L. Newton's Second Law often appears in the form

$$\begin{aligned}\text{Force} &= \text{rate of change of momentum}\\ &= \frac{d}{dt}\{mv\}\end{aligned}$$

where m is the mass at time t, and v the velocity at time t. When the mass does not change with time, as in our problem, this becomes

$$\begin{aligned}\text{Force} &= m\frac{dv}{dt} = ma\\ &= \text{mass} \times \text{acceleration}\end{aligned}$$

which is a more familiar form of Newton's Law.

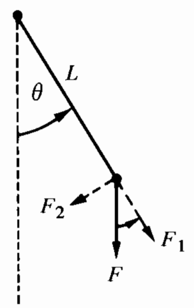

Figure 1.4

In our pendulum problem, the mass is M (which is constant) and the velocity of the moving bob is $L\, d\theta/dt$. The only force to be considered is F_2, so Newton's Law yields the equation

$$F_2 = M\frac{d}{dt}\left\{L\frac{d\theta}{dt}\right\} = ML\frac{d^2\theta}{dt^2}$$

By trigonometry, we find from Figure 1.4 that $F_2 = -Mg\sin\theta$, so that, after canceling M, we arrive at the desired differential equation for a swinging pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{1.3}$$

Restating in terms of the function ϕ, we obtain the following:

A mathematical model for the motion of this pendulum is a function ϕ, defined and continuous on the interval $t \geq 0$, which has a continuous second derivative and is such that

$$\phi''(t) = -\frac{g}{L}\sin\phi(t) \qquad \textit{for all } t \geq 0 \tag{1.4}$$

$$\begin{aligned}\phi(0) &= \theta_0\\ \phi'(0) &= 0\end{aligned} \tag{1.5}$$

How do the questions we asked at the beginning of our problem translate into questions about this model? The first, which was

How does the nature of the motion depend on the mass of the bob or the length of the rod?

becomes

How does the graph of the function ϕ depend on the number M or the number L?

Since the number M does not appear in (1.4), we see that the motion is independent of M. We are not ready yet to explore the nature of the dependence upon L.

The second question about reality,

What will be the speed at the bottom of the swing and how long will it take the pendulum to reach the bottom?

becomes

For what value of t, say t_0, is it true that

$$\phi(t_0) = 0 \tag{1.6}$$

and what is the value of $L\phi'(t_0)$?

Finally, the last question, which was

Will the pendulum eventually stop swinging, and if so, how long will this take?

becomes

Is there a number T such that

$$\phi'(t) = \phi(t) = 0 \tag{1.7}$$

for all $t > T$?

We have not yet presented the mathematical techniques needed to answer the questions about the function ϕ. However, here are several of the results that one can obtain.

There is a number $t_0 > 0$ such that

$$0 < \phi(t) \qquad \textit{for all } t,\ 0 < t < t_0 \tag{1.8}$$

and

$$\phi(t_0) = 0$$

Moreover,

$$t_0 = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{1.9}$$

and the value of $L\phi'$ at t_0 is given by

$$V_0 = 2\sqrt{gL}\,\sin(\tfrac{1}{2}\theta_0) \tag{1.10}$$

Furthermore, for all $t > 0$,

$$\phi(t+4t_0) = \phi(t) \tag{1.11}$$

and $\lim_{t\to\infty} \phi(t)$ does not exist.

These statements are next translated back into statements about reality, or rather into statements about the idealized physical pendulum from which our problem arose. (In each case that follows, you should check back to see exactly how the mathematical statement answers one of the questions asked about the mathematical model, and how it in turn relates back to the original physical reality.)

Thus, from (1.8) we see that the length of time it takes the pendulum bob to reach the bottom of its first swing is t_0, the number given in formula (1.9). Moreover, the velocity which it has reached at that moment is the number V_0 given in formula (1.10). Finally, the motion of the pendulum is periodic, repeating exactly after the time interval $4t_0$. It never stops moving nor does it ever slow down.

These are the results which the mathematical model predicts about the behavior of the real pendulum. The next step is to compare them with the results of experimentation with real pendulums. For example, we can test (1.8) and (1.9) by measuring the length of time it takes the pendulum bob to descend to the bottom of its swing and comparing this with the value of t_0 predicted from (1.9) for various choices of the initial angle θ_0 and the pendulum length L. We could also do the same for formula (1.10) for the velocity V_0. The degree of agreement is some sort of measure of the degree to which the model is a true reflection of reality. It is also a measure of the confidence one could place in the predictions of the model in areas where its agreement with reality has not been tested by experiment.

Regarding the latter, one must never lose sight of the fact that a model is not the same as reality. Although a model may behave in many ways like the real situation which gave rise to it, it will also predict behavior which does not occur. For example, (1.11) says that once set moving, a pendulum will never stop, and any experiment will show that this is false. Furthermore, the values of t_0 and V_0 given in formulas (1.9) and (1.10) will not match exactly the values obtained from measurement. A partial answer is to say that our model was constructed for an "ideal" pendulum, and that requirements such as a weightless, perfectly rigid rod and a frictionless pivot cannot be satisfied in practice. It is, of course, possible to start from a less ideal physical picture, permitting the rod to have mass and to be slightly elastic, the bob to have nonzero radius, and the pivot to have frictional forces, etc. When all this is done, the resulting mathematical model will be immensely complicated, possibly beyond the limits of presently understood techniques of mathematical solution. Nevertheless, it would still be our contention that this last model would predict kinds of behavior not exhibited by real pendulums and that in no sense can we identify such a model with reality, whatever indeed that is.

This discussion suggests that it is often useful to consider several different models for the same physical situation. Perhaps you have noticed that the formula (1.9) for the quarter-period t_0 of a pendulum depends in a complicated way upon the initial angle θ_0. If you recall the frequently quoted statement that the period of a pendulum does not depend upon the amplitude (which is determined by the angle θ_0), then you may wonder where the error lies. The facts are easily checked, and a simple experiment will show you that the length of the period of a pendulum is visibly different for $\theta_0 = 10°$ and for $\theta_0 = 80°$.

The explanation is that in many discussions of the motion of a pendulum, a much cruder mathematical model is used than that set forth in (1.4) and (1.5). In the cruder model, the differential equation (1.4) is replaced by the equation

$$\phi''(t) = -\frac{g}{L}\phi(t) \tag{1.12}$$

or, in more familiar notation,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

The predictions which result from this model are different from those of the previous model, but are more easily obtained. We will find out that

$$t_0 = \frac{\pi}{2}\sqrt{\frac{L}{g}} \tag{1.13}$$

$$V_0 = \sqrt{gL}\,\theta_0 \tag{1.14}$$

and it is evident that the value of t_0 does not involve θ_0 at all.

The connection between these two models for the motion of a pendulum is a mathematical one and not a physical one, although it can be given a physical interpretation. When θ is small, $\sin\theta$ is very nearly the same number as θ. Thus, if the motion of the pendulum is confined to a small neighborhood of the bottom of its arc, then the term $\sin\phi(t)$ which occurs in (1.4) is nearly equal to $\phi(t)$; this replacement turns (1.4) into (1.12). For the same reason, one should expect that the predicted values of t_0 and V_0 given in formulas (1.9) and (1.10) turn into those in (1.13) and (1.14) as θ_0 tends toward zero.

Another simple illustration may help here. Physicists speak of light sometimes as a wave and sometimes as a particle. Which is it? The answer, of course, is that light is neither. Both words describe specific mathematical models for light, but neither *is* light; one must not confuse a model with reality itself. Both of these models successfully predict ("explain") some of the observed phenomena of light; both predict behavior that light does not exhibit. A model is most useful when it imitates nature closely, but there will always be aspects of reality it does not reproduce, and it will always predict

events that do not in fact occur. The skill of a scientist lies in knowing how far and in which contexts to rely on a particular mathematical model. The final moral is clear; there is never a unique *correct* mathematical model for an aspect of reality. Those "laws of nature" that are not merely tautologies, akin to the statement that there are three feet to every yard, are models, and as such they are subject to revision and replacement by others. None is "correct" or "true" in some eternal sense. (At this time, we have moved far onto the quicksands of metaphysics; those interested in further speculation are encouraged to read H. Poincaré, *Science and Hypothesis*. Dover Publications, New York, 1952.)

The Rocket Flight Problem

The force of gravity is not constant, but depends upon the distance between objects; in the case of the earth, it also depends upon local rock densities and other factors. In dealing with objects at large distances from the earth, it is sufficient to regard the earth as a sphere, and to assume that the force on an object of mass M far above the atmosphere is given by

$$F = \frac{kM}{x^2} \tag{1.15}$$

where k is a negative constant and x is the distance from the object to the center of the earth. This relationship is usually described by saying that the force of gravity varies inversely as the square of the distance. The constant k is negative because the force is an attraction and acts to decrease the distance x.

During the initial powered-flight period, the motion of a rocket obeys rather complicated laws, since both the force of the exhaust and the mass of the rocket are changing. After burnout, things are much simpler: since the mass is then constant, the rocket is coasting, and its velocity is slowly being decreased by the effects of the earth's gravity.

If we assume that the rocket is moving directly away from the earth and we know its height and velocity at burnout, what will its future motion be? Will it continue moving upward and never stop, or will it reach a maximum height and then fall back?

Under the conditions we have described, an appropriate mathematical model for the motion of the rocket (now considered to be a particle of mass M) will be a function f such that $f(t) = x$ is the distance from the center of the earth to the rocket at time t. Let $t = 0$ correspond to the instant of burnout. Then, the distance from the earth at this moment is $R_0 = f(0)$, and the vertical velocity of the rocket at this moment is $V_0 = f'(0)$. We know both R_0

and V_0, and we want to predict the shape of the graph of f for $t > 0$. The final step in constructing the model is to produce a differential equation which the function f must satisfy. Again, this comes by the use of Newton's Second Law. From Figure 1.5 and formula (1.15), we obtain first

$$M\frac{d^2x}{dt^2} = \frac{kM}{x^2}$$

We can find the value of k by using the fact that on the surface of the earth, the acceleration of gravity is $-g$, which is 32 ft/sec^2. Thus, we put $d^2x/dt^2 = -g$ and $x = R$ (the radius of the earth), cancel M, and find $k = -gR^2$. This gives us the final form of the rocket flight equation:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{x^2} \tag{1.16}$$

We combine all the preceding discussion to formulate the mathematical model for our rocket flight problem.

The function f is defined and has a continuous second derivative f'' (that is, f is of class C^2) for all $t \geq 0$, and obeys the differential equation

$$f''(t) = -\frac{gR^2}{(f(t))^2} \qquad \textit{for all } t > 0$$

In addition, $f(0) = R_0$ and $f'(0) = V_0$.

The questions which we raised about the motion of the rocket are now translated into the following questions about the function f in our model.

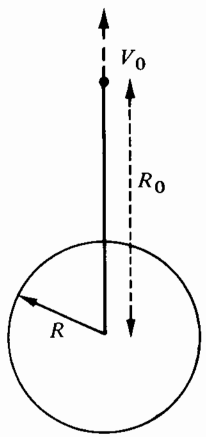

Figure 1.5

Is the function f monotonic increasing for all $t > 0$ and unbounded, or does f have a maximum value; and if so, what is it, and for what value of t is it achieved? More generally, what is the shape of the graph of f?

Without entering into the methods used to answer these questions, which will be discussed in Section 2.3, it may be proved that the graph of f has one of two typical shapes. The first, shown in Figure 1.6, occurs whenever $V_0 \geq R\sqrt{2g/R_0}$, and the second, shown in Figure 1.7, occurs when $V_0 < R\sqrt{2g/R_0}$.

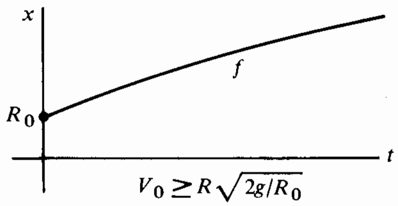

Figure 1.6

These facts about the function f can be translated back to yield useful statements about the expected behavior of the rocket. Thus, if the distance and velocity of the rocket at burnout are such that $V_0 \geq R\sqrt{2g/R_0}$, then the model predicts that the rocket will continue to move upward forever; it will recede indefinitely from the earth, eventually reaching any assigned distance. On the other hand, if $V_0 < R\sqrt{2g/R_0}$, the rocket will reach a maximum height and then fall back toward the earth, reaching it after a specific length of time. The most frequently examined case occurs when R_0, the distance at burnout, is approximately the same as R, the radius of the earth. In this case, the condition for nonreturn becomes $V_0 \geq \sqrt{2gR}$. Taking $g = 32$ ft/sec^2 and $R = 4000$ miles, computation gives the critical escape velocity as $V_0 = 40/\sqrt{33} = 6.9$ mi/sec. Thus, a rocket that at burnout is traveling vertically at about 7 miles per second will never fall back to the earth.

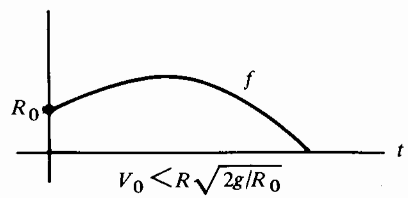

Figure 1.7

The Two-Tank Problem

Suppose that tanks I and II are placed as shown in Figure 1.8 and provided with inlet and outlet pipes as indicated. Each tank contains 100 gallons of liquid, and the flow capacity of each pipe is as shown on the diagram. The left-hand pipe admits pure water to tank I. Suppose that at the moment that all pipes are opened, tank II contains 10 grams of a toxic substance such as rat poison (Warfarin), and tank I contains only pure water. How long will it be until the liquid flowing out of the final outlet pipe on the right is virtually free of the poison (that is, until the level of concentration decreases to 0.001 g/gal)?

As a starting point in building a model for this situation, let x be the amount of the toxic substance in tank I at time t, and y the amount in tank II at time t. For example, we know that at $t=0$, when the system starts, $x=0$ and $y=10$. The rates of change of x and y are governed by a pair of related differential equations which we shall derive shortly:

$$\frac{dx}{dt} = \frac{2}{100}y - \frac{6}{100}x \qquad (1.17)$$

$$\frac{dy}{dt} = \frac{6}{100}x - \frac{6}{100}y$$

Thus a model for the two-tank situation is provided by a pair of continuous functions $\phi(t)$ and $\psi(t)$ such that $x=\phi(t)$ and $y=\psi(t)$ satisfy the equations (1.17), and in addition, are such that $\phi(0)=0$, $\psi(0)=10$. The basic question which we must ask of the model is

For what value of t is it true that $\psi(t) < 0.1$?

Other questions about the model are suggested in the exercises. Again, we have not yet discussed the techniques needed to obtain an answer to questions about this model, and we therefore postpone further discussion until later.

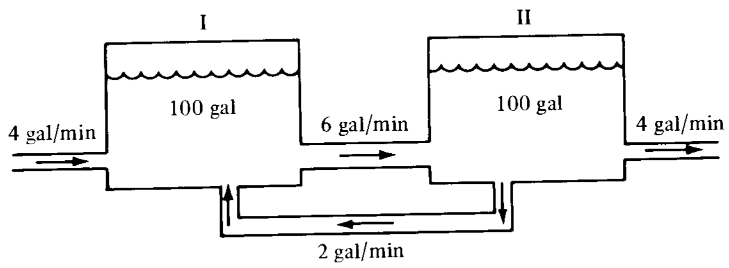

Figure 1.8

Some general comments about the differential equations (1.17) are, however, in order. It should first be noted that the process of going from reality to the mathematical model is not a logical exercise like the proof of a theorem in geometry. The concept of deduction that is involved is of a different sort. We must present an argument that suggests that equations (1.17) are an appropriate mathematical image of the process of mixing that takes place in the pair of tanks. The procedure is often a combination of precise mathematical reasoning and arguments based on physical intuition. In the creation of a model, there is room for wild guesses and even luck. The role of rigorous mathematics is the deduction of consequences of the model, and the ultimate test of a model is how well its predictions match what is seen in reality. Quantum mechanics is accepted not because it is a rigorous mathematical deduction from observed phenomena, but because it enables one to calculate the results of measurements which can be verified.

We can arrive at formula (1.17) as follows. Looking at the diagram in Figure 1.8, we suppose that liquid is flowing as indicated. At a particular moment of time, we suppose that x and y are the amounts of the toxic substance in tanks I and II in grams. Its concentration in each tank in grams per gallon is $x/100$ and $y/100$. During a short interval of time Δt, what change will take place? Looking at each pipe in turn, we find

$4\,\Delta t$ gallons of pure water enters tank I,
$2\,\Delta t$ gallons enters tank I carrying the toxic substance at concentration $y/100$,
$6\,\Delta t$ gallons leaves tank I at concentration $x/100$,
$6\,\Delta t$ gallons enters tank II at concentration $x/100$,
$2\,\Delta t$ gallons leaves tank II at concentration $y/100$,
$4\,\Delta t$ gallons leaves tank II at concentration $y/100$.

From these findings, we can calculate the total change in the amount of the toxic substance in each tank at the end of the Δt-long interval.

$$\Delta x = 0 + 2\,\Delta t(y/100) - 6\,\Delta t(x/100)$$

$$\Delta y = 6\,\Delta t(x/100) - (2+4)(\Delta t)(y/100)$$

Dividing by Δt, we are immediately led to impose the relations given in (1.17) upon the functions that govern the values of x and y.

This is an example of one of the methods used in building a model. It may be noted that in building a model, we have assumed that mixing is slow enough to allow us to use $x/100$ as the concentration of the liquid leaving tank I while we are adding the new toxic substance, but fast enough that we need not worry about how long it takes for the new material to diffuse throughout the tank for the next step. Perhaps these assumptions seem inconsistent. One might therefore desire a different model which takes account of the true speed of diffusion and the unequal concentration in different parts of either tank. Such a model, although a better reflection of reality, is likely to be so difficult to work with that we may not be able to learn much from it.

In the exercises that follow, we ask you to examine a number of models and to attempt to build others. There are few guidelines that can be given. Remember that there is no uniquely correct model of any portion of nature, although some models are more useful than others.

Exercises

1. Each of the space-time diagrams in Figure 1.9 describes the straight-line motion of one or more points. Give an equivalent verbal description of the events.

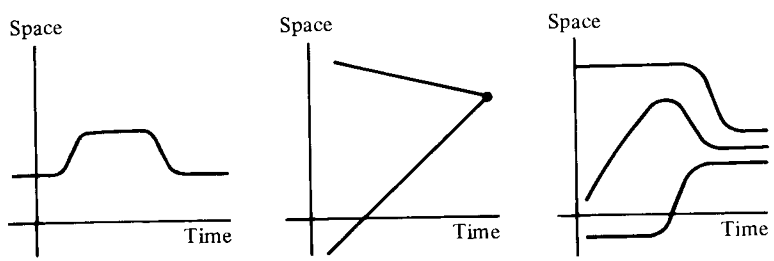

Figure 1.9

2. The graph of f' is shown in Figure 1.10. Given that $f(0) = 1$, draw the graph of f'' and f.

3. If the function $F(t) = (3t, t^3 - t^2, t^2 - 4t)$ describes a motion in space, find the velocity, speed, and acceleration at the points $(-3, -2, 5)$ and $(6, 4, -4)$.

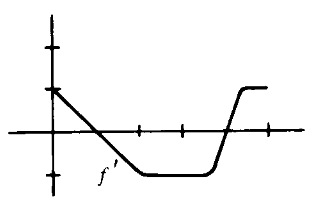

Figure 1.10

4. Can a space-time diagram of a straight-line motion turn out to be a circle?

5. Verify analytically that for values of θ_0 near zero, formula (1.10) is approximately the same as formula (1.14).

6. According to either of the models for a simple pendulum, what is the effect of lengthening the rod? What happens if you make it twice or four times as long? How could you test your answer?

7. When $\theta_0 = \frac{1}{2}\pi$, calculate the predicted value of V_0 by each of the models for a simple pendulum. Can you design an experiment that would tell you which better represents reality? Is this easier to do if $\theta_0 = \pi$?

8. Construct simple pendulums in which the rod is replaced by (a) a string, (b) a long rubber band, (c) a heavy rod whose mass is comparable to that of the bob. Can you decide if either of the models in the text represents any of these real pendulums?

9. (If you have enough physics experience.) Can you devise a mathematical model for the pendulum shown in Figure 1.11 similar to either of those in the text?

10. A company which makes radios has two factories and five warehouses, all of which are in different cities. The various costs of shipping a radio from either factory to any of the warehouses are known. We are given the available storage room at each warehouse, and the maximum production capacity of each factory. Devise a model which could enable you to answer a question such as "What plan for producing and shipping radios will be the cheapest?" [*Note:* The simplest model will be an algebraic one, not involving functions or differential equations at all.]

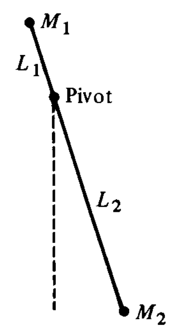

Figure 1.11

11. A lake is being polluted constantly by the addition of a fixed amount of a certain chemical per day. At one end of the lake, pure water is entering at a constant rate, and at the other end, water is flowing out at the same rate. Can you construct a model for this lake which will enable you (after you have learned to solve differential equations) to discuss the future behavior of the lake and to predict when the pollution level will exceed a certain threshold?

12. How would you modify the model for Exercise 11 if the amount of chemical being added per day varies in a known way? If the amount of water flowing in and out per day is not a constant, but also changes with time?

13. It is observed that a quantity of material A left by itself emits radiation and after a while is found to be a mixture of substance A and substance B. It is conjectured that the atoms of substance A are "unstable," and that they randomly turn into atoms of substance B, emitting radiation as they do so. If the quantity of material is large, so that there are many, many atoms involved, it is conjectured that the rate at which substance A turns into substance B can be regarded as proportional to the amount of A still present. Construct a mathematical model which would enable you to answer questions such as the following: "If it takes 10^5 years for 1 gram of A to turn into 0.5 gram A and 0.5 gram B, then how long will it take 1 gram of A to become 0.8 gram A and 0.2 gram B?" What is the connection here with carbon dating of human artifacts?

14. Construct a model for human population growth. (Since no answer to this question can be either simple or unique, it is an ideal topic for class discussion; an optimal answer may be a list of the factors that should be taken into account in building such a model and some of the functions which behave in the required way.)

15. (a) Use your experience and physical intuition to guess the shape of the functions ϕ and ψ in the Two-Tank Problem.
    (b) How do you think these graphs would change if we had started with 10 grams of the toxic substance in tank II and 50 grams in tank I? (Everything else stays the same.)
    (c) Under the circumstances holding in part (b), what would be the effect of changing the flow in the two pipes connecting tank I and tank II from 2 gal/min and 6 gal/min to 12 gal/min and 16 gal/min?

16. Can you devise a model for a system consisting of three tanks and a number of interconnecting pipes, and one or more pollutants?

1.2 First-Order Equations

Having seen how physical problems can give rise to differential equations, we shall begin a systematic study of these equations. From the mathematical point of view, a differential equation is one in which the unknown is a function, rather than a number as in the case of an algebraic equation; the unknown function is to be determined from the identities and relations which it and its derivatives are known to satisfy. When we are dealing with functions of one (independent) variable, the equations are called **ordinary differential equations**; when we are dealing with functions of several variables and the identities involve partial differentiation, then they are called **partial differential equations**. As an example, the study of heat distribution leads to the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Functions U that satisfy this identity are called *potential* or *harmonic* functions. Until Chapter 7, we shall be concerned only with ordinary differential equations.

Differential equations are classified in various ways. The term **order** is used to refer to the highest-order derivative of the unknown function that appears in the identity to be solved. Thus, a first-order equation may involve the unknown function and its first derivative, but will not involve its second derivative or higher derivatives. Since time is so often the independent variable in physical applications, we choose to label points in the plane as (t, y) rather than (x, y). Then, the standard form of a first-order differential equation is

$$\frac{dy}{dt} = f(t, y) \tag{1.18}$$

where f is a function of two variables defined on a region D in the TY-plane.

A solution of (1.18) on an interval I is a function ϕ such that (see Figure 1.12)

1. $\phi'(t)$ exists for all $t \in I$.
2. The graph of ϕ lies in the region D.
3. For each $t \in I$, $y = \phi(t)$ satisfies the equation (1.18), meaning that

$$\phi'(t) = f(t, \phi(t)) \qquad \text{for all } t \in I$$

As an example, the function $y = 3e^{t^2}$ is a solution of the equation

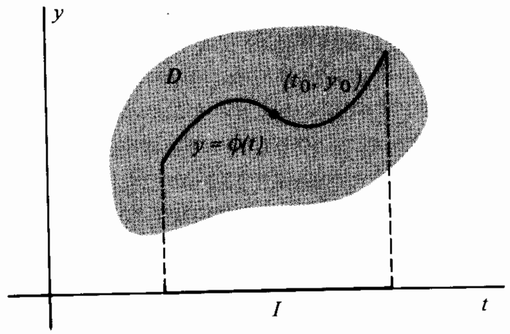

Figure 1.12

$dy/dt = 2ty$ on the interval $-5 < t < 5$ because for every such t,

$$\frac{d}{dt}(3e^{t^2}) = (3e^{t^2})(2t) = 6te^{t^2}$$

and

$$2ty = (2t)(3e^{t^2}) = 6te^{t^2}$$

In this case, there is nothing special about the choice of the interval I, and in fact $3e^{t^2}$ is a solution of the given equation on the whole T-axis, $-\infty < t < \infty$.

A more typical example is provided by the equation

$$\frac{dy}{dt} = -\frac{t}{y} \tag{1.19}$$

Here, the function f is defined for any choice of t and y, with $y \neq 0$. Thus, D could be taken to be any region in the upper half-plane. The function given by $y = \sqrt{1-t^2}$ is defined only on the interval $-1 < t < 1$; but on that interval, it is a solution of the equation, for

$$\begin{aligned}\frac{dy}{dt} + \frac{t}{y} &= \tfrac{1}{2}(1-t^2)^{-1/2}(-2t) + \frac{t}{\sqrt{1-t^2}} \\ &= 0\end{aligned}$$

It is also easily checked that $y = \sqrt{2-t^2}$ is also a solution of this same equation—this time on the larger interval $(-\sqrt{2}, \sqrt{2})$. Indeed, for any positive value of A, the function $y = \sqrt{A-t^2}$ is a solution of this equation on the interval $-\sqrt{A} < t < \sqrt{A}$.

There is a simple geometric relationship between a first-order differential equation $dy/dt = f(t, y)$ and the functions that are its solutions. Moreover,

this relationship is the basis for a simple and effective method for finding out something about the solutions of an equation; and it also lies at the heart of the procedures by which high-speed computers obtain approximate solutions of differential equations.

Since dy/dt is associated with the concept of slope, we can use the given differential equation to construct a direction field or tangent field in the region D where f is defined. We proceed as follows. At each point $p = (t, y)$ in D, we evaluate f and use this value $f(p) = f(t, y)$ as the slope of a short line segment which we draw centered on p. The totality of such line segments for all points p in D is called the **direction field** or tangent field for the differential equation. In Figure 1.13, we have drawn a portion of the tangent field defined by the equation $dy/dt = -y$, and in Figure 1.14, the field defined by $dy/dt = y - t$. (A special trick which makes it easier to construct a tangent field, called the method of isoclines, is given in the exercises—see Exercise 4.)

What does the direction field for an equation have to do with solutions of the equation? The fact that $y = \phi(t)$ satisfies the equation $dy/dt = f(t, y)$ means that at any point p_0 on the curve that is the graph of ϕ, the slope of the curve must coincide with the slope of the line segment in the direction field passing through p_0. In other words, the curve must be tangent to each of the line segments whose midpoints it passes through. In Figure 1.14, we have drawn a number of the solution curves to show this tangency property.

It is now evident that this procedure can be used in reverse to find out something about the solutions of a given differential equation. In fact, given a carefully drawn tangent field, a skilled person can often produce reasonably

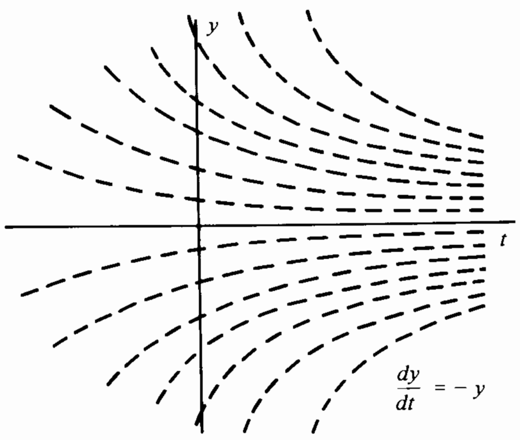

Figure 1.13

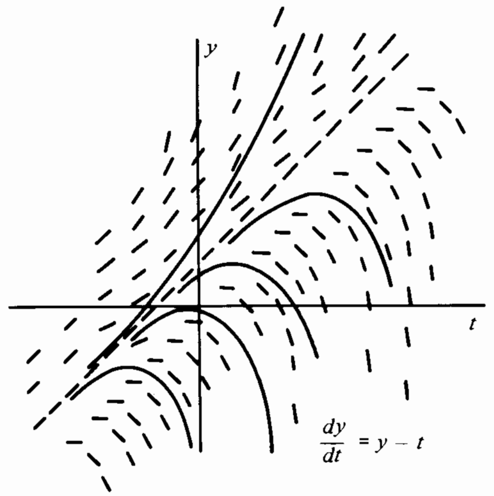

Figure 1.14

accurate sketches of many of the solutions by drawing curves that have this property of being tangent to, and therefore of flowing along with, the directions of the field. Such freehand sketches often reveal something of the general qualitative nature of a solution—indicating, for example, whether it is increasing, oscillating, tending to a limit, etc. (For accuracy, this eyeball method must be combined with the calculating ability of a digital computer.)

Although it is possible to invent a differential equation that has only one (or even no) real solution, in general a differential equation will have an infinite number of solutions. For example, a differential equation associated with a swinging pendulum will have as many solutions as there are ways of starting the motion, one for each initial angle θ_0 and each initial velocity. In a physical situation, we are usually looking for only one of the possible solutions of the equation, one which satisfies certain additional restrictions associated with the real problem from which the model came.

In first-order equations, these normally take the simple form of requiring that y be some specific value y_0 when t is a particular moment t_0 (often $t_0 = 0$). This condition is called an **initial condition**. We therefore seek a function ϕ which is a solution of the given equation $dy/dt = f(t, y)$ and which in addition obeys $\phi(t_0) = y_0$. Translated into geometric language, this merely means that the graph of ϕ must go through the point (t_0, y_0). Of course, this point (t_0, y_0) must be in the region D. Such a problem is usually called an *initial-value problem.*

We can summarize this geometric approach to the approximate solution of first-order equations as follows.

To solve the differential equation $dy/dt = f(t, y)$ with given initial condition $y = y_0$ when $t = t_0$, first construct the tangent field in the region D where f is defined. Then, construct a curve in D which passes through the point (t_0, y_0) and is "tangent" to the field everywhere. This curve will be the graph of a solution.

In most cases, and certainly in all the most useful ones, there will be one and only one solution curve passing through an interior point of the given region D. However, as we shall see, this depends upon the nature of the defining function f and may fail if the function f isn't sufficiently smooth.

A physical problem may involve two or more interrelated quantities that depend upon time. In this case, a standard model for the situation may be a system of differential equations. This was the case, for example, in the Two-Tank Problem in Section 1.1 [see formula (1.17)]. Systems of equations are also classified by order, and the standard form for a system of two first-order differential equations would be

$$\frac{dx}{dt} = f(t, x, y) \qquad (1.20)$$
$$\frac{dy}{dt} = g(t, x, y)$$

The general approach is quite similar to that of a single equation. We assume that both f and g are defined in a region D in TXY-space. A solution of the system (1.20) is a pair of functions ϕ and ψ, each defined on the same T-interval I, such that ϕ' and ψ' are also defined there. Furthermore, when $t \in I$, the points $(t, \phi(t), \psi(t))$ must all lie in D, and $x = \phi(t)$, $y = \psi(t)$ must satisfy the equations as an identity; that is,

$$\phi'(t) = f(t, \phi(t), \psi(t))$$
$$\psi'(t) = g(t, \phi(t), \psi(t))$$

for all $t \in I$.

It is possible to give a geometric interpretation here. The pair of functions f and g can be used to create a direction field on the region D, and the set of points $(t, \phi(t), \psi(t))$ trace a curve in 3-space which must be tangent to the direction field in order for $x = \phi(t)$ and $y = \psi(t)$ to be a solution.

Exercises

1. Verify that for $C > 0$, each of the functions $y = \sqrt{C-t^2}$ is a solution of the differential equation (1.19). What about the functions $y = -\sqrt{C-t^2}$? Sketch some of these solutions for several choices of C.

2. Show that the functions $y = 1/(t-C)$ are solutions of the equation $dy/dt = -y^2$ on certain intervals which depend upon the choice of C. Sketch these solutions for $C = 0, \pm 1, \pm 2$.

3. Verify that for each C, the function $y = (1+Ce^t)/(1-Ce^t)$ is a solution on an appropriate interval of the equation $dy/dt = \frac{1}{2}(y^2-1)$. Sketch the graphs of these solutions for $C = 0$, $C = 1$, and $C = -1$.

4. The **isocline** method for constructing the direction field for the equation $dy/dt = f(t, y)$ consists in drawing line segments of slope m at all the points on the level curve of f with equation $m = f(t, y)$. Apply this method to the equation $dy/dt = 2y$ and sketch several of the solution curves for the differential equation.

5. By the isocline method sketch the direction field for the equation $dy/dt = t+y$ and draw several of the solutions.

6. Do the same as in Exercise 5 for the equation $dy/dt = t/(t-y)$.

7. Do the same as in Exercise 5 for the equation $dy/dt = t^2+y^2$. Do you think that the solution passing through the point $(0, 1)$ can be extended to arbitrarily large values of t?

8. Among the solutions discussed in Exercise 1, can you find one which satisfies the condition $y = 2$ when $t = 1$? for which $y = -1$ when $t = 3$?

9. Among the solutions discussed in Exercise 3, is there one which passes through the point $t = 1$, $y = 5$? Is there one such that $y = 2$ when $t = 0$?

10. A class of functions is described by the formula $y = Ct/\sqrt{C^2-t^2}$ for each real number C. Is there a function in this family such that $y = 2$ when $t = 1$? such that $y = 1$ when $t = 2$?

11. (a) Verify that the functions given by $x = e^{4t}+e^{-6t}$, $y = e^{4t}-e^{-6t}$ are solutions of the system of equations

$$\frac{dx}{dt} = 5y - x, \qquad \frac{dy}{dt} = 5x - y$$

(b) Verify that all functions of the form $x = Ae^{4t}+Be^{-6t}$ and $y = Ae^{4t}-Be^{-6t}$ are solutions of the same system.

12. (a) Find a solution of the system in Exercise 11 which obeys the initial condition $t = 0$, $x = 1$, $y = 3$.
(b) Do the same for the initial condition $t = 1$, $x = 1$, $y = 3$.

13. Show that $x = -t^3/3$ and $y = 3/t$ are solutions of the system of first-order equations $dx/dt = xy$, $dy/dt = 3/(xy)$. Do you think there are others?

14. Show that for every choice of A, the functions $x = t+At^2$, $y = 1+2At-At^2$ are solutions of the system $dx/dt = x+y-t$, $dy/dt = 2y/t-2x/t^2$. Do you think there are other solutions?

1.3 Higher-Order Equations

A differential equation may involve higher-order derivatives, as in the case of the first two illustrations of mathematical models given in Section 1.1. One obvious reason is that many models involve Newton's Laws of Motion and therefore forces and accelerations which involve second derivatives with respect to time.

Let f be a function defined in a region D in 3-space. Then, with f we can associate a general second-order differential equation whose standard form is

$$\frac{d^2y}{dt^2} = f(t, y, y') \tag{1.21}$$

(Here, we have written y' for dy/dt. Another common notation is $\dot{y}$.)

A function ϕ is a solution of the equation (1.21) on an interval I of the T-axis if the following conditions hold:

1. $\phi(t)$, $\phi'(t)$, and $\phi''(t)$ are defined for all $t \in I$.
2. For each $t \in I$, the point $(t, \phi(t), \phi'(t))$ lies in the region D.
3. $y = \phi(t)$ satisfies the equation (1.21) on I, meaning that the identity

$$\phi''(t) = f(t, \phi(t), \phi'(t))$$

holds for all $t \in I$.

As an example, consider the equation

$$y'' - 2y' + y = 0$$

which is an equivalent way to write

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

This can be rewritten as

$$\frac{d^2y}{dt^2} = 2\frac{dy}{dt} - y$$

which has the form (1.21) with $f(t, y, y') = 2y' - y$. The set D can be taken to be all of 3-space. Then, the function $\phi(t) = e^t$ is easily seen to be a solution of the equation on the interval $-\infty < t < \infty$. For, $y' = e^t$ and $y'' = e^t$, so that $y'' - 2y' + y = e^t - 2e^t + e^t = 0$ for all values of t.

It is possible with second-order differential equations, as with first-order equations, to attach a geometric interpretation to the relationship between an equation and one of its solutions, but it is far less useful either as a practical guide in finding solutions or as a theoretical tool. If we look at a particular solution function ϕ which satisfies the equation (1.21), then the knowledge of the coordinates $t = t_0$, $y = y_0$ of a point on the graph of ϕ and the knowledge of the slope y_0' of this graph at (t_0, y_0) enables us to use f to compute the value $f(t_0, y_0, y_0')$, which is the value of $\phi''(t_0)$. Thus, we can determine the curvature of the graph of ϕ at any point where we know its slope. This does not lead to a practical scheme for obtaining freehand sketches of solutions to (1.21).

A technique to single out one among the infinite number of solutions of a second-order equation is to specify initial conditions; this means that the desired solution $y = \phi(t)$ must have prescribed values A and B for y and $y' = dy/dt$ corresponding to $t = t_0$. In terms of the geometric interpretation given above, this means that we have asked for a solution of the given equation which passes through a specified point (t_0, A), having there the assigned slope B.

For example, the differential equation

$$\frac{d^2y}{dt^2} = -\frac{t}{1+t}y' + \frac{1}{1+t}y$$

has for solutions the functions

$$y = Ae^{-t} + Bt$$

for any choice of the numbers A and B. Suppose we want a solution such that at $t = 1$, $y = 3$ and $y' = 2$. We have

$$y' = \frac{dy}{dt} = -Ae^{-t} + B$$

Setting $t = 1$, $y = 3$, and $y' = 2$, we obtain the algebraic equations

$$3 = Ae^{-1} + B$$

$$2 = -Ae^{-1} + B$$

These can be solved to find $B = \frac{5}{2}$ and $A = \frac{1}{2}e$.

A second type of condition which picks one of the solutions of an equation from all the rest occurs in the study of higher-order equations; it has no analogy in the study of first-order equations. Sometimes, the nature of the problem that gave rise to a particular second-order differential equation leads to the requirement that the unknown function ϕ must satisfy a condition like $\phi(t_1) = a_1$, $\phi(t_2) = a_2$, where t_1 and t_2 are different moments of time. (For example, we may ask that the moving object be at specified locations at the start and at the end of its motion.) This type of condition is called a **two-point boundary-value problem** and is of great significance in the theory and application of differential equations. As a statement about the graph of a solution, it merely says that the graph of ϕ passes through two specified points, but says nothing about the slope at either point.

For example, if we return to the solutions $Ae^{-t} + Bt$ of the previous example and ask that $y = 2$ when $t = 0$, and $y = 3$ when $t = 1$, then substitution again gives a set of algebraic equations:

$$2 = Ae^0 + B(0) = A$$

$$3 = Ae^{-1} + B$$

from which we find $A = 2$ and $B = 3 - 2/e$. Thus the desired solution is $y = 2e^{-t} + (3 - 2/e)t$.

Finally, we point out a mathematical fact about differential equations of higher order that is of great importance for creating a general theory. It simplifies the general viewpoint and even helps in the discussion of an individual equation. The first special case of the general principle is that any second-order equation can be regarded as a system of two first-order equations. In general, a single nth-order equation can be regarded as a special system of n first-order equations.

To see how this is possible, suppose we have the second-order equation

$$y'' = \frac{d^2y}{dt^2} = f(t, y, y') \tag{1.22}$$

Now, put $x = dy/dt$ so that $dx/dt = d^2y/dt^2$. Then, equation (1.22) is clearly equivalent to the system

$$\frac{dx}{dt} = f(t, y, x)$$

$$\frac{dy}{dt} = x$$

As another example, the system of first-order equations

$$\frac{dx}{dt} = f(t, y, z, x)$$

$$\frac{dz}{dt} = x$$

$$\frac{dy}{dt} = z$$

is equivalent to the general third-order equation

$$\frac{d^3y}{dt^3} = f(t, y, y', y'') \tag{1.23}$$

Exercises

1. (a) Verify that the functions $y = Ae^{4t} + Be^{-6t}$ are solutions of the equation $d^2y/dt^2 = 24y - 2\,dy/dt$ for every choice of A and B.
   (b) Find a solution such that $y = 2$ and $y' = 3$ when $t = 0$.
2. (a) Show that the functions $y = A/(t+B)$ are solutions of the equation $d^2y/dt^2 = 2(y')^2/y$ for every A and B with $A \neq 0$. Can you guess some other solutions?
   (b) Can you find a solution which satisfies the initial conditions $y = 2$, $y' = 3$, when $t = 0$? Are there any admissible initial conditions you cannot satisfy?
3. (a) The equation $y'' = (5/2t)y' - (3/2t^2)y$ has solutions of the form $y = t^\alpha$. Find two of these, ϕ and ψ.
   (b) Show that every function $y = A\phi + B\psi$ is also a solution, and find such a solution that satisfies the initial conditions $y = 3$, $y' = 2$, when $t = 1$.
4. Can you find a solution of the equation discussed in Exercise 1 which satisfies the following two-point boundary condition: when $t = 0$, $y = 0$ and when $t = 1$, $y = 1$?
5. Can you find a solution of the equation discussed in Exercise 2 which satisfies the following two-point boundary condition: when $t = 0$, $y = 1$ and when $t = 1$, $y = 3$?

6. Can you find a solution of the equation in Exercise 3 which obeys the two-point condition $t = 1$, $y = 2$; $t = 4$, $y = 1$?
7. Check the equivalence of the equation (1.23) to the system of three first-order equations given.
8. Write D for d/dt so that, for example,

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 5x = D^2x - 4Dx + 5x = (D^2 - 4D + 5)x$$

   Show that the system $dx/dt = 3x+2y$, $dy/dt = 7x-5y$ can be written in the form

$$(D-3)x = 2y$$
$$(D+5)y = 7x$$

9. Using the previous exercise, show that the functions x and y in this particular system must satisfy the identity

$$\phi'' + 2\phi' - 29\phi = 0$$

10. If x and y are functions satisfying the system $dx/dt = x+y$, $dy/dt = x-y$, then show that x and y each must satisfy the relation $\phi'' = 2\phi$.
11. Show that a general nth-order differential equation is equivalent to a special system of n first-order equations.
12. Show that the equation $t^2y'' + Aty' + By = 0$ has $y = t^\alpha$ and $y = t^\beta$ as solutions whenever $A = 1 - \alpha - \beta$ and $B = \alpha\beta$.

1.4 Differential Equations and Families of Functions

The collection of solutions of a differential equation is a parametrized family of functions or curves. Conversely, given such a family, it is often possible to find a differential equation which characterizes this particular family. For example, consider the family of all straight lines in the plane. If we exclude the vertical lines, the family can be described by the general formula $y = Ax + B$, where A and B can be any pair of real numbers. (Such a family is sometimes said to have two **degrees of freedom**, or to be a two-parameter family.) It should be evident that this same family can be described as the set of all solutions of the second-order differential equation $d^2y/dx^2 = 0$.

A second illustration will show better how one can go from a formula for a family of curves to a differential equation describing the family. Suppose we are given the curves whose equations have the form

$$y = \frac{x^2+A}{x+A} \tag{1.24}$$

This is a one-parameter family of curves, and for each choice of A, a new curve results. (In Figure 1.15, we have shown a number of these curves for various choices of A.) We pose the problem: Is there a single differential equation which has all of these curves among its solutions? Such an equation will be a single relationship composed of x, y, dy/dx, and perhaps higher derivatives, which is satisfied by functions of the form (1.24), no matter what value A has.

In effect, we want to eliminate A from equation (1.24) by means of differentiation. One approach is to solve for A, first writing $y(x+A) = x^2+A$, and then writing

$$A = \frac{x^2-xy}{y-1}$$

Now, differentiate this with respect to x, remembering that y is a function

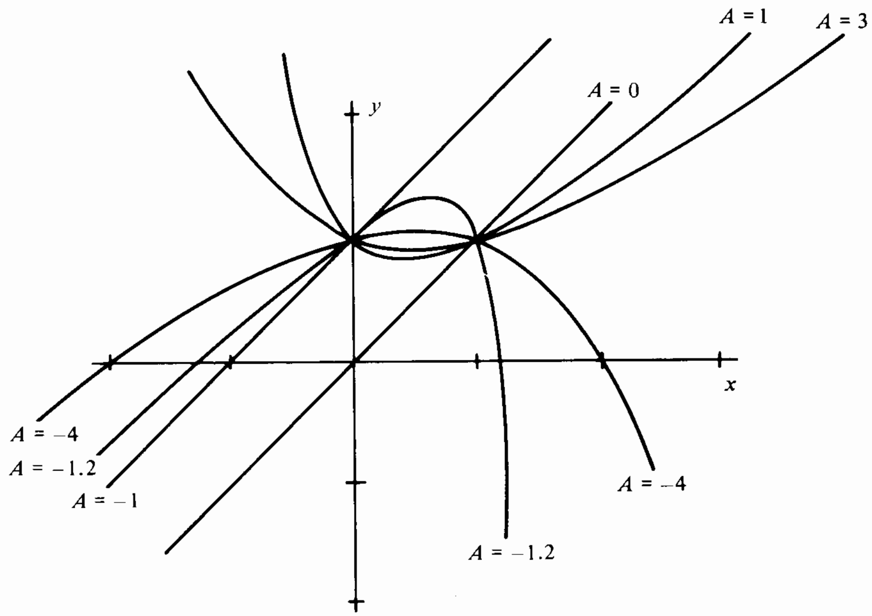

Figure 1.15

of x and that A is merely a number. We obtain

$$0 = \frac{(y-1)\left(2x - x\dfrac{dy}{dx} - y\right) - (x^2-xy)\dfrac{dy}{dx}}{(y-1)^2}$$

If we solve this for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{(2x-y)(y-1)}{x^2-x} \tag{1.25}$$

This is the desired differential equation. It can be verified directly that each of the functions given by (1.24) is a solution of this equation; one need only substitute for y and compare both sides of (1.25).

Another example may be helpful. Consider the family of all circles that pass through the origin and have their centers on the X-axis. The equation of such a circle is

$$(x-c)^2 + y^2 = c^2$$

We seek a differential equation that is satisfied by each of these curves. First, transform the circle equation to the form

$$x^2 + y^2 = 2cx$$

and then differentiate with respect to x, obtaining

$$2x + 2y\frac{dy}{dx} = 2c$$

This enables us to eliminate c, obtaining first

$$x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$$

and then

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$$

which is the desired differential equation.

When a family of curves is defined by a formula that contains n independent parameters, one expects that the associated differential equation will be of the nth order. For example, let us find a differential equation for the two-parameter family of parabolas

$$y = A(x-B)^2$$

If we differentiate this twice, we obtain

$$y' = 2A(x-B)$$

$$y'' = 2A$$

We can then use these to eliminate both A and B, obtaining first

$$A = \tfrac{1}{2}y''$$

$$x - B = \frac{y'}{y''}$$

and then

$$y = \tfrac{1}{2}y''\left(\frac{y'}{y''}\right)^2$$

which finally yields the desired equation

$$y'' = \frac{(y')^2}{2y}$$

Since the family of curves was a two-parameter family, we are not surprised to find that the appropriate equation is of second order.

A very important application of this process is in the study of **orthogonal families** of curves. Two families are called orthogonal if each curve in one family is perpendicular to every curve in the other family at every crossing point. As an illustration, consider the following two families of hyperbolas:

$$\mathcal{F}_1: \quad x^2 - y^2 = A$$

$$\mathcal{F}_2: \quad xy = B$$

These have been sketched in Figure 1.16; the dashed curves belong to family $\mathcal{F}_1$, and each is orthogonal to every curve of $\mathcal{F}_2$ which it crosses.

To prove this, we find the appropriate differential equation for each family. Using the techniques explained in the preceding examples, we first work with $\mathcal{F}_1$. If we differentiate the formula for a curve of the family, we immediately arrive at the desired equation:

$$\mathcal{F}_1: \quad y' = \frac{x}{y} \tag{1.26}$$

If we apply the same process to $\mathcal{F}_2$, we arrive at a different differential equation:

$$\mathcal{F}_2: \quad y' = -\frac{y}{x} \tag{1.27}$$

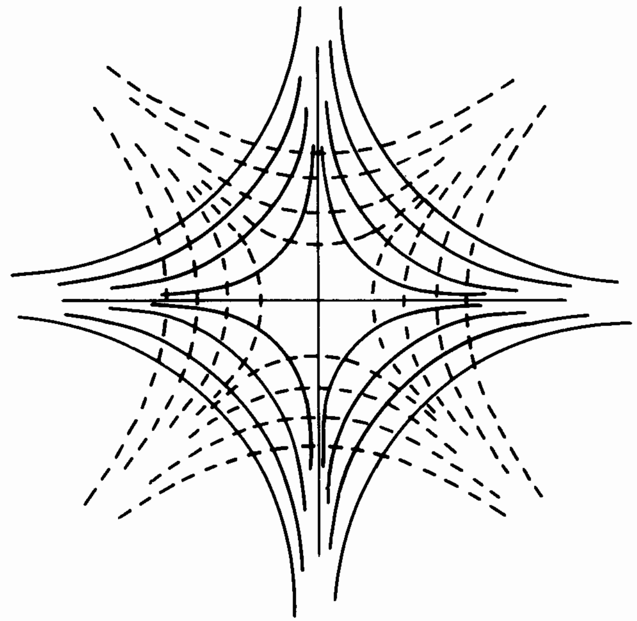

Figure 1.16

The mutual orthogonality property of the two families is at once evident from (1.26) and (1.27), for we observe that the product of the expressions for each of the slopes is -1: using m_1 for the slope of any curve in $\mathcal{F}_1$ and m_2 for the slope of any curve in $\mathcal{F}_2$, we have

$$m_1 m_2 = \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right) = -1$$

(Recall that two lines are perpendicular if the product of their slopes is -1.)

In many physical problems, it is useful to be able to find the family of curves that is orthogonal to a given family of curves. For example, in a force field, the flow lines along which a free particle will move will often be the curves that are orthogonal to the family of equipotential curves of the force field. In perhaps a more familiar context, the paths of steepest ascent of a mountain will be the curves that are orthogonal to the contour lines on its topographic map.

To see how such a problem is solved, let us find the orthogonal family (also called the **orthogonal trajectories**) associated with the following family of ellipses:

$$\tfrac{1}{4}x^2 + y^2 = A$$

(These could be the contour lines for a single mountain peak, or the force field for a single charged particle.) If we ask for the differential equation for this family, we are led to the first-order equation

$$y' = -\frac{x}{4y} \tag{1.28}$$

We can immediately write down the differential equation for the orthogonal family, merely by taking the negative reciprocal of the right-hand side, obtaining

$$y' = \frac{4y}{x} \tag{1.29}$$

While we are not yet in a position to solve this equation to find the formula for the associated family of curves, we can verify directly (Exercise 12) that it is given by the simple equation

$$y = Cx^4 \tag{1.30}$$

We give a number of additional illustrations of this among the exercises.

Exercises

1. Find a differential equation satisfied by all cubic curves $y = Ax^3 + Bx^2 + Cx + D$.
2. What differential equation is satisfied by the family $y = e^{Ct}$?
3. Find a differential equation for the family of all lines that go through the point $(-\frac{1}{2}, -1)$.
4. Find a differential equation for the family of cubic curves given by $y = (t-c)^3$.
5. What differential equation is satisfied by the two-parameter family $w = Ae^{By}$?
8. What differential equation is satisfied by $y = a\tan(at+b)$?
7. Find a differential equation for the family of all circles which are tangent to $y = x$ at $(0,0)$.
6. What is a differential equation for the family of all circles going through the origin?

9. Find a differential equation for the family of all lines that are tangent to the parabola $y = x^2$. Does $y = x^2$ also satisfy that differential equation? Draw a picture showing various members of the family.
10. Verify equations (1.26) and (1.27) for the families $\mathcal{F}_1$ and $\mathcal{F}_2$.
11. Verify equation (1.28).
12. Show that the differential equation for the family (1.30) is indeed that in (1.29).
13. Show that the family of cubic curves with general formula $y = Ax^3$ is orthogonal to the family of ellipses with formula $\frac{1}{3}x^2 + y^2 = B$.
14. (a) Find the differential equation for the family of curves with formula $1/x + 1/y = A$.
    (b) Write the differential equation for the orthogonal family.
    (c) Verify that this family has the formula $x^3 - y^3 = B$.
15. (a) Find the differential equation for the family of all circles with centers at $(0, 2)$.
    (b) Find the differential equation for the orthogonal trajectories.
    (c) Use your geometric intuition to identify this family, and verify that it leads to the equation found in part (b).

1.5 Theory versus Experiment

We have seen that the task of formulating a mathematical model for a physical phenomenon (the motion of a pendulum, for example) can lead to a differential equation. Different ways of approaching the phenomenon, different physical approximations, may result in different models. In the pendulum example of Section 1.1, we arrived at the equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{1.31}$$

In order to answer some of the questions raised about the motion of the pendulum, we must find a solution of this equation satisfying the initial conditions $\theta = \theta_0$, $d\theta/dt = 0$, when $t = 0$.

It would certainly be disconcerting if there were to be no such solution! Since an actual pendulum really moves, this would mean that our model is quite useless, and we would have to construct a new model. This means that the only useful models are those which have solutions; an important area of mathematical study has to do with proving that certain very general classes of differential equations do have solutions, even when we do not have any techniques for finding these solutions.

Nor is the existence of solutions the only reasonable requirement that we might wish for in constructing a differential equation that is to lead to a useful model. Suppose that we take an actual pendulum, displace it an angle θ_0, and release it from rest. Experience suggests that if we could repeat this experiment exactly, we would obtain precisely the same motion. The time of descent of the bob and the speed attained would be the same each time. This is really a philosophical assumption, connected with the hypothesis of determinism. If we accept this assumption, with regard to a mathematical model for the motion of a pendulum it means that a differential equation such as (1.31) should have exactly one solution satisfying a given set of initial conditions. If ϕ and ψ are solutions of (1.31) for which $\phi(0) = \psi(0) = \theta_0$ and $\phi'(0) = \psi'(0) = 0$, then we must have $\phi(t) = \psi(t)$ for all $t > 0$. The model must also predict that repetitions of the same experiment have the same results. The corresponding mathematical study is concerned with the uniqueness of solutions of differential equations which have assigned initial conditions or boundary condition.

Finally, experience also suggests a third requirement for models. We know that in practice we cannot repeat an experiment in exactly the same way. However, if all the initial conditions are almost the same, then we expect the outcomes to be almost the same. If the pendulum is released at an angle very nearly θ_0, then the time of descent ought to be very nearly that obtained when the angle was exactly θ_0. The solutions of our model ought to have the same behavior. Stating this in mathematical language, we say that the solutions of a differential equation for a physical situation ought to depend in a continuous way upon the numerical values imposed in the initial conditions. For brevity, we refer to this as "continuity of the solution."

We digress for a moment to observe that there are areas of modern physics where it has seemed necessary to relax some of these assumptions in order to build adequate models. Especially at the atomic and nuclear level, there seem to be phenomena that may violate both the determinism and the continuity hypotheses.

Summarizing the three mathematical requirements that seem desirable for a differential equation used to model a physical situation, we have

1. *Existence:* The equation has a solution satisfying the given set of initial conditions.

2. *Uniqueness:* Two solutions of the equation that satisfy the same initial conditions are identical.
3. *Continuity:* Solutions depend continuously on the initial conditions.

Much of the research of mathematicians who study the theory of differential equations has dealt with the effort to show that wider and wider classes of equations satisfy these three requirements. For example, they have succeeded in showing that equations so complicated that there is no known method for finding formulas for their solutions satisfy all three requirements. This accomplishment is particularly important for applications; faced with such an equation, an applied mathematician can turn to a high-speed computer and calculate an approximation to the solution, knowing by requirements 2 and 3 that there is only one exact solution and that his values must be close to the true values.

We shall explain the meaning of certain of the basic theorems of this type without attempting to prove them now. The first, dealing with first-order equations, is the following:

THEOREM 1.1 *Let D be an open convex set in the TY-plane. Let $f(t, y)$ be defined for all (t, y) in D, and suppose that f and its partial derivative f_y are continuous in D. Let $p_0 = (t_0, y_0)$ be any interior point of D. Then the differential equation $dy/dt = f(t, y)$ has a unique solution $y = \phi(t)$ which satisfies the initial condition $\phi(t_0) = y_0$. Its graph passes through p_0, and ϕ is defined on an interval I containing t_0. Moreover, if t is in I, then the value of $\phi(t)$ is a continuous function of the initial conditions (that is, of the point p_0.)*

A **convex** set in the plane is one that contains the line segment joining any two points of the set. For example, the set of points inside an ellipse is convex. An **interior point** of D is one that is the center of a disk consisting entirely of points of D; this will not be true of a point located on the boundary of a set.

To give a simple illustration of how Theorem 1.1 can be used, consider the equation

$$\frac{dy}{dt} = t^2 + te^{-y} \tag{1.32}$$

Here, we have $f(t, y) = t^2 + te^{-y}$, which is defined and continuous in the whole TY-plane. Since $f_y(t, y) = 0 - te^{-y}$, which is also continuous everywhere, we conclude that (1.32) can be solved uniquely to satisfy any given initial condition. For example, there is exactly one solution of (1.32) that satisfies the condition that y shall be 103 when $t = 1.71$.

When would an equation fail to satisfy the hypotheses of this theorem? The standard example is the equation

$$\frac{dy}{dt} = 3y^{2/3} \tag{1.33}$$

Here, $f(t, y) = 3y^{2/3}$, which is continuous for all t and y. However, $f_y(t, y) = 2y^{-1/3} = 2/\sqrt[3]{y}$. This is not continuous or even defined at points (t, y) with $y = 0$. Thus, the theorem would not apply if we were trying to discuss solutions of (1.33) on a region D that contained any points of the T-axis. What makes this equation interesting is that although it has solutions for any given initial condition, the uniqueness condition breaks down completely. If the point $p_0 = (t_0, y_0)$ is not on the horizontal T-axis, then there is exactly one solution curve passing through p_0. However, if p_0 is of the form $(t_0, 0)$, then there are infinitely many solutions of the equation that pass through p_0.

For example, if $p_0 = (0, 0)$, then two solutions are easily found. Both $y = 0$ for all t and $y = t^3$ for all t are solutions, and both fulfill the requirement $y = 0$ when $t = 0$. Less obvious is that each of the functions defined by

$$y = \begin{cases} 0 & -\infty < t < b \\ (t-b)^3 & b \le t < \infty \end{cases} \tag{1.34}$$

is a continuous function having a continuous derivative satisfying the equation (1.34) and the assigned initial condition. (See Figure 1.17.)

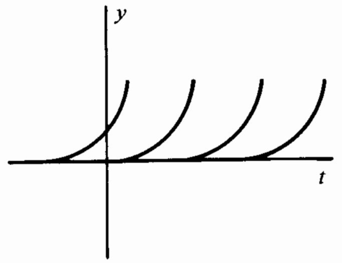

Figure 1.17

This example shows that uniqueness is not automatic, and that an adequate theory of differential equations is needed to enable us to know conditions which will ensure that solutions for assigned initial conditions are unique.

A similar result can be stated for second-order equations.

THEOREM 1.2 *Let D be a convex set in 3-space. Let f and its partial derivatives be defined and continous in D. Let (t_0, A, B) be an interior point of D. Then, the differential equation $d^2y/dt^2 = f(t, y, y')$ has a unique solution $y = \phi(t)$ which satisfies the initial conditions $y = \phi(t_0) = A$, $y' = \phi'(t_0) = B$. The function ϕ is defined on an interval I containing t_0, and if $t \in I$, then the value of $\phi(t)$ depends continuously on the initial values A and B.*

To illustrate this, consider the following equation, which is a model for a pendulum swinging in a resisting medium such as oil:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta - k\left(\frac{d\theta}{dt}\right)^2 \tag{1.35}$$

If we set $y = \theta$ and $y' = d\theta/dt$, then this takes the standard form $y'' = f(t, y, y') = -(g/L)\sin y - k(y')^2$. To check the hypotheses, we compute the partial derivatives of f. Since we are dealing with a complicated situation, it is convenient to introduce another notation for partial derivatives that has many advantages, and use f_1, f_2, and f_3 to indicate the function obtained by differentiating f with respect to the first variable, the second variable, or the third variable, respectively. In our particular case, we have

$$\begin{aligned} f_1(t, y, y') &= 0 \\ f_2(t, y, y') &= -\frac{g}{L}\cos y \\ f_3(t, y, y') &= -2ky' \end{aligned} \tag{1.36}$$

Since all these are continuous everywhere, we conclude that equation (1.35) has a unique solution corresponding to any assignment of initial conditions.

Theorems 1.1 and 1.2 are commonly called the **Existence and Uniqueness theorems**. Their proofs are not particularly difficult but do involve some ideas that are often taken up in advanced calculus, and so we have postponed their treatment to Section 3.6. It should be remarked that the hypotheses placed on the function f are unnecessarily strong, and better theorems have been discovered which show that existence and uniqueness hold even under weaker requirements on f. (The example $y' = 3y^{2/3}$ should remind you that *some* restrictions are needed.)

As has been mentioned earlier, once it is known that a particular differential equation has exactly one solution obeying a given set of initial conditions, then one may have much more confidence in using approximate numerical methods or other techniques that describe general properties of the solution, especially in those cases where there is no known method for obtaining formulas for the exact solutions.

It should also be stated that there are interesting physical problems which give rise to differential equations that do not satisfy the hypotheses of Theorems 1.1 and 1.2, and which in fact do not have unique solutions.

Exercises

1. Apply Theorem 1.1 to the equation $dy/dt = 1+y^2$. What is the region D in which one might expect to find solutions? Are there any points in this region which should not be used as initial points?

2. Do the same as in Exercise 1 for the equation $dy/dt = \sqrt{y^2-1}$. (Note that $y = \phi(t) \equiv 1$ is a solution.)

3. Check that the equations (1.34) do provide solutions of the differential equation $dy/dt = 3y^{2/3}$ and that Figure 1.17 is correct.

4. Use Theorem 1.2 to discuss the existence of solutions to the equation $d^2y/dt^2 = a(t)+b(t)\,y+c(t)(dy/dt)$.

5. Do the same as in Exercise 4 for the equation

$$y'' = \frac{y}{4-(y')^2}$$

   Are there any initial conditions that cannot be expected to be usable for solutions?

6. Do the same as in Exercise 4 for the equation $y'' = (t-y)/(t-y')$.

7. Speculative question: Suppose a differential equation is a model for a certain type of chemical reaction. Could the fact that the equation does not have a solution indicate that the reaction cannot take place? Would the fact that the equation has a solution guarantee that the reaction *does* take place?

8. Why can't the family of curves shown in Figure 1.18 be the solution curves for the differential equation $y' = f(t, y)$, where f is a polynomial in t and y?

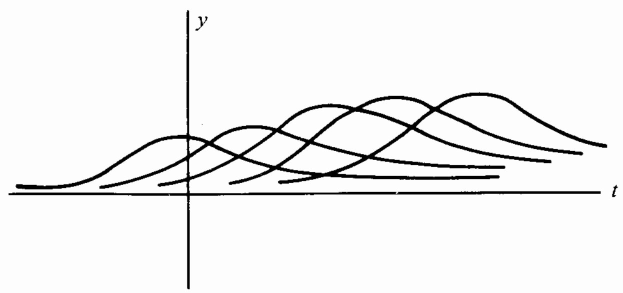

Figure 1.18

9. The parabola $y = x^2$ and the line $y = 2x-1$ are both solutions of the equation $y' = 2x-2\sqrt{x^2-y}$ and obey the initial conditions $x = 1$, $y = 1$. Does this contradict Theorem 1.1?