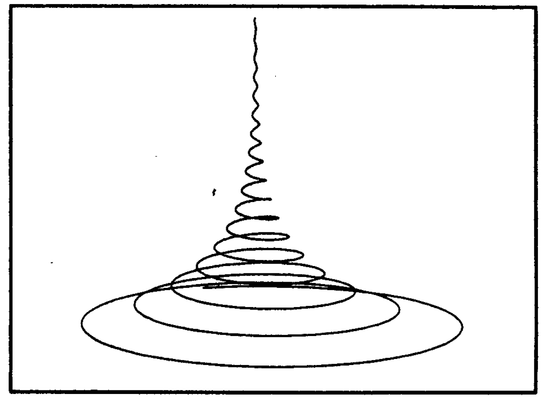
CHAPTER 4



Second-Order Equations

We are ready to move on to differential equations of higher order. We will start with those of second order. These are especially important since so many of the equations that arise in science and engineering are of second order.

In Section 4.3 we will completely solve homogeneous, second-order, linear equations with constant coefficients, and in Section 4.4 we will apply this to the analysis of harmonic motion. We then examine methods of solution for inhomogeneous equations and apply these methods to the study of forced harmonic motion.

1 Definitions and Examples

A second-order differential equation is an equation involving the independent variable t and an unknown function y along with its first and second derivatives. We will assume it is possible to solve for the second derivative, in which case the equation has the form

$$y'' = f(t, y, y').$$

A solution to such an equation is a twice continuously differentiable function $y(t)$ such that

$$y''(t) = f(t, y(t), y'(t)).$$

Many problems in physics give rise to models that are second-order equations. For example, in the study of motion, almost all models start with Newton's second law,

$$F = ma.$$

Here we are modeling the displacement $x(t)$ of a body from some reference point. The derivative of x is the velocity v and the second derivative is the acceleration

a. The force F acting on the body is usually a function of time t, the displacement x, and the velocity v, or $F = F(t, x, v)$. Thus Newton's second law gives us the differential equation

$$m\frac{d^2x}{dt^2} = F(t, x, dx/dt),$$

an equation of second order. We will discuss an important example in this section.

Linear equations

We will spend most of our time discussing ***linear equations***. These have the special form

$$y'' + p(t)y' + q(t)y = g(t). \tag{1.1}$$

The ***coefficients*** p, q, and g can be arbitrary functions of the independent variable t, but y, y', and y'' must all appear to first order. This means we do not allow products of these to occur, nor any powers higher than 1, nor any complicated functions like $\cos y'$. For example, when the coefficients p and q are positive constants, equation (1.1) is the equation for the harmonic oscillator, which we will derive later in this section. On the other hand, the equation for the angular displacement of a pendulum bob is

$$\theta'' + k\sin\theta = 0.$$

Because of the $\sin\theta$ term this equation is nonlinear.

The function $g(t)$ on the right side of equation (1.1) is called the ***forcing term*** since it often arises from an external force. For example, such is the case in the equation of the harmonic oscillator. If the forcing term is equal to 0, the resulting equation is said to be ***homogeneous***. Thus the equation

$$y'' + p(t)y' + q(t)y = 0. \tag{1.2}$$

will be called the homogeneous equation associated to (1.1).

An example—the vibrating spring

An important example of a second-order differential equation occurs in the model of the motion of a vibrating spring. The mathematical principles behind the vibrating spring appear in many areas of science and engineering. The differential equation that we derive here is the paradigm of oscillatory behavior.

The situation is illustrated in Figure 1. We consider the spring suspended from a beam. In Figure 1(a) we see the spring with no mass attached. It is assumed to be in equilibrium, so there is no motion. This is called the ***spring equilibrium***. The position of the bottom of the spring is the reference point from which we measure displacement, so it corresponds to $x = 0$. We will orient our measurements by making x positive below the spring equilibrium.

In Figure 1(b) we have attached a weight of mass m to the spring. This weight has stretched the spring until it is once more in equilibrium at $x = x_0$. This is called the ***spring-mass equilibrium***. At this point there are two forces acting on the mass. There is the force of gravity mg, and there is the restoring force of the spring, which we denote by $R(x)$ since it depends on the distance x that the spring is stretched.

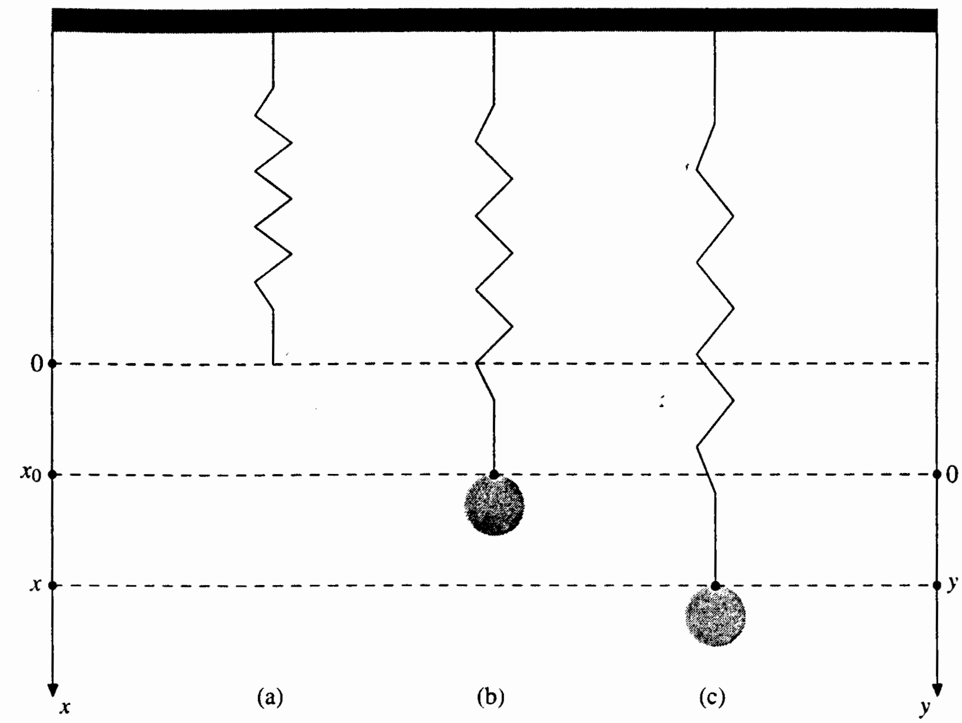


Figure 1 Analysis of a vibrating spring.

The fact that we have equilibrium at $x = x_0$ means that the total force on the weight is 0,

$$R(x_0) + mg = 0. \tag{1.3}$$

In Figure 1(c) we have stretched the spring further. The weight is no longer in equilibrium so it is most likely moving. Its velocity is

$$v = x'. \tag{1.4}$$

Now, in addition to gravity and the restoring force, there is a damping force D, which is the resistance to the motion of the weight due to the medium (air?) through which the weight is moving and perhaps to something internal to the spring. The damping force depends on a lot of factors, such as the shape of the body, but the major dependence is on the velocity. Hence we will write it as $D(v)$. To be complete, we will allow for an external force $F(t)$ as well.

Let $a = v' = x''$ denote the acceleration of the weight. According to Newton's second law,

$$\begin{aligned} ma &= \text{total force acting on the weight} \\ &= R(x) + mg + D(v) + F(t). \end{aligned}$$

Since $a = x''$, and $v = x'$, we get the second-order differential equation

$$mx'' = R(x) + mg + D(x') + F(t). \tag{1.5}$$

To discover the form of the restoring force we resort to experimentation. For many springs, it turns out that the restoring force is proportional to the displacement. This experimental fact is referred to as Hooke's law. It says that

$$R(x) = -kx. \tag{1.6}$$

We use the minus sign because the restoring force is acting to decrease the displacement, and this allows us to say that the ***spring constant*** k is positive. It is important to realize that Hooke's law is an experimental fact. There are some springs for which it is not true, even for small displacements. It is not true for a bungee cord, for example. Furthermore, for any spring Hooke's law is valid only for small displacements. Assuming Hooke's law, (1.5) becomes

$$mx'' = -kx + mg + D(x') + F(t). \tag{1.7}$$

Assuming, for the moment, that there is no external force and that the weight is at spring-mass equilibrium where $x = x_0$, and $x' = x'' = 0$, then the damping force is $D = 0$, and we have (see (1.3))

$$0 = R(x_0) + mg = -kx_0 + mg \quad \text{or} \quad mg = kx_0. \tag{1.8}$$

EXAMPLE 1.9 ◆ In an experiment, a 4 kg weight is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 49 cm. What is the spring constant?

This example will give us an opportunity to discuss units. We will use the ***International System***, in which the unit of length is the meter (abbreviated m), that of time is the second (abbreviated s), and the unit of mass is the kilogram (abbreviated kg). Other units are derived from these. For example, the unit for velocity is m/s, and that for acceleration is m/s^2. Thus, the acceleration due to gravity near the earth is $g = 9.8\,\text{m/s}^2$. According to this system, the unit for force is $\text{kg}\cdot\text{m/s}^2$, but this is called a Newton (abbreviated N).

We can determine the spring constant in our example by solving (1.8) for k,

$$k = \frac{mg}{x_0}.$$

According to our data, the mass $m = 4\,\text{kg}$, and $x_0 = 49\,\text{cm} = 0.49\,\text{m}$. Using $g = 9.8\,\text{m/s}^2$, we find that the spring constant is

$$k = \frac{4\,\text{kg} \times 9.8\,\text{m/s}^2}{0.49\,\text{m}} = 80\,\text{kg/s}^2.$$ ◆

Using (1.8) to substitute for mg in equation (1.7), it becomes

$$mx'' = -k(x - x_0) + D(x') + F(t).$$

This motivates us to introduce the new variable $y = x - x_0$. Notice that y is the displacement of the weight from the spring-mass equilibrium (see Figure 1). Since $y' = x'$ and $y'' = x''$, our equation becomes

$$my'' = -ky + D(y') + F(t). \tag{1.10}$$

The damping force $D(v)$ always acts against the velocity. Hence we can write it as

$$D(v) = -\mu v, \tag{1.11}$$

where $\mu = \mu(v)$ is a non-negative function of the velocity. Again it is an experimental fact that for objects of many shapes and for small velocities, the damping force is proportional to the velocity. In such cases μ is a nonnegative constant, called the ***damping constant***. Again there are examples when the dependence of D on v is more complicated.

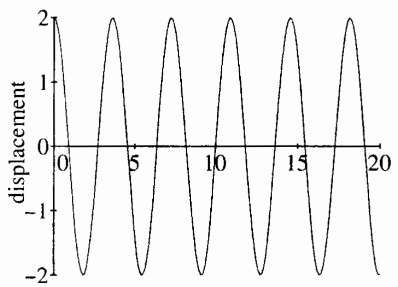


Figure 2 A vibrating spring with no damping.

If we use (1.11) in equation (1.10) it becomes

$$my'' = -ky - \mu y' + F(t) \tag{1.12}$$

or

$$my'' + \mu y' + ky = F(t). \tag{1.13}$$

If μ is a constant, this is a second-order, linear differential equation for the displacement $y(t)$. We can compute the solution numerically. With $k = 3$, $m = 1$, no damping ($\mu = 0$), and no external force ($F(t) = 0$), a solution is plotted in Figure 2. In Figures 3 and 4, we see the results with damping—in Figure 3 $\mu = 0.4$, and in Figure 4 $\mu = 4$.

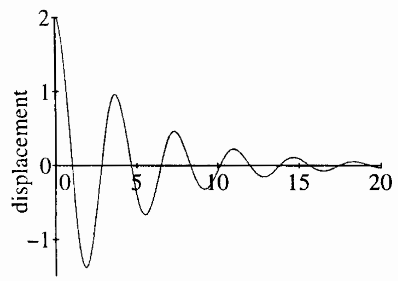


Figure 3 A vibrating spring with small damping.

Let's look at the case when the spring is undamped ($\mu = 0$) and unforced ($F(t) = 0$). Then our equation reduces to

$$y'' = -\frac{k}{m}y. \tag{1.14}$$

Can we solve this equation? We will develop systematic methods to solve such equations later, but for now we can only use our knowledge of calculus. It might help to look at the solution plotted in Figure 2. To solve equation (1.14) we must find a function whose second derivative is a negative multiple of itself. When we think in those terms (and look at Figure 2) we are led to consider the sine and cosine. With a little experimentation we can discover that

$$\cos\left(\sqrt{k/m}\,t\right) \quad \text{and} \quad \sin\left(\sqrt{k/m}\,t\right)$$

are solutions to (1.14). In fact, direct substitution shows that any function of the form

$$y(t) = a\cos\left(\sqrt{k/m}\,t\right) + b\sin\left(\sqrt{k/m}\,t\right) \tag{1.15}$$

where a and b are constants is a solution to (1.14).

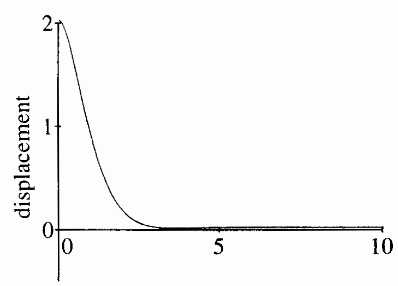


Figure 4 A vibrating spring with large damping.

From (1.15) we see that our solutions are periodic functions. If we introduce the ***natural frequency***

$$\omega_0 = \sqrt{k/m},$$

the solution can be written as

$$y(t) = a\cos\omega_0 t + b\sin\omega_0 t. \tag{1.16}$$

If

$$T = 2\pi/\omega_0 = 2\pi\sqrt{m/k}$$

$$\cos\omega_0(t+T) = \cos(\omega_0 t + \omega_0 T) = \cos(\omega_0 t + 2\pi) = \cos(\omega_0 t)$$

and the same is true for $\sin\omega_0(t+T)$. Hence $y(t+T) = y(t)$, so y is periodic and T is the ***period***.

Existence and uniqueness

The existence and uniqueness results for second-order equations are very similar to those for first-order equations. We will state a result for linear equations, which we will find quite useful.

THEOREM 1.17 Suppose the functions $p(t)$, $q(t)$, and $g(t)$ are continuous on the interval (α, β). Let t_0 be any point in (α, β). Then for any real numbers y_0 and y_1 there is one and only one function $y(t)$ defined on (α, β), which is a solution to

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{for } \alpha < t < \beta,$$

and satisfies the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$.

The major difference between this result and the corresponding theorem for first-order linear equations in Section 7 of Chapter 2 is that an initial condition is needed not only for the function y, but also for its derivative y'. It is important to notice that we can be sure that a solution exists, and furthermore that it exists over the entire interval where the coefficients are defined and continuous.

Structure of the general solution

We will use Theorem 1.17 to find the form of the general solution to a homogeneous linear equation. It is based on the following result, which is the defining feature of linearity.

PROPOSITION 1.18 Suppose that y_1 and y_2 are both solutions to the equation

$$y'' + p(t)y' + q(t)y = 0. \tag{1.19}$$

Then the function $y = C_1y_1 + C_2y_2$, is also a solution to (1.19) for any constants C_1 and C_2.

Proof We notice that $y' = C_1y_1' + C_2y_2'$ and $y'' = C_1y_1'' + C_2y_2''$. Consequently, by simply rearranging the terms we get

$$\begin{aligned} y'' + py' + qy &= \left(C_1y_1'' + C_2y_2''\right) + p\left(C_1y_1' + C_2y_2'\right) + q\left(C_1y_1 + C_2y_2\right) \\ &= C_1\left(y_1'' + py_1' + qy_1\right) + C_2\left(y_2'' + py_2' + qy_2\right) \\ &= 0. \end{aligned}$$

As an example, direct substitution will show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are solutions to the linear equation $y'' - y' - 2y = 0$. In light of Proposition 1.18, we know that any linear combination $y(t) = C_1e^{-t} + C_2e^{2t}$ is also a solution. Again this can be checked by direct substitution.

The general linear combination, $y = C_1y_1 + C_2y_2$, of two solutions y_1 and y_2, contains two arbitrary constants. One might be led to expect that this is the general solution. This is often the case, but not always. This will be the content of our main theorem. However, to state it we need some terminology.

DEFINITION 1.20 Two functions u and v are said to be ***linearly independent*** if neither is a constant multiple of the other. If one is a constant multiple of the other they are said to be ***linearly dependent***.

Thus, the functions $u(t) = t$ and $v(t) = t^2$ are linearly independent. It is true that $v(t) = tu(t)$, but the factor t is not a constant. On the other hand $u(t) = \sin t$ and $v(t) = -4 \sin t$ are obviously linearly dependent.

We are going to use Theorem 1.17 to prove the following result, which will provide us with our solution strategy for homogeneous equations.

THEOREM 1.21 Suppose that y_1 and y_2 are linearly independent solutions to the equation

$$y'' + p(t)y' + q(t)y = 0. \tag{1.22}$$

Then the general solution to (1.22) is

$$y = C_1 y_1 + C_2 y_2,$$

where C_1 and C_2 are arbitrary constants.

Theorem 1.21 will be proved after some discussion of the result. We will find it advantageous to define some more terminology.

DEFINITION 1.23 A ***linear combination*** of the two functions u and v is any function of the form

$$w = Au + Bv,$$

where A and B are constants.

With this definition we can express Proposition 1.18 by saying that a linear combination of two solutions is also a solution. Theorem 1.21 says that the general solution is the general linear combination of the solutions y_1 and y_2, provided that y_1 and y_2 are linearly independent. Because of this result we will say that two linearly independent solutions form a ***fundamental set of solutions***.

Notice that Theorem 1.21 defines a strategy to be used in solving homogeneous equations. It says that it is only necessary to find two linearly independent solutions to find the general solution. That is what we will do in what follows.

EXAMPLE 1.24 ◆ Find a fundamental set of solutions to the equation for simple harmonic motion,

$$x'' + \omega^2 x = 0.$$

It can be shown by substitution that

$$x_1(t) = \cos \omega t \quad \text{and} \quad x_2(t) = \sin \omega t$$

are solutions. (See equation (1.14) and what follows.) It is clear that these functions are not multiples of each other, so they are linearly independent. It follows from

Theorem 1.21 that x_1 and x_2 are a fundamental set of solutions. Therefore, every solution to the equation for simple harmonic motion is a linear combination of x_1 and x_2. ◆

To prove Theorem 1.21, we need to know a little more about the impact of linear independence. The best way to determine if two given functions are linearly independent is by simple observation. For example, in Example 1.24 it is pretty obvious that $\cos \omega t$ and $\sin \omega t$ are not multiples of each other and therefore are linearly independent. However, we will need a way of making this determination in more difficult cases. The ***Wronskian*** of two functions u and v is defined to be

$$W(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t).$$

The relationship of the Wronskian to linear independence is summed up in the next two propositions.

PROPOSITION 1.25 Suppose the functions u and v are solutions to the linear, homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then the Wronskian of u and v is either identically equal to zero on (α, β) or it is never equal to zero there.

Proof To prove this result, we differentiate the Wronskian $W = uv' - vu'$. We get

$$W' = u'v' + uv'' - v'u' - vu'' = uv'' - vu''.$$

Since u and v are solutions to $y'' + py' + qy = 0$, we can solve for their second derivatives and substitute. We get

$$\begin{aligned} W' &= u\left(-pv' - qv\right) - v\left(-pu' - qu\right) \\ &= -p(uv' - vu') \\ &= -pW. \end{aligned}$$

This is a separable first-order equation for W. If t_0 is a point in (α, β), the solution is

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)\,ds} \quad \text{for } \alpha < t < \beta.$$

If $W(t_0) = 0$, then $W(t) = 0$ for $\alpha < t < \beta$. On the other hand, if $W(t_0) \neq 0$, then $W(t) \neq 0$, since the exponential term is never zero.

Consider the solutions $x_1(t) = \cos \omega t$ and $x_2(t) = \sin \omega t$ we found in Example 1.24. The Wronskian of x_1 and x_2 is

$$W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t) = \omega \cos^2 \omega t + \omega \sin^2 \omega t = \omega.$$

Thus for these two solutions the Wronskian is never equal to zero. This is always the case for a fundamental set of solutions, as we will prove in the next result.

PROPOSITION 1.26 Suppose the functions u and v are solutions to the linear, homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1.27}$$

in the interval (α, β). Then u and v are linearly dependent if and only if their Wronskian is identically zero in (α, β).

Proof Suppose first that u and v are linearly dependent in (α, β). Then one of them is a constant multiple of the other. Suppose $u = Cv$. Then $u' = Cv'$ as well, so

$$W(t) = u(t)v'(t) - v(t)u'(t) = Cv(t)v'(t) - v(t)Cv'(t) = 0.$$

Conversely, suppose that $W(t) = 0$ for $\alpha < t < \beta$. It remains to show that u and v are linearly dependent. First, if $v(t) = 0$ for $\alpha < t < \beta$, then $v = 0u$, so u and v are linearly dependent. Suppose, therefore, that v is not identically equal to 0 on (α, β). Suppose that $v(t_1) \neq 0$. Since v is continuous, there is an interval (c, d) containing t_1 on which $v \neq 0$. On this interval we have

$$\frac{d}{dt}\frac{u}{v} = \frac{u'v - uv'}{v^2} = \frac{-W}{v^2} = 0.$$

Hence, on the interval (c, d), u/v is equal to a constant C, or $u = Cv$. In particular, at t_1 we have $u(t_1) = Cv(t_1)$ and $u'(t_1) = Cv'(t_1)$. By Proposition 1.18 both u and Cv are solutions to the differential equation $y'' + py' + qy = 0$. Since they have the same initial conditions at t_1, it follows from Theorem 1.17 that $u = Cv$ everywhere in (α, β). Consequently, u and v are linearly dependent.

Let's restate the results of these two propositions to highlight the points we will need.

PROPOSITION 1.28 Suppose the functions u and v are solutions to the linear, homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). If $W(t_0) \neq 0$ for some t_0 in the interval (α, β), then u and v are linearly independent in (α, β). On the other hand, if u and v are linearly independent in (α, β), then $W(t)$ never vanishes in (α, β).

Proof If $W(t_0) \neq 0$ for some t_0 in the interval (α, β), then by Proposition 1.26, u and v cannot be linearly dependent. Hence they are linearly independent.

If u and v are linearly independent in (α, β), then by Proposition 1.26 the Wronskian is not identically equal to 0. By Proposition 1.25 it is never equal to 0 in (α, β).

Now we are ready to prove Theorem 1.21.

Proof of Theorem 1.21 Suppose that y_1 and y_2 are linearly independent solutions to the equation $y'' + py' + qy = 0$, and suppose that $y(t)$ is any solution. We need to find the constants C_1 and C_2 such that $y = C_1y_1 + C_2y_2$. Let t_0 be any point in (α, β). We choose the constants so that

$$\begin{aligned} y(t_0) &= C_1y_1(t_0) + C_2y_2(t_0), \quad \text{and} \\ y'(t_0) &= C_1y_1'(t_0) + C_2y_2'(t_0). \end{aligned} \tag{1.29}$$

This system is solvable provided the determinant

$$\det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0.$$

However, this determinant will be recognized as the Wronskian W of y_1 and y_2. Since y_1 and y_2 are linearly independent, by Proposition 1.28 we know that $W(t_0) \neq 0$. Therefore we can find C_1 and C_2 solving (1.29).

We know that y and $C_1y_1 + C_2y_2$ are both solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$, and by (1.29) they have the same initial conditions at t_0. By the uniqueness part of Theorem 1.17, we have

$$y(t) = C_1y_1(t) + C_2y_2(t) \quad \text{for } \alpha < t < \beta.$$

Initial value problems

Let's give some thought to formulating the initial value problem for the second-order equation $y'' = F(t, y, y')$. We see in Theorem 1.17 that to determine a solution y uniquely it is necessary to specify both $y(t_0)$ and $y'(t_0)$. While the theorem applies only to linear equations, this is true for all second-order equations.

EXAMPLE 1.30 ◆ Find the solution to the equation for simple harmonic motion $x'' + 4x = 0$, with initial conditions $x(0) = 4$ and $x'(0) = 2$.

We know from Example 1.24 that the general solution has the form

$$x(t) = a\cos 2t + b\sin 2t,$$

where a and b are arbitrary constants. Substituting the initial conditions we get

$$4 = x(0) = a, \quad \text{and} \quad 2 = x'(0) = 2b.$$

Thus, $a = 4$ and $b = 1$ and our solution is

$$x(t) = 4\cos 2t + \sin 2t.$$

◆

EXERCISES

In Exercises 1–4, show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify, again by direct substitution, that any linear combination of the two given solutions is also a solution.

1. $y'' - y' - 6y = 0, \quad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}$
2. $y'' + 4y = 0, \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
3. $y'' - 2y' + 2y = 0, \quad y_1(t) = e^t\cos t, \quad y_2(t) = e^t\sin t$
4. $y'' + 4y' + 4y = 0, \quad y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t}$

In Exercises 5–8, use Definition 1.20 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the given differential equation. In addition, calculate the Wronskian and use it to explain the independence of the given solutions.

5. $y'' - y' - 2y = 0, \quad y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}$
6. $y'' + 9y = 0, \quad y_1(t) = \cos 3t, \quad y_2(t) = \sin 3t$
7. $y'' + 4y' + 13y = 0, \quad y_1(t) = e^{-2t}\cos 3t, \quad y_2(t) = e^{-2t}\sin 3t$
8. $y'' + 6y' + 9y = 0, \quad y_1(t) = e^{-3t}, \quad y_2(t) = te^{-3t}$
9. Show that the functions

$$y_1(t) = t^2 \qquad \text{and} \qquad y_2(t) = t|t|$$

are linearly independent on $(-\infty, +\infty)$. Next, show that the Wronskian of the two functions is identically zero on the interval $(-\infty, +\infty)$. Why doesn't this result contradict Proposition 1.26?

10. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0$, then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.
11. Show that $y_1(t) = \cos 4t$ and $y_2(t) = \sin 4t$ form a fundamental set of solutions for $y'' + 16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.
12. Show that $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions for $y'' + 2y' + 5y = 0$, then find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.
13. Show that $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ form a fundamental set of solutions for $y''+8y'+16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.
14. Unfortunately, Theorem 1.21 does not show us how to find two independent solutions. However, there is a technique that can be used to find a second solution when one solution is known.

(a) Show that $y_1(t) = t^2$ is a solution of

$$t^2y'' + ty' - 4y = 0. \tag{1.31}$$

(b) Let $y_2(t) = vy_1(t) = vt^2$, where v is a yet to be determined function of t. Note that if $y_2/y_1 = v$, and v is nonconstant, then y_1 and y_2 are independent. Show that the substitution $y_2 = vt^2$ reduces equation (1.31) to the separable equation

$$5v' + tv'' = 0. \tag{1.32}$$

Solve equation (1.32) for v, form the solution $y_2 = vt^2$, then state the general solution of equation (1.31).

Use the technique shown in Exercise 14 to find the general solution of the second order equations in exercises 15–18.

15. $t^2y'' - 2ty' + 2y = 0, \quad y_1(t) = t$
16. $t^2y'' + ty' - y = 0, \quad y_1(t) = t$
17. $t^2y'' - 3ty' + 3y = 0, \quad y_1(t) = t$
18. $t^2y'' + 4ty' + 2y = 0, \quad y_1(t) = 1/t$

.2 Second-Order Equations and Systems

A ***planar system*** of first-order equations is a set of two first-order differential equations involving two unknown functions. It might be written as

$$x' = f(t, x, y)$$
$$y' = g(t, x, y),$$

where f and g are functions of the independent variable t and the two unknowns x and y.

There is a close connection between higher-order equations and first-order systems. In this section, we will begin to explore that connection. Then we will explore ways to visualize solutions to second-order equations. One of those ways involves the use of the phase plane, which is really a way of visualizing solutions to planar systems of equations. We will begin a systematic study of first-order systems in Chapter 8.

Second-order equations and planar systems

Let's look again at the second order equation

$$y'' = F(t, y, y'). \tag{2.1}$$

If we introduce a new variable $v = y'$, then, in terms of v, equation (2.1) can be written $v' = F(t, y, v)$. We see that the functions y and v are related by the system of first-order equations

$$\begin{aligned} y' &= v \\ v' &= F(t, y, v) \end{aligned} \tag{2.2}$$

If y is a solution to the second-order equation (2.1), and we set $v = y'$, then the pair of functions y and v solve the first-order system (2.2). The converse is also true. If the pair of functions y and v solve the first-order system (2.2), then y is a solution to the second-order equation (2.1). To see this, notice that by the first equation in (2.2), $y' = v$. Differentiating this and using the second equation in (2.2) we get

$$y'' = v' = F(t, y, v) = F(t, y, y'),$$

which is equation (2.1).

Thus, the second-order equation (2.1) and the first-order system (2.2) are equivalent in the sense that a solution of either leads to a solution of the other. This fact is important for two reasons. First, when solving higher-order equations numerically, it is often necessary to solve the equivalent first-order system, since numerical methods are typically designed only to solve first-order systems. Second, the equivalence allows us to study first-order systems and deduce from them results about higher-order equations. This procedure will be illustrated when we begin the study of first-order systems in Chapter 8.

EXAMPLE 2.3 ◆ Let's look first at one of the most important examples, the single, second-order, linear equation. The general such equation has the form

$$y'' + p(t)y' + q(t)y = F(t). \tag{2.4}$$

To find an equivalent system we introduce the new variable $v = y'$. Then solving (2.4) for y'' we see that $v' = y'' = F(t) - p(t)y' - q(t)y = F(t) - p(t)v - q(t)y$. Hence, y and v solve the system

$$\begin{aligned} y' &= v \\ v' &= F(t) - p(t)v - q(t)y. \end{aligned}$$ ◆

Visualization of solutions

The simplest and most obvious way to visualize a solution to a second-order differential equation is to graph it. We have already seen examples of this in Figures 2, 3, and 4 in Section 4.1.

Sometimes it is of interest to graph both the solution y and its first derivative, which is especially true if the derivative has physical significance, which is often the case in applications. For example, if we are solving for a displacement y, then $y' = v$ is the velocity. If we are solving for the charge Q on a condenser, then the derivative $Q' = I$ is the current. In both of these cases it might be of interest to see graphs of both the solution and its derivative.

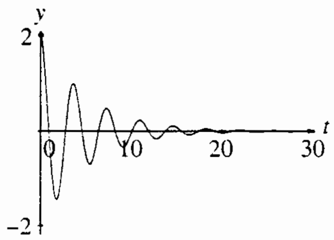


Figure 1 The graph of the displacement.

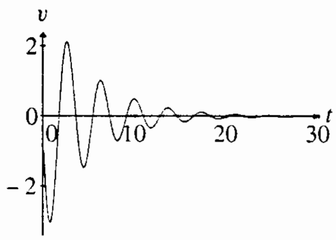


Figure 2 The graph of the velocity.

Let's use as an example a damped unforced spring. The equation is

$$my'' + \mu y' + ky = 0.$$

Let's use $m = 1$, $\mu = 0.4$, and $k = 3$. We will look at the solution y which satisfies the initial conditions $y(0) = 2$ and $v(0) = y'(0) = -1$. The graph of the displacement y alone is shown in Figure 1, and the graph of the velocity alone is in Figure 2. Sometimes it is useful to see both plotted together. This is shown in Figure 3.

Another type of figure that is often useful is the plot of the curve

$$t \to (y(t), v(t))$$

in the yv-plane. The yv-plane is called the ***phase plane***, and this is called a ***phase plane plot***. For our example the phase plane plot is shown in Figure 4. Notice how this curve spirals into the origin. This gives an interesting visual interpretation of the effect of the damping. Missing from a phase plane plot is any indication of the dependence on t. However, it does show nicely the interplay between the displacement and the velocity.

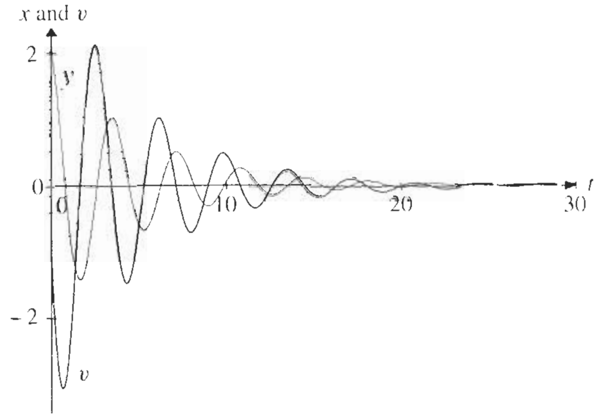


Figure 3 The displacement y and the velocity v.

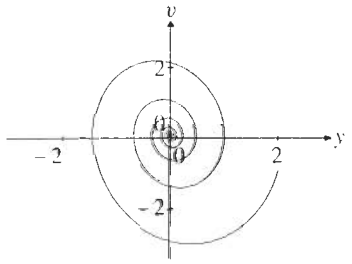


Figure 4 The phase plane plot of y and v.

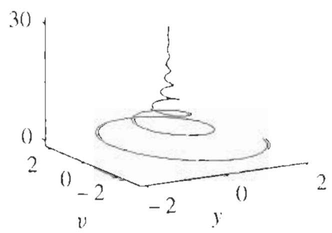


Figure 5 A 3D plot of $t \to (y(t), v(t), t)$.

Also illuminating is a three dimensional (3D) plot of the three variables t, y, and v. These can be plotted in either of the orders

$$t \to (t, y(t), v(t)) \quad \text{or} \quad t \to (y(t), v(t), t).$$

Which is more effective is often a matter of taste. The latter is shown in Figure 5. The 3D plots show the interplay of all three variables. However, some people find them difficult to understand. Often they become clearer simply by changing the orientation slightly.

Finally, there is the ***composite plot***, shown in Figure 6. In this one figure we see plots of y and v versus t, the phase plane plot, and the 3D plot. The relationships among all of these become clearer by looking at this figure. The 3D plot is central to the figure, and it is shown plotted in blue. This curve will be seen to be the same as the curve in Figure 5. The plot of the displacement y versus t is shown on the back. It is the projection of the 3D plot onto the back panel. Similarly, the plot of v versus t is on the right panel, and it too is a projection of the 3D plot. Finally the phase plane plot is shown on the bottom, and, again, it is the projection of the 3D plot onto the bottom panel.

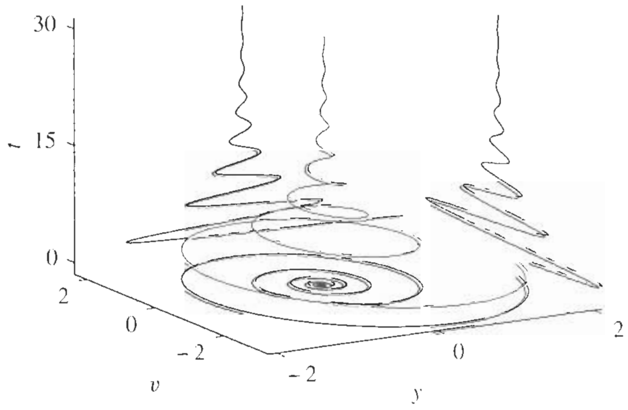


Figure 6 The composite plot of the solution.

We see that there are many possible ways to represent solutions. In particular situations, one of these might be much better than the others. There might also be ad hoc representations that are better than any of those shown here.

EXERCISES

In Exercises 1–6, use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equations (planar system).

1. $y'' + 2y' - 3y = 0$
2. $4y'' + 4y' + y = 0$
3. $y'' + 3y' + 4y = 2\cos 2t$
4. $y'' + 2y' + 2y = \sin 2\pi t$
5. $y'' + \mu(t^2 - 1)y' + y = 0$
6. $y'' + cy' - ay + by^3 = A\cos\omega t$
7. Sometimes the physical situation aids in the selection of substitution variables. For example, in an *LRC* circuit, the current is the derivative of the charge. In the *LRC* circuit governed by the equation
$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$
show that the substitution $I = Q'$ transforms the equation into the planar system
$$Q' = I,$$
$$I' = -\frac{R}{L}I - \frac{1}{LC}Q + \frac{1}{L}E(t).$$
8. In general, when changing a second order equation to a planar system, the choice of variables for substitution is arbitrary. If
$$y'' = 2y' - 3y + 2\cos 3t,$$
show that the substitutions $x_1 = y$ and $x_2 = y'$ lead to the planar system
$$x_1' = x_2,$$
$$x_2' = 2x_2 - 3x_1 + 2\cos 3t.$$

In Exercises 9–16, you are given the mass, damping, and spring constants of an undriven spring-mass system
$$my'' + \mu y' + ky = 0.$$
You are also given initial conditions. Use a numerical solver to

(i) provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t), and

(ii) provide a combined plot of both position and velocity versus time, and

(iii) provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

In each exercise, choose a viewing window that highlights the important features of the solutions.

9. $m = 1$ kg, $\mu = 0$ kg/s, $k = 4$ kg/s, $y(0) = -2$ m, $y'(0) = -2$ m/s

10. $m = 1\,\text{kg}$, $\mu = 0\,\text{kg/s}$, $k = 9\,\text{kg/s}^2$, $y(0) = 3\,\text{m}$, $y'(0) = 2\,\text{m/s}$

11. $m = 1\,\text{kg}$, $\mu = 2\,\text{kg/s}$, $k = 1\,\text{kg/s}^2$, $y(0) = -3\,\text{m}$, $y'(0) = -2\,\text{m/s}$

12. $m = 4\,\text{kg}$, $\mu = 4\,\text{kg/s}$, $k = 1\,\text{kg/s}^2$, $y(0) = 3\,\text{m}$, $y'(0) = 1\,\text{m/s}$

13. $m = 1\,\text{kg}$, $\mu = 0.5\,\text{kg/s}$, $k = 4\,\text{kg/s}^2$, $y(0) = 2\,\text{m}$, $y'(0) = 0\,\text{m/s}$

14. $m = 1\,\text{kg}$, $\mu = 2\,\text{kg/s}$, $k = 1\,\text{kg/s}^2$, $y(0) = -1\,\text{m}$, $y'(0) = -5\,\text{m/s}$

15. $m = 1\,\text{kg}$, $\mu = 3\,\text{kg/s}$, $k = 1\,\text{kg/s}^2$, $y(0) = -1\,\text{m}$, $y'(0) = -5\,\text{m/s}$

16. $m = 1\,\text{kg}$, $\mu = 0.2\,\text{kg/s}$, $k = 1\,\text{kg/s}^2$, $y(0) = -3\,\text{m}$, $y'(0) = -2\,\text{m/s}$

17. Consider carefully the graphs of v and y versus t, shown in Figure 3.

(a) Why do the peaks of the curve $t \to y(t)$ occur where the curve $t \to v(t)$ crosses the t-axis? What is the physical significance of this fact?

(b) Do the peaks of the curve $t \to v(t)$ occur where the curve $t \to y(t)$ crosses the t-axis?

18. Consider carefully the phase plane plot of the spring-mass system given in Figure 4.

(a) What physical configuration of the spring-mass system is represented by the points where the solution curve $t \to (y(t), v(t))$ crosses the y-axis?

(b) What physical configuration of the spring-mass system is represented by the points where the solution curve $t \to (y(t), v(t))$ crosses the v-axis?

(c) What physical significance can be attached to the fact that the solution curve $t \to (y(t), v(t))$ spirals towards the origin with the passage of time?

If your software supports 3D capability, sketch the solution curve $t \to (y(t), v(t), t)$ for the spring-mass system with constants and initial conditions given in the indicated exercise.

19. Exercise 9

20. Exercise 10

21. Exercise 15

22. Exercise 16

In Exercises 23–28, you are given the inductance, resistance, and capacitance of a driven *LRC* circuit

$$LQ'' + RQ' + \frac{1}{C}Q = 2\cos 2t.$$

You are also given initial conditions. Use a numerical solver to

(i) provide separate plots of the charge on the capacitor versus time (Q vs. t) and the current in the circuit versus time (I vs. t), and

(ii) provide a combined plot of both charge and the current versus time, and

(iii) provide a plot of the current versus the charge (I vs. Q) in the QI phase plane.

See Exercise 7 for aid in setting up the system. In each exercise, choose a viewing window that highlights the important features of the solutions.

23. $L = 1\,\text{H}$, $R = 0\,\Omega$, $C = 1\,\text{F}$, $Q(0) = -3\,\text{C}$, $I(0) = -2\,\text{A}$

24. $L = 1\,\text{H}$, $R = 0\,\Omega$, $C = 1/4\,\text{F}$, $Q(0) = 1\,\text{C}$, $I(0) = 2\,\text{A}$

25. $L = 1\,\text{H}$, $R = 5\,\Omega$, $C = 1\,\text{F}$, $Q(0) = 1\,\text{C}$, $I(0) = 2\,\text{A}$

26. $L = 2\,\text{H}$, $R = 4\,\Omega$, $C = 1\,\text{F}$, $Q(0) = -1\,\text{C}$, $I(0) = -2\,\text{A}$

27. $L = 1\,\text{H}$, $R = 0.5\,\Omega$, $C = 1\,\text{F}$, $Q(0) = 1\,\text{C}$, $I(0) = 2\,\text{A}$

28. $L = 1\,\text{H}$, $R = 0.2\,\Omega$, $C = 1\,\text{F}$, $Q(0) = -11\,\text{C}$, $I(0) = -1\,\text{A}$

If your software supports 3D capability, sketch the solution curve $t \to (Q(t), I(t), t)$ for the *LRC* circuit with constants and initial conditions given in the indicated exercise.

29. Exercise 23

30. Exercise 24

31. Exercise 27

32. Exercise 28

Linear, Homogeneous Equations with Constant Coefficients

This a class of equations which we can solve easily. They are equations of the form

$$y'' + py' + qy = 0, \tag{3.1}$$

where p and q are constants. If $p \geq 0$ and $q > 0$, this is the equation for unforced harmonic motion, which we will discuss in the next section. We show there that it includes the equation for the unforced motion of a vibrating spring, and the equation for the behavior of an *RLC* circuit.

Remember the solution strategy we devised in Section 4.1 using Theorem 1.21. It is only necessary to find two linearly independent solutions, which we call a fundamental set of solutions. The general solution is the general linear combination of these.

The key idea

The analogous first-order, linear, homogeneous equation with constant coefficients is the equation

$$y' + py = 0.$$

This is the exponential equation. It is separable and easily solved. Its general solution is

$$y(t) = Ce^{-pt},$$

where C is an arbitrary constant.

Motivated by the fact that the first-order equation has an exponential solution, let's see if we can find an exponential solution to the second-order equation (3.1). We will look for a solution of the type

$$y(t) = e^{\lambda t},$$

where λ is a constant, as yet unknown. Inserting this function into our differential equation, we obtain

$$\begin{aligned} y'' + py' + qy &= \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} \\ &= (\lambda^2 + p\lambda + q)e^{\lambda t}. \end{aligned}$$

Since $e^{\lambda t} \neq 0$, we will have solution to (3.1) if and only if

$$\lambda^2 + p\lambda + q = 0. \tag{3.2}$$

This is called the ***characteristic equation*** for the differential equation in (3.1). The polynomial $\lambda^2 + p\lambda + q$ is called the ***characteristic polynomial*** for the equation. A root of the characteristic equation is called a ***characteristic root***. If λ is a characteristic root, then $y = e^{\lambda t}$ is a solution to the differential equation.

It is illuminating to write the differential equation and its characteristic equation in proximity,

$$\begin{aligned} y'' + py' + qy &= 0, \\ \lambda^2 + p\lambda + q &= 0. \end{aligned}$$

This indicates clearly how to pass from the differential equation to its characteristic equation.

Since the characteristic equation is a quadratic equation, its roots are given by the quadratic formula

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are three cases to consider:

1. two distinct real roots if $p^2 - 4q > 0$;
2. two distinct complex roots if $p^2 - 4q < 0$;
3. one repeated real root if $p^2 - 4q = 0$.

We will look at each of these in what follows. Our way will be guided by Theorem 1.21 in Section 1. We know that if we find a fundamental set of solutions, then the general solution is the general linear combination of these. Thus we need to find two linearly independent solutions.

Distinct real roots

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$ are both solutions. Since the roots are not equal, the solutions are not constant multiples of each other. Hence, they are linearly independent, and by Theorem 1.21 we have the following result.

PROPOSITION 3.3 If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two distinct real roots λ_1 and λ_2, then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},$$

where C_1 and C_2 are arbitrary constants.

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.

EXAMPLE 3.4 ◆ Find the general solution to the equation

$$y'' - 3y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 1$.

Letting $y = e^{\lambda t}$ and inserting this into our differential equation, we obtain

$$0 = y'' - 3y' + 2y = (\lambda^2 - 3\lambda + 2)e^{\lambda t}.$$

Thus, the characteristic equation is

$$0 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1),$$

with solutions $\lambda_1 = 2$ and $\lambda_2 = 1$. According to Proposition 3.3, the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^t. \tag{3.5}$$

To find the particular solution for the initial conditions $y(0) = 2$ and $y'(0) = 1$, we differentiate the general solution,

$$y'(t) = 2C_1 e^{2t} + C_2 e^t. \tag{3.6}$$

Then we substitute $t = 0$ into (3.5) and (3.6) to obtain the system of equations

$$\begin{aligned} 2 &= y(0) = C_1 + C_2 \\ 1 &= y'(0) = 2C_1 + C_2. \end{aligned}$$

The solutions are $C_1 = -1$ and $C_2 = 3$, so the solution to our initial value problem is

$$y(t) = -e^{2t} + 3e^t.$$ ◆

Complex numbers

Before going into complex roots, let's spend a little time reviewing complex arithmetic. A complex number is one of the form $z = x + iy$, where x and y are real numbers. The number i satisfies $i^2 = -1$. In the complex number $z = x + iy$, the real number x is called the ***real part*** of z and is denoted by $x = \operatorname{Re} z$. The real number y is called the ***imaginary part*** of z and is denoted by $y = \operatorname{Im} z$.

Complex addition and multiplication satisfy the usual rules for real numbers. For example,

$$(3 + 5i) + (2 - 3i) = (3 + 2) + (5 - 3)i = 5 + 2i.$$

In multiplication it is necessary to use $i^2 = -1$. For example,

$$\begin{aligned} (3 + 5i) \cdot (4 - 2i) &= 3(4 - 2i) + 5i(4 - 2i) \\ &= 12 - 6i + 20i - 10i^2 \\ &= 12 + 14i + 10 \\ &= 22 + 14i. \end{aligned}$$

The ***complex conjugate*** of the complex number $z = x + iy$ is the number $\bar{z} = x - iy$. Notice that conjugation affects only the imaginary part of the complex number, replacing the imaginary part with its negative. In particular, we see that

$$\bar{z} = z \quad \text{if and only if } z \text{ is a real number.}$$

We can solve the two equations $z = x + iy$ and $\overline{z} = x - iy$ for x and y obtaining

$$x = \operatorname{Re} z = \frac{z + \overline{z}}{2} \quad \text{and} \qquad (3.7)$$
$$y = \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

The process of conjugation preserves algebraic combinations. Suppose that $z = x + iy$ and $w = u + iv$ are complex numbers. Then the following facts are proved directly from the definition of the conjugate.

$$\overline{z + w} = \overline{z} + \overline{w}$$
$$\overline{z - w} = \overline{z} - \overline{w}$$
$$\overline{zw} = \overline{z} \cdot \overline{w}$$
$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

The ***absolute value*** of a complex number $z = x + iy$ is the real number

$$|z| = \sqrt{x^2 + y^2}.$$

The absolute value of a complex number has many uses. It is also called the magnitude of the number, and for many purposes that name is highly illuminating. Notice that

$$z\overline{z} = |z|^2. \qquad (3.8)$$

Formula (3.8) provides the secret to computing the reciprocal of a complex number. We have

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}. \qquad (3.9)$$

For example,

$$\frac{1}{4 - 3i} = \frac{4 + 3i}{|4 - 3i|^2} = \frac{4 + 3i}{25}.$$

Knowing how to compute reciprocals and products, we can also compute quotients. Thus, using (3.9) we have

$$\frac{w}{z} = w \cdot \frac{1}{z} = \frac{w\overline{z}}{|z|^2}.$$

Other important properties of the absolute value follow most easily from (3.8). If z and w are two complex numbers, then

$$|zw| = |z||w| \quad \text{and} \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}. \qquad (3.10)$$

Most people feel more comfortable with complex numbers when they are represented as points in the complex plane. We will identify the complex number $z = x + iy$ with the point in the plane having cartesian coordinates (x, y). This identification is illustrated in Figure 1. Under this identification the x-axis contains

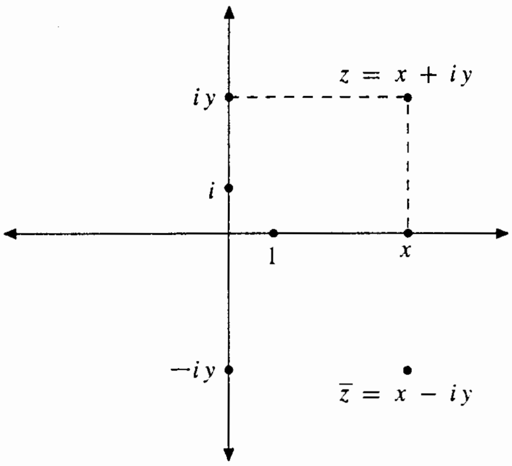


Figure 1 The complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$.

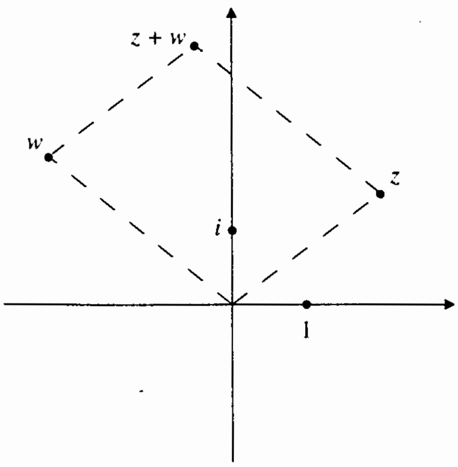
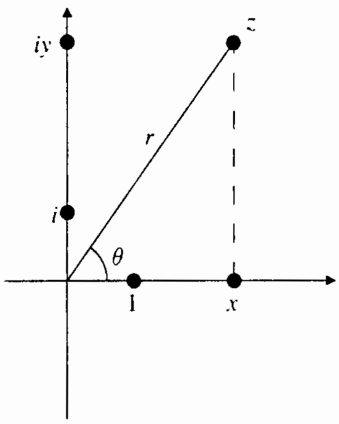


Figure 2 The sum of two complex numbers.

the real numbers and is therefore called the ***real axis***. Similarly the y-axis is called the ***imaginary axis***. Figure 1 also shows the complex conjugate $\bar{z} = x - iy$. This is the reflection of z in the real axis.

Figure 2 provides the geometric interpretation of complex addition. If z and w are complex numbers, then the sum $z + w$ corresponds to the fourth vertex of the parallelogram with the other three vertices at the origin 0, z, and w.



3 Polar coordinates for the ex number $z = x + iy$.

Polar coordinates have an especially appealing interpretation for complex numbers. (See Figure 3.) The complex number $z = x + iy$ has polar coordinates $r \geq 0$ and θ, with $-\pi < \theta \leq \pi$, defined by the standard equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then

$$r = \sqrt{x^2 + y^2} = |z| \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

Hence we can identify r with $|z|$. The angle θ is called the ***argument*** of z. Using the polar coordinates we can write

$$z = r\cos\theta + ir\sin\theta = r[\cos\theta + i\sin\theta]. \tag{3.11}$$

Equation (3.11) takes on a different character after we introduce the complex exponential. For a real number θ we define

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.12}$$

We will refer to this formula as ***Euler's formula***. The definition is motivated by the Taylor series for the function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Inserting $x = i\theta$, and using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right). \end{aligned}$$

The real part is the Taylor series for $\cos\theta$ and the imaginary part is the Taylor series for $\sin\theta$. Thus $e^{i\theta} = \cos\theta + i\sin\theta$.

As a result of Euler's formula, we can rewrite equation (3.11) as

$$z = re^{i\theta}. \tag{3.13}$$

This very concise formula expresses the polar coordinates of a complex number. Its usefulness is enhanced after we learn about the full complex exponential and its properties. We define the exponential of a complex number by assuming that the addition formula for the exponential is still valid. The ***exponential*** of the complex number $z = x + iy$ is defined to be

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y). \tag{3.14}$$

Thus, e^{x+iy} is the complex number with real part $e^x\cos y$ and imaginary part $e^x\sin y$.

The complex exponential satisfies all the familiar rules for real exponents. We set these rules down in the following proposition. The proof is left to the exercises.

PROPOSITION 3.15 The complex exponential satisfies the following properties:

1. $e^{z_1+z_2} = e^{z_1}e^{z_2}$
2. $e^{z_1-z_2} = e^{z_1}/e^{z_2}$
3. $(e^z)^r = e^{rz}$ for any real number r
4. $\overline{e^z} = e^{\bar{z}}$
5. $|e^z| = e^{\operatorname{Re} z}$
6. $\dfrac{d}{dt}\{e^{\lambda t}\} = \lambda e^{\lambda t}$ for any complex number λ

Proof As we said previously, the complete proof will be in the exercises. For now we will only verify the last property in the special case when $\lambda = i$. The derivative of such a complex valued function is computed by differentiating the real and imaginary parts in the ordinary way. We have

$$\begin{aligned}
\frac{d}{dt}\{e^{it}\} &= \frac{d}{dt}\{\cos t + i\sin t\} \quad \text{by definition of } e^{it}\\
&= \frac{d}{dt}\cos t + i\frac{d}{dt}\sin t\\
&= -\sin t + i\cos t\\
&= i(\cos t + i\sin t)\\
&= ie^{it}.
\end{aligned}$$

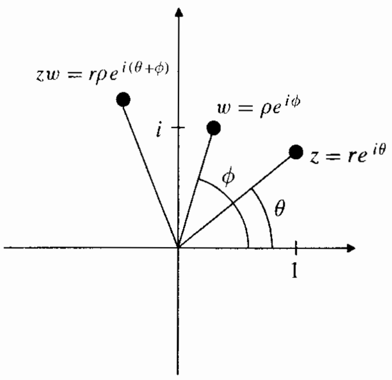


Figure 4 The product of two complex numbers.

Proposition 3.15 allows us to make a geometric interpretation of the product of two complex numbers. Suppose that in polar coordinates we have $z = re^{i\theta}$ and $w = \rho e^{i\phi}$. Then by Part (1) of Proposition 3.15, the product is

$$zw = re^{i\theta}\cdot\rho e^{i\phi} = r\rho e^{i(\theta+\phi)}.$$

We automatically get the polar coordinates of the product. We see that $|zw| = r\rho$ and that the argument of zw is $\theta + \phi$, the sum of the arguments of z and w. This means that the effect of multiplying z by $w = \rho e^{i\phi}$ is to multiply the absolute value of z by $\rho = |w|$, and to rotate z by ϕ, the argument of w. See Figure 4.

Complex roots

If the roots to the characteristic equation (3.2) are complex, then, since the coefficients of the characteristic polynomial are real, the roots are complex conjugates. They have the form $\lambda = a + ib$ and $\overline{\lambda} = a - ib$. The corresponding solutions are

$$\begin{aligned} z(t) &= e^{(a+ib)t} = e^{at}\left(\cos(bt) + i\sin(bt)\right), \quad \text{and} \\ \overline{z}(t) &= e^{(a-ib)t} = e^{at}\left(\cos(bt) - i\sin(bt)\right). \end{aligned} \tag{3.16}$$

Clearly the solutions z and $\overline{z}$ are not multiples of each other, so they are linearly independent. Hence using Theorem 1.21, we see that the general solution is

$$y(t) = C_1 z(t) + C_2 \overline{z}(t). \tag{3.17}$$

The solutions z and $\overline{z}$ are complex valued. Such solutions are often preferred (for example, in circuit analysis). However, we are aiming for real valued solutions.

Notice from (3.16) that z and $\overline{z}$ are complex conjugates. Written in terms of their real and imaginary parts, we have

$$z(t) = y_1(t) + iy_2(t) \quad \text{and} \quad \overline{z}(t) = y_1(t) - iy_2(t), \tag{3.18}$$

where

$$y_1(t) = \operatorname{Re} z(t) = e^{at}\cos(bt) \quad \text{and} \quad y_2(t) = \operatorname{Im} z(t) = e^{at}\sin(bt). \tag{3.19}$$

From (3.7), we have

$$y_1(t) = \frac{1}{2}\left(z(t) + \overline{z}(t)\right) \quad \text{and} \quad y_2(t) = \frac{1}{2i}\left(z(t) - \overline{z}(t)\right).$$

Thus, by Proposition 1.18, $y_1(t)$ and $y_2(t)$, the real and imaginary parts of $z(t)$, are solutions. Furthermore, these are real valued solutions. Clearly they are not constant multiples of each other, so they are linearly independent. Hence by Theorem 1.21, they form a fundamental set of solutions, and the general solution can be written as

$$y(t) = A_1 y_1(t) + A_2 y_2(t) = A_1 e^{at}\cos(bt) + A_2 e^{at}\sin(bt),$$

where A_1 and A_2 are arbitrary constants.

We summarize our discussion in the following proposition.

PROPOSITION 3.20 Suppose the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two complex conjugate roots, $\lambda = a + ib$ and $\overline{\lambda} = a - ib$.

1. The functions

$$z(t) = e^{(a+ib)t} \quad \text{and} \quad \overline{z}(t) = e^{(a-ib)t}$$

form a complex valued fundamental set of solutions, so the general solution is

$$y(t) = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t},$$

where C_1 and C_2 are arbitrary complex constants.

2. The functions

$$y_1(t) = e^{at}\cos(bt) \quad \text{and} \quad y_2(t) = e^{at}\sin(bt)$$

form a real valued fundamental set of solutions, so the general solution is

$$y(t) = e^{at}(A_1 \cos bt + A_2 \sin bt),$$

where A_1 and A_2 are constants.

In either case, the constants can be determined in the usual way to find the solution to an initial value problem.

EXAMPLE 3.21 ◆ Find the general solution to the system

$$y'' + 2y' + 2y = 0.$$

Find the solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 3$.

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. Its roots are $\lambda = -1 \pm i$. The corresponding solutions (given in (3.19) with $a = -1$ and $b = 1$) are

$$y_1(t) = e^{-t}\cos t \quad \text{and} \quad y_2(t) = e^{-t}\sin t.$$

By Proposition 3.20, the general solution is given by

$$y(t) = C_1 y_1(t) + C_2 y_2(t) = e^{-t}(C_1 \cos t + C_2 \sin t). \tag{3.22}$$

To find the solution with the initial conditions $y(0) = 2$ and $y'(0) = 3$, we differentiate (3.22),

$$y'(t) = -e^{-t}(C_1 \cos t + C_2 \sin t) + e^{-t}(-C_1 \sin t + C_2 \cos t). \tag{3.23}$$

At $t = 0$, equations (3.22) and (3.23) become

$$2 = y(0) = C_1$$
$$3 = y'(0) = -C_1 + C_2.$$

This system has solutions $C_1 = 2$ and $C_2 = 5$. Therefore, the solution to the initial value problem is

$$y(t) = e^{-t}(2\cos t + 5\sin t).$$

◆

Repeated roots

If the roots of the characteristic equation (3.2) are repeated, then it becomes

$$0 = \lambda^2 + p\lambda + q = (\lambda - \lambda_1)^2$$

where λ_1 is the repeated root. By the quadratic formula,

$$\lambda_1 = \left(-p \pm \sqrt{p^2 - 4q}\right)/2 = -p/2. \tag{3.24}$$

In order for λ_1 to be a double root, we must have $p^2 - 4q = 0$. This value of λ gives one solution to the differential equation, namely

$$y_1 = e^{\lambda_1 t}.$$

To use Theorem 1.21, we need to find another solution that is not a constant multiple of this one. We will use a method of finding the second solution that can be used in many other circumstances when multiple roots give rise to cases that need to be handled separately.

First we point out that in terms of the characteristic roots, the characteristic polynomial can be written as

$$\begin{aligned}\lambda^2 + p\lambda + q &= (\lambda - \lambda_1)(\lambda - \lambda_2)\\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.\end{aligned}$$

Thus, $p = -(\lambda_1 + \lambda_2)$ and $q = \lambda_1\lambda_2$. This means that the characteristic roots determine the differential equation. The key to our method is to perturb the differential equation slightly. Instead of looking at our case where $\lambda_2 = \lambda_1$, we look at the equation where $\lambda_2 = \lambda_1 + s$, for s a small real number.

The equation with these characteristic roots is

$$y'' + (p - s)y' + (q + \lambda_1 s)y = 0, \tag{3.25}$$

where $p = -2\lambda_1$ and $q = \lambda_1^2$. The characteristic roots for this equation are λ_1 and $\lambda_1 + s$, so the functions $e^{\lambda_1 t}$ and $e^{(\lambda_1+s)t}$ are solutions. In particular, for $s \neq 0$ the function

$$u_s(t) = \frac{1}{s}\left[e^{(\lambda_1+s)t} - e^{\lambda_1 t}\right]$$

is a solution to (3.25). Let's rewrite that as

$$u_s'' + (p - s)u_s' + (q + \lambda_1 s)u_s = 0. \tag{3.26}$$

We now let s tend to 0 in (3.26). Assuming for the moment that everything makes sense, and in particular that $u(t) = \lim_{s\to 0} u_s(t)$ exists, then in the limit (3.26) becomes

$$u'' + pu' + qu = 0. \tag{3.27}$$

If everything makes sense, we have discovered a solution to our original equation. Let's find out what u is by computing the limit

$$u(t) = \lim_{s\to 0} u_s(t) = \lim_{s\to 0} \frac{1}{s}\left[e^{(\lambda_1+s)t} - e^{\lambda_1 t}\right].$$

The limit can be computed using either the limit quotient definition of the derivative or l'Hôpital's rule to get

$$u(t) = te^{\lambda_1 t}.$$

We suspect that u is actually a solution to (3.27), but that needs verification. There are two ways to do this. The easy way is simply to verify it by direct substitution. The harder way is to show that the limit in (3.26) does make sense, and that the limit is (3.27). We will leave both of these as exercises.

Thus, our second solution is

$$y_2(t) = u(t) = te^{\lambda_1 t}.$$

Since $y_2 = ty_1$ and t is not a constant, y_2 is not a constant multiple of y_1. They are linearly independent. Theorem 1.21 now gives us the form of the general solution, and we summarize our discussion in the following proposition.

PROPOSITION 3.28 If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has only one double root λ_1, then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = (C_1 + C_2 t)\, e^{\lambda_1 t},$$

where C_1 and C_2 are arbitrary constants.

The constants C_1 and C_2 can be found from initial conditions to solve initial value problems.

EXAMPLE 3.29 ◆ Find the general solution to

$$y'' - 2y' + y = 0.$$

Find the solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = -1$.

The characteristic equation is

$$0 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$

so $\lambda = 1$ is a double root. Hence, e^t and te^t form a fundamental set of solutions to this differential equation, and the general solution is

$$y(t) = C_1 e^t + C_2 te^t. \tag{3.30}$$

To find the solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = -1$, we differentiate y,

$$y'(t) = C_1 e^t + C_2 te^t + C_2 e^t. \tag{3.31}$$

At $t = 0$, equations (3.30) and (3.31) become

$$2 = y(0) = C_1$$
$$-1 = y'(0) = C_1 + C_2.$$

This system has solutions $C_1 = 2$ and $C_2 = -3$, so the solution to the initial value problem is

$$y = 2e^t - 3te^t.$$

◆

EXERCISES

The equations in Exercises 1–6 have distinct, real, characteristic roots. Find the general solution in each case.

1. $y'' - y' - 2y = 0$
2. $y'' + 5y' + 6y = 0$
3. $y'' + y' - 12y = 0$
4. $2y'' - y' - y = 0$
5. $6y'' + y' - y = 0$
6. $6y'' + 5y' - 6y = 0$

The equations in Exercises 7–12 have complex characteristic roots. Find the general solution in each case.

7. $y'' + y = 0$
8. $y'' + 4y = 0$
9. $y'' + 4y' + 5y = 0$
10. $y'' + 2y' + 17y = 0$
11. $y'' + 2y = 0$
12. $y'' + 2y' + 2y = 0$

The equations in Exercises 13–18 have repeated, real, characteristic roots. Find the general solution in each case.

13. $y'' - 4y' + 4y = 0$
14. $y'' - 6y' + 9y = 0$
15. $4y'' + 4y' + y = 0$
16. $4y'' + 12y' + 9y = 0$
17. $16y'' + 8y' + y = 0$
18. $y'' + 8y' + 16y = 0$

In Exercises 19–28, find the solution of the given initial value problem.

19. $y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2$
20. $y'' - 2y' + 17y = 0, \quad y(0) = -2, \quad y'(0) = 3$
21. $y'' + 25y = 0, \quad y(0) = 1, \quad y'(0) = -1$
22. $y'' + 10y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -1$
23. $y'' - 2y' - 3y = 0, \quad y(0) = 2, \quad y'(0) = -3$
24. $y'' - 4y' - 5y = 0, \quad y(1) = -1, \quad y'(1) = -1$
25. $8y'' + 2y' - y = 0, \quad y(-1) = 1, \quad y'(-1) = 0$
26. $4y'' + y = 0, \quad y(1) = 0, \quad y'(1) = -2$
27. $y'' + 12y' + 36y = 0, \quad y(1) = 0, \quad y'(1) = -1$
28. $y'' - 4y' + 13y = 0, \quad y(0) = 4, \quad y'(0) = 0$
29. Given that the characteristic equation $\lambda^2 + p\lambda + q = 0$ has a double root, $\lambda = \lambda_1$, show, by direct substitution, that $y = te^{\lambda_1 t}$ is a solution of $y'' + py' + qy = 0$.
30. We need to complete some details in the proof of Proposition (3.28). Recall that we made the following definition.

$$u_s(t) = \frac{1}{s}\left[e^{(\lambda_1 + s)t} - e^{\lambda_1 t}\right] \tag{3.32}$$

(a) Use l'Hôpital's rule to show that $u(t) = \lim_{s \to 0} u_s(t) = te^{\lambda_1 t}$.

(b) Show that $u'(t) = \lim_{s \to 0} u_s'(t)$. Again, l'Hôpital's rule is helpful.

(c) Show that $u''(t) = \lim_{s\to 0} u_s''(t)$.

(d) Show that the limit of equation (3.26) is equation (3.27), so that u really is a solution to (3.27).

31. Use Euler's identity to prove that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

32. Verify equation 3.10. That is, show that if z and w are complex numbers, then

$$|zw| = |z||w| \quad \text{and} \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

33. Prove property (1) of Proposition 3.15. The addition formulas for the sine and cosine will be useful.

34. Prove property (2) of Proposition 3.15. This will be easy if you use what you learned in the previous exercise.

35. Prove property (4) of Proposition 3.15.

36. Prove property (5) of Proposition 3.15.

37. Prove property (6) of Proposition 3.15. Remember that $\lambda = \mu + i\nu$ is a complex number.

4.4 Harmonic Motion

In Section 4.1 we derived the equation for the motion of a vibrating spring. It is (see equation (1.13))

$$my'' + \mu y' + ky = F(t), \tag{4.1}$$

where the constant coefficients are m, the mass, μ, the damping constant, and k, the spring constant, and the function $F(t)$ is an external force.

In Section 3.4 of Chapter 3 we derived differential equations that modeled simple RLC circuits. If the circuit consisted of a resistor of resistance R, a condenser of capacitance C, and an inductor of inductance L and had a source voltage $E(t)$, then the current I satisfies the equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}. \tag{4.2}$$

The coefficients R, C, and L are all constants.

It is interesting to compare equations (4.1) and (4.2). They are almost identical, differing only in the letters chosen to represent the coefficients and the unknown functions. If we compare the coefficients, we see that the inductance L acts like the mass m, the resistance R like the damping constant, and the reciprocal of the capacitance $1/C$ like the spring constant. Finally, the derivative of the source voltage acts like the external force on the spring.

It is important to keep these analogies in mind. Physically it means that the two phenomena have similar behavior. Mathematically it means that when we discover facts about solutions to one of these equations, we also have related facts about the others. For example, if we have a circuit without resistance ($R = 0$) and no source voltage, then equation (4.2) simplifies to

$$L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0. \tag{4.3}$$

This corresponds to the equation for an unforced, undamped spring. Earlier we discovered solutions to that equation (see equation (1.15)). Replacing the mass m with the inductance L, and the spring constant k with the reciprocal of the capacitance $1/C$, we see that any function of the form

$$I(t) = a\cos(t/\sqrt{LC}) + b\sin(t/\sqrt{LC})$$

is a solution to (4.3).

If we divide equations (4.1) and (4.2) by their leading coefficient (L or m), they become

$$\frac{d^2y}{dt^2} + \frac{\mu}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{1}{m}F(t), \quad \text{and}$$

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt}.$$

If we make the identifications $c = \mu/2m$, $\omega_0 = \sqrt{k/m}$, $f(t) = F(t)/m$, and $x = y$ in the first equation, or $c = R/2L$, $\omega_0 = \sqrt{1/LC}$, $f(t) = (dE/dt)/L$, and $x = I$ in the second, we get the equation

$$x'' + 2cx' + \omega_0^2 x = f(t), \tag{4.4}$$

where $c \geq 0$ and $\omega_0 > 0$ are constants. We will refer to this equation as the equation for ***harmonic motion***. It includes the vibrating spring and the arbitrary *RLC* circuit, but there are many other phenomena that lead to this equation.

It is common to use the terminology of the vibrating spring when discussing harmonic motion. In particular, c is called the ***damping*** constant, and f is the ***forcing term***.

In this section we will study unforced harmonic motion. This means that $f(t) = 0$ in (4.4), so we will be studying the homogeneous equation

$$x'' + 2cx' + \omega_0^2 x = 0, \tag{4.5}$$

where $c \geq 0$ and $\omega_0 > 0$ are constants. This is the type of equation we considered in the previous section, so we are in a position to analyze unforced harmonic motion.

Simple harmonic motion

In the special case when there is no damping (so $c = 0$) the motion is called ***simple harmonic motion***. Equation (4.5) simplifies to

$$x'' + \omega_0^2 x = 0. \tag{4.6}$$

The characteristic equation is

$$\lambda^2 + \omega_0^2 = 0.$$

The characteristic roots are $\lambda = \pm i\omega_0$. According to Proposition 3.20, the general solution is

$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t, \qquad (4.7)$$

where a and b are constants.

If we define $T = 2\pi/\omega_0$, so that $\omega_0 T = 2\pi$, then the periodicity of the trigonometric functions implies that $x(t+T) = x(t)$ for all t. Thus, the solution x is itself periodic with period T. For this reason, ω_0 is called the ***natural frequency*** of the spring. We will continue with this terminology even in the damped case.

EXAMPLE 4.8 ◆ Suppose we have simple harmonic motion with a natural frequency $\omega_0 = 4$. Find the solution with initial values $x(0) = 1$ and $x'(0) = 0$.

From (4.7) we see that the general solution is

$$x = a\cos 4t + b\sin 4t.$$

The initial condition $x(0) = 1$ becomes $a = 1$, and $x'(0) = 0$ becomes $4b = 0$. Hence the solution to the initial value problem is $x(t) = \cos 4t$. The graph of x is given in Figure 1. ◆

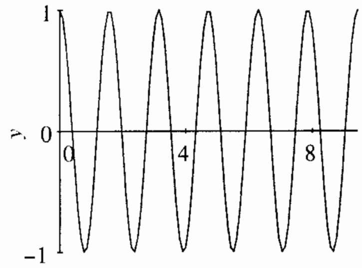


Figure 1 The undamped oscillation in Example 4.8.

In the case of a vibrating spring without friction, we see that the mass on the spring oscillates up and down with the natural frequency $\omega_0 = \sqrt{k/m}$. Note that the natural frequency increases as the spring constant increases, and it decreases as the mass increases.

Amplitude and phase angle

It is frequently convenient to put the solutions in (4.7) into another form that is more convenient and more revealing of the nature of the solution.

Consider the vector (a, b) in the plane (see Figure 2). We will write this in polar coordinates. Assuming that $(a, b) \neq (0, 0)$, there is a positive number A, which is the ***length*** of (a, b), and an angle ϕ in the interval $(-\pi, \pi]$, called the ***polar angle***, such that

$$a = A\cos\phi \quad \text{and} \quad b = A\sin\phi. \qquad (4.9)$$

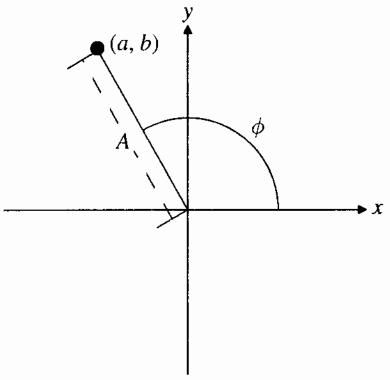


Figure 2 Polar coordinates.

Examples are shown in Figure 5.

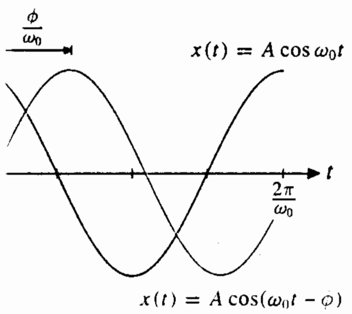
If we substitute these equations into (4.7), it becomes

$$\begin{aligned} x(t) &= a\cos(\omega_0 t) + b\sin(\omega_0 t) \\ &= A\cos\phi\cos(\omega_0 t) + A\sin\phi\sin(\omega_0 t) \\ &= A\cos(\omega_0 t - \phi). \end{aligned} \qquad (4.10)$$

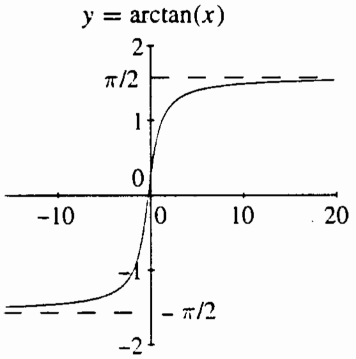
Thus, we see that the general solution to the second-order equation (4.6) can be written as

$$x(t) = A\cos(\omega_0 t - \phi). \qquad (4.11)$$

This expression for the solution makes it clear that undamped harmonic motion is a pure sinusoidal oscillation. The parameter A is the ***amplitude*** of the oscillation.



3 The phase angle shifts
raph of the cosine



4 The arctangent takes
s between $-\pi/2$ and $\pi/2$.

Since the cosine term oscillates between ± 1, the limits of the oscillation in (4.11) are $\pm A$. The parameter ϕ represents the ***phase*** of the oscillation. A positive phase shifts the graph of the cosine to the right. This effect is shown in Figure 3. It can best be understood by writing (4.11) as

$$x(t) = A\cos\left(\omega_0\left(t - \frac{\phi}{\omega_0}\right)\right).$$

Notice that there are still two undetermined constants in (4.11). Now they are the amplitude A and the phase ϕ.

This is all very well and good, but it is not very useful unless we can find the constants A and ϕ when we know a and b. Equations (4.9) tell us how to compute a and b if we know A and ϕ. They also can be solved for A and ϕ. First, if we square each equation and add, we get

$$A^2 = a^2 + b^2 \quad \text{or} \quad A = \sqrt{a^2 + b^2}. \tag{4.12}$$

Next, if we divide the second equation by the first, we get

$$\tan\phi = \frac{b}{a}. \tag{4.13}$$

While it is tempting to "solve" this last equation for ϕ and write $\phi = \arctan(b/a)$, this would be a mistake. As Figure 4 shows, the arctan takes values between $-\pi/2$ and $\pi/2$, whereas we know that ϕ can be any angle between $-\pi$ and π. The range $-\pi/2 < \phi < \pi/2$ corresponds to the points (a, b) where $a = A\cos\phi > 0$. These are points (a, b) in the right half-plane. How do we compute ϕ when $a < 0$?

When (a, b) is in the left half-plane, then $(-a, -b)$ is in the right half-plane, and clearly $\arctan(b/a) = \arctan(-b/-a)$. Thus $\arctan(b/a)$ is measuring the polar angle of $(-a, -b)$, which differs by $\pm\pi$ from the polar angle of (a, b) (see Figure 5). There are three cases depending on the signs of a and b, as shown in Figure 5. Following the angles in Figure 5, we see that the polar angle is given by

$$\phi = \begin{cases} \arctan(b/a), & \text{if } a > 0; \\ \arctan(b/a) + \pi, & \text{if } a < 0 \text{ and } b > 0; \\ \arctan(b/a) - \pi, & \text{if } a < 0 \text{ and } b < 0. \end{cases}$$

XAMPLE 4.14 ◆ A mass of 4 kg is attached to a spring with a spring constant of $k = 169\,\text{kg/s}^2$. It is then stretched 10 cm from the spring-mass equilibrium and set to oscillating with an initial velocity of 130 cm/s. Assuming it oscillates without damping, find the frequency, amplitude, and phase of the vibration.

The differential equation is

$$4y'' + 169y = 0 \quad \text{or} \quad y'' + 42.25y = 0.$$

The natural frequency is $\omega_0 = \sqrt{42.25} = 6.5$ and the general solution is

$$y(t) = C_1\cos 6.5t + C_2\sin 6.5t.$$

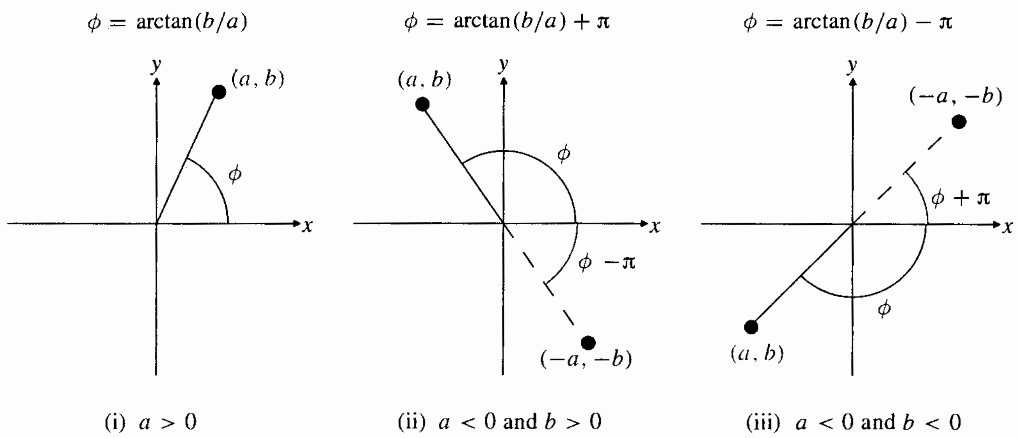


Figure 5 Polar coordinates. There are three cases for finding the angle.

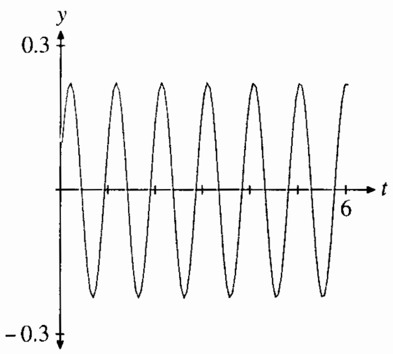


Figure 6 The undamped motion in Example 4.14.

The initial conditions are $y(0) = 0.1$ m and $y'(0) = 1.3$ m/s. The specific solution satisfying these initial conditions is

$$y = 0.1 \cos 6.5t + 0.2 \sin 6.5t.$$

The amplitude of vibration is

$$A = \sqrt{0.01 + 0.04} = \frac{\sqrt{5}}{10} \approx 0.2236 \text{ m}.$$

The phase is $\phi = \arctan(2) \approx 1.1071$. Hence we can write the solution as $y(t) = \sqrt{5}/10 \cos(6.5t - \phi)$.

The solution is plotted in Figure 6. ◆

Damped harmonic motion

Now $c > 0$. The differential equation

$$x'' + 2cx' + \omega_0^2 x = 0$$

has the characteristic equation

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0.$$

The roots are

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \quad \text{and} \quad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}. \tag{4.15}$$

We have three cases to consider depending on the sign of the discriminant $c^2 - \omega_0^2$.

1. $c < \omega_0$. This is the ***underdamped*** case. The roots in (4.15) are distinct complex numbers. Hence the general solution is

$$x(t) = e^{-ct}\left[C_1 \cos \omega t + C_2 \sin \omega t\right],$$

where

$$\omega = \sqrt{\omega_0^2 - c^2}.$$

2. $c > \omega_0$. This is the ***overdamped case***. Now the roots in (4.15) are distinct and real. Further, since $\sqrt{c^2 - \omega_0^2} < \sqrt{c^2} = c$, we have $\lambda_1 < \lambda_2 < 0$. The general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

3. $c = \omega_0$. This is the ***critically damped*** case, and in this case, the root in (4.15) is a double root,

$$\lambda = -c.$$

The general solution is

$$x(t) = C_1 e^{-ct} + C_2 t e^{-ct}.$$

In all of the cases the solution decays to zero as $t \to \infty$ due to the exponential term in the solution, and the fact that $c > 0$. In the critically damped case, this follows since, for $c > 0$, $\lim_{t\to\infty} t/e^{ct} = 0$ by l'Hôpital's rule.

In the underdamped case the cosine and sine terms cause the solution to oscillate with frequency ω as it converges to zero. Notice that this frequency is always smaller than the natural frequency of the spring. In the other two cases there is no oscillation.

Let's look at specific examples of damping phenomena.

XAMPLE 4.16 ◆ Consider the spring in Example 4.14 with damping constant $\mu = 12.8$ kg/s. Find the solution with initial conditions $y(0) = 0.1$ m and $y'(0) = 1.3$ m/s.

Since $m = 4$, $k = 169$ and $\mu = 12.8$, the differential equation is

$$4y'' + 12.8y' + 169y = 0 \quad \text{or} \quad y'' + 3.2y' + 42.25y = 0.$$

The characteristic polynomial $\lambda^2 + 3.2\lambda + 42.25$ has roots $-1.6 \pm i\sqrt{42.25 - 1.6^2}$ $= -1.6 \pm 6.3i$. Thus, $\omega = 6.3$ and the general solution is

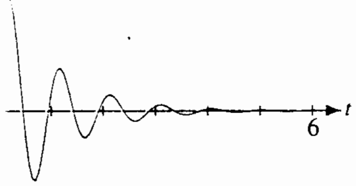
$$y(t) = e^{-1.6t}(C_1 \cos 6.3t + C_2 \sin 6.3t).$$

For the initial conditions we have

$$0.1 = y(0) = C_1$$
$$1.3 = y'(0) = -1.6\,C_1 + 6.3\,C_2.$$

Hence $C_1 = 0.1$ and $C_2 = 1.46/6.3 \approx 0.2317$. The solution is

$$y(t) = e^{-1.6t}(0.1 \cos 6.3t + 0.2317 \sin 6.3t).$$



7 The underdamped
n in Example 4.16.

This can also be written as

$$y(t) = 0.2524e^{-1.6t} \cos(6.3t - 1.1634).$$

The underdamped motion in Example 4.16 is plotted in Figure 7. The decaying oscillatory motion that is seen there is typical of underdamped motion. As an example, consider the springs of a car, which are damped by shock absorbers. If the shocks are very old, when you press the front of the car down and release it, the car will bounce up and down for a while with decreasing amplitude until it reaches equilibrium. This is because as the shocks wear out, the damping constant decreases. When it decreases to the point that the motion is underdamped, the front end will bounce. ◆

EXAMPLE 4.17 ◆ Consider the spring in Example 4.14 with damping constant $\mu = 77.6\,\text{kg/s}$. Find the solution with initial conditions $y(0) = 0.1$ m and $y'(0) = 1.3$ m/s.

Since $m = 4$, $k = 169$ and $\mu = 77.6$, the differential equation is

$$4y'' + 77.6y' + 169y = 0 \quad \text{or} \quad y'' + 19.4y' + 42.25y = 0.$$

The characteristic polynomial $\lambda^2 + 19.4\lambda + 42.25$ has roots

$$-9.7 \pm \sqrt{9.7^2 - 42.25} = -9.7 \pm 7.2.$$

Set $\lambda_1 = -16.9$ and $\lambda_2 = -2.5$. This is the overdamped case. The general solution is

$$y(t) = C_1 e^{-16.9t} + C_2 e^{-2.5t}.$$

For the initial conditions we have

$$0.1 = y(0) = C_1 + C_2$$
$$1.3 = y'(0) = -16.9C_1 - 2.5C_2.$$

Hence $C_1 = -31/288$, and $C_2 = 299/1440$. The solution is given by

$$y(t) = -\frac{31}{288}e^{-16.9t} + \frac{299}{1440}e^{-2.5t}.$$

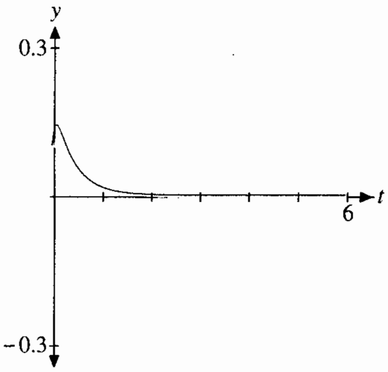


Figure 8 The overdamped motion in Example 4.17.

The overdamped motion in Example 4.17 is plotted in Figure 8. Typical of such behavior, the displacement decreases to its equilibrium without the oscillation we see in Example 4.16. It is possible with properly chosen initial conditions that the displacement passes through the equilibrium just once before decaying to it. An example of overdamped motion is provided by the springs of a car with brand new shock absorbers. When you press the front of the car down and release it, the car simply rises to its original position without any oscillation. ◆

EXAMPLE 4.18 ◆ For the spring in Example 4.14, find the value of the damping constant μ for which there is critical damping. Find the solution with initial conditions $y(0) = 0.1$ m and $y'(0) = 1.3$ m/s.

Critical damping occurs when $c = \omega_0$. Since $c = \mu/2m$ and $\omega_0 = \sqrt{k/m}$, we need $\mu = 2m\sqrt{k/m} = 2\sqrt{mk} = 52\,\text{kg/s}$ With this value of μ, the equation becomes

$$4y'' + 52y' + 169y = 0 \quad \text{or} \quad y'' + 13y' + 42.25y = 0.$$

The characteristic polynomial is $\lambda^2 + 13\lambda + 42.25 = (\lambda + 6.5)^2$. The general solution is

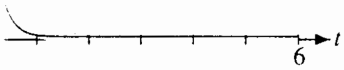
$$y(t) = C_1 e^{-6.5t} + C_2 t e^{-6.5t}.$$

For the initial conditions we have

$$0.1 = y(0) = C_1$$
$$1.3 = y'(0) = -6.5\,C_1 + C_2.$$

Hence $C_1 = 0.1$, and $C_2 = 1.95$. The solution is

$$y(t) = 0.1e^{-6.5t} + 1.95te^{-6.5t}.$$



The critically damped motion in Example 4.18 is plotted in Figure 9. By its definition, the damping constant for critical damping is the dividing line that separates underdamping from overdamping. The motion itself looks very similar to overdamping. It simply decays to its equilibrium. Once more it is possible that it will cross the equilibrium just once. ◆

9 The critically damped ı in Example 4.18.

Designers of shock absorbers aim for a damping constant that is just a little larger than critical damping. If the damping constant is too large, the shocks prevent the springs from absorbing bounces, and the ride becomes bumpy. If the damping constant gets below critical damping, the car oscillates. So just a little above critical damping is the right goal. This prevents the oscillations of the underdamped case, and it allows the springs to absorb the bounces. In addition it allows for the damping constant to decrease a little before it becomes underdamped.

EXERCISES

In Exercises 1–6,

(i) use a computer or calculator to plot the graph of the given function, and

(ii) place the solution in the form $y = A\cos(\omega t - \phi)$ and compare the graph of your answer with the plot found in part (i).

1. $y = \cos 2t + \sin 2t$

2. $y = \cos t - \sin t$

3. $y = \cos 4t + \sqrt{3}\sin 4t$

4. $y = -\sqrt{3}\cos 2t + \sin 2t$

5. $y = 0.2\cos 2.5t - 0.1\sin 2.5t$

6. $y = 0.2\cos 6.3t - 0.5\sin 6.3t$

In Exercises 7–10, place each equation in the form $y = Ae^{-ct}\cos(\omega t - \phi)$, then, on one plot, place the graphs of $y = Ae^{-ct}\cos(\omega t - \phi)$, $y = Ae^{-ct}$, and $y = -Ae^{-ct}$, the last two using a different linestyle and/or color than the first.

7. $y = e^{-t/2}(\cos 5t + \sin 5t)$

8. $y = e^{-t/4}(\sqrt{3}\cos 4t - \sin 4t)$

9. $y = e^{-0.1t}(0.2\cos 2t + 0.1\sin 2t)$

10. $y = e^{-0.2t}(\cos 4.2t - 1.2\sin 4.2t)$

11. A 0.2 kg mass is attached to a spring having a spring constant 5 kg/s^2. The system is displaced 0.5 m from its equilibrium position and released from rest. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

12. A 0.1 kg mass is attached to a spring having a spring constant 3.6 kg/s^2. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

13. The undamped system

$$\frac{2}{5}x'' + kx = 0, \quad x(0) = 2, \quad x'(0) = v_0$$

is observed to have period $\pi/2$ and amplitude 2. Find k and v_0.

14. Consider the undamped oscillator

$$mx'' + kx = 0, \quad x(0) = x_0, \quad x'(0) = v_0.$$

Show that the amplitude of the resulting motion is $\sqrt{x_0^2 + mv_0^2/k}$.

15. A 2 μF capacitor (1 $\mu\text{F} = 1 \times 10^{-6}$ F) is charged to 20 V and then connected across a 6 μH inductor (1 $\mu\text{H} = 1 \times 10^{-6}$ H), forming an *LC* circuit.

(a) Find the initial charge on the capacitor.

(b) At the time of connection, the initial current is zero. Assuming no resistance, find the amplitude, frequency, and phase of the current. Plot the graph of the current versus time. [Use equations (4.2) and (4.3) in Section 3.4.]

16. A 1 kg mass, when attached to a large spring, stretches the spring a distance of 4.9 m.

(a) Calculate the spring constant.

(b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3$ kg/s. The system is allowed to come to rest. Then the mass is displaced 1 m in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 m/s in the downward direction. Find the position of the mass as a function of time and plot the solution.

17. Prove that an overdamped solution of $my'' + \mu y' + ky = 0$ can cross the time axis no more than once, regardless of the initial conditions. Use a numerical solver to create a plot of an overdamped system that crosses the time axis one time and a second plot where the plot does not cross the time axis.

18. A 50 g mass (1 kg = 1000 g) stretches a spring 20 cm (1 m = 100 cm). Find a damping constant μ so that the system is critically damped. If the mass is displaced 15 cm from its equilibrium position and released from rest, find the position of the mass as a function of time and plot the solution.

19. Prove that a critically damped solution of $my'' + \mu y' + ky = 0$ can cross the time axis no more than once, regardless of the initial conditions. Use a numerical solver to create a plot of a critically damped system that crosses the time axis one time and a second plot where the plot does not cross the time axis.

20. A spring mass system is modeled by the equation

$$x'' + \mu x' + 4x = 0.$$

(a) Show that the system is critically damped when $\mu = 4$ kg/s.

(b) Suppose that the mass is displaced upward 2 m and given an initial velocity of 1 m/s. Use a numerical solver to plot the solution for $\mu = 4, 4.2, 4.4, 4.6, 4.8, 5$. Plot all solution curves on one figure. What is special about the critically damped solution in comparison to the other solutions? Why would you want to adjust the spring on a screen door so that it was critically damped?

21. If $\mu > 2\sqrt{km}$, the system $mx'' + \mu x' + kx = 0$ is overdamped. The system is allowed to come to equilibrium. Then the mass is given a sharp tap, imparting an instantaneous downward velocity v_0.

(a) Show that the position of the mass is given by

$$x(t) = \frac{v_0}{\gamma} e^{-\mu t/(2m)} \sinh \gamma t, \quad \text{where} \quad \gamma = \frac{\sqrt{\mu^2 - 4mk}}{2m}.$$

(b) Show that the mass reaches its lowest point at

$$t = \frac{1}{\gamma} \tanh^{-1} \frac{2m\gamma}{\mu},$$

a time independent of the initial conditions.

(c) Show that, in the critically damped case, the time it takes the mass to reach its lowest point is given by $t = 2m/\mu$.

22. A 100 g mass (1 kg = 1000 g) is hung from a spring having spring constant $9.8\,\text{kg/s}^2$. The system is placed in a viscous medium that imparts a force of 0.3 N when the mass is moving at 0.2 m/s. Assume that the force applied by the medium is proportional, but opposite, the mass's velocity. The mass is displaced 10 cm from its equilibrium position and released from rest. Find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

23. A capacitor (0.008 F) is charged to 50 V then connected in series with an inductor (4 H) and a resistor (20 Ω). Initially, there is no current in the circuit. Find the amplitude, frequency, and phase of the current and plot its graph.

24. A 10 kg mass stretches a spring 1 m. The system is placed in a viscous medium that provides a damping constant $\mu = 20$ kg/s. The system is allowed to attain equilibrium. Then a sharp tap to the mass imparts an instantaneous downward velocity of 1.2 m/s. Find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

25. A capacitor (0.02 F) is charged to 1 V then connected in series with an inductor (10 H) and a resistor (40 Ω). Initially, there is no current in the circuit. Find the amplitude, frequency, and phase of the charge on the capacitor and plot its graph.

Inhomogeneous Equations; the Method of Undetermined Coefficients

We now turn to the solution of inhomogeneous linear equations. These are equations of the form

$$y'' + py' + qy = f, \tag{5.1}$$

where $p = p(t)$, $q = q(t)$, and $f = f(t)$ are functions of the independent variable. Remember that f is called the inhomogeneous term, or the forcing term.

Our solution strategy comes from the understanding of the structure of the general solution, which is contained in the next theorem.

THEOREM 5.2 Suppose that y_p is a particular solution to the inhomogeneous equation (5.1), and that y_1 and y_2 form a fundamental set of solutions to the associated homogeneous equation

$$y'' + py' + qy = 0. \tag{5.3}$$

Then the general solution to the inhomogeneous equation (5.1) is given by

$$y = y_p + C_1 y_1 + C_2 y_2$$

where C_1 and C_2 are arbitrary constants.

Notice that the general solution can be written as $y = y_p + y_h$, where $y_h = C_1 y_1 + C_2 y_2$ is the general solution to the corresponding homogeneous equation. Thus, to find the general solution to the inhomogeneous equation (5.1) we first find the general solution, y_h, to the corresponding homogeneous equation (5.3). Next, we find a particular solution, y_p, to the inhomogeneous equation. The general solution to the inhomogeneous equation is then $y = y_p + y_h$.

Proof Suppose that y is a solution to (5.1). We are given that y_p is also a solution, so we have the two equations

$$y'' + py' + qy = f, \quad \text{and}$$
$$y_p'' + py_p' + qy_p = f,$$

Subtracting we get

$$(y - y_p)'' + p(y - y_p)' + q(y - y_p) = 0.$$

Therefore, $y - y_p$ is a solution to the associated homogeneous equation (5.3). Since y_1 and y_2 form a fundamental set of solutions, there are constants C_1 and C_2 such that

$$y - y_p = C_1 y_1 + C_2 y_2.$$

Consequently,

$$y = y_p + C_1 y_1 + C_2 y_2$$

as promised.

For equations with constant coefficients, we already know how to solve the homogeneous equation, so in this section we will concentrate on finding one particular solution to the inhomogeneous equation.

The method of undetermined coefficients

We will be looking at the inhomogeneous equation

$$y'' + py' + qy = f, \tag{5.4}$$

where p and q are constants and $f = f(t)$ is a function of the independent variable. The method of undetermined coefficients only works if the coefficients are constants. Second-order equations are the most important applications of the method, but the method works for higher-order equations in exactly the same way that it does for second-order equations. For simplicity and convenience we will emphasize the second-order case.

The method of undetermined coefficients is based on the fact that there are some situations where the form of the forcing term in (5.4) allows us to almost guess the form of a particular solution. Let's highlight the key idea.

> If the forcing term f has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

Exponential forcing terms

The easiest example of such a forcing term is an exponential function $f(t) = e^{at}$. Then $f'(t) = ae^{at}$, which is also an exponential function. The method is illustrated by our first example.

EXAMPLE 5.5 ◆ Find a particular solution to the equation

$$y'' - y' - 2y = 2e^{-2t}. \tag{5.6}$$

The forcing term is $f(t) = 2e^{-2t}$. We look for a particular solution with the same form as f, or

$$y(t) = ae^{-2t},$$

where a is an as yet undetermined coefficient. Its derivatives are

$$y'(t) = -2ae^{-2t} \quad \text{and} \quad y''(t) = 4ae^{-2t}.$$

If we insert these expressions into the left-hand side of (5.6), we get

$$y'' - y' - 2y = 4ae^{-2t} - (-2ae^{-2t}) - 2(ae^{-2t}) = 4ae^{-2t}.$$

In order for y to be a solution of (5.6), we must have

$$4ae^{-2t} = 2e^{-2t}.$$

Equating the coefficients of e^{-2t} gives $4a = 2$, or $a = 1/2$. Consequently,

$$y(t) = e^{-2t}/2$$

is a particular solution to (5.6). We encourage the readers to check by direct substitution that y is a solution to (5.6). ◆

Trigonometric forcing terms

Next, consider a forcing term of the form

$$f(t) = A\cos\omega t + B\sin\omega t.$$

The derivative of f has the same general form, so we will look for solutions of the form

$$y(t) = a\cos\omega t + b\sin\omega t,$$

where a and b are as yet undetermined coefficients.

EXAMPLE 5.7 ◆ Compute a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin 3t. \tag{5.8}$$

We look for a particular solution of the form

$$y = a\cos 3t + b\sin 3t.$$

As we will see, we will need both the cosine and the sine terms, even though only a sine term appears in the forcing term.

The derivatives of y are

$$y' = -3a\sin 3t + 3b\cos 3t \quad \text{and}$$
$$y'' = -9a\cos 3t - 9b\sin 3t.$$

Insert these expressions into the left-hand side of (5.8).

$$\begin{aligned} y'' + 2y' - 3y &= (-9a\cos 3t - 9b\sin 3t) + 2(-3a\sin 3t + 3b\cos 3t) \\ &\quad - 3(a\cos 3t + b\sin 3t) \\ &= (-12a + 6b)\cos 3t + (-6a - 12b)\sin 3t. \end{aligned} \tag{5.9}$$

Equating the coefficients of the sine and cosine terms in (5.8) and (5.9) gives two equations for a and b,

$$-12a + 6b = 0$$
$$-6a - 12b = 5,$$

with solutions $a = -1/6$ and $b = -1/3$. Hence, a particular solution is

$$y = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t. \tag{5.10}$$

◆

The complex method

There is another way to find a particular solution in situations where the forcing function contains a trigonometric term. We will illustrate the method by solving the same equation as we did in Example 5.7.

EXAMPLE 5.11 ◆ Use the complex method to find a particular solution to the equation

$$y'' + 2y' - 3y = 5\sin 3t. \tag{5.12}$$

Notice that the right-hand side of (5.12) is $5\sin 3t = \text{Im}(5e^{3it})$. Instead of solving (5.12) directly as we did in Example 5.7, we will look for a solution to

$$z'' + 2z' - 3z = 5e^{3it} \tag{5.13}$$

using the techniques of Example 5.5. If $z(t) = x(t) + iy(t)$ is that solution, then formally we have

$$\begin{aligned} z'' + 2z' - 3z &= (x + iy)'' + 2(x + iy)' - 3(x + iy) \\ &= (x'' + 2x' - 3x) + i(y'' + 2y' - 3y). \end{aligned} \tag{5.14}$$

On the other hand, expanding the right-hand side of (5.13) using Euler's formula, we get

$$z'' + 2z' - 3z = 5e^{3it} = 5[\cos 3t + i \sin 3t]. \tag{5.15}$$

Equating the imaginary parts of (5.14) and (5.15), we see that $y(t) = \operatorname{Im} z(t)$ is a solution to (5.12). We notice in passing that $x(t) = \operatorname{Re} z(t)$ is a solution to the equation

$$x'' + 2x' - 3x = 5\cos 3t.$$

Our solution to (5.13) should have the same form as the forcing term, so we try $z(t) = ae^{3it}$. Substituting this into the left-hand side of (5.13), we get

$$z'' + 2z' - 3z = (3i)^2 ae^{3it} + 2(3i)ae^{3it} - 3ae^{3it} = -6(2 - i)ae^{3it}.$$

For z to be a solution of (5.13) we must have $-6(2 - i)a = 5$. Therefore,

$$a = -\frac{1}{6}\frac{5}{2-i} = -\frac{1}{6}\frac{5}{2-i}\frac{2+i}{2+i} = -\frac{1}{6}\frac{10+5i}{5} = -\frac{2+i}{6}.$$

Hence the complex solution is

$$\begin{aligned} z(t) &= -\frac{2+i}{6}e^{3it} \\ &= -\frac{1}{6}(2+i)(\cos 3t + i \sin 3t) \\ &= -\frac{1}{6}\left\{[2\cos 3t - \sin 3t] + i[\cos 3t + 2\sin 3t]\right\}. \end{aligned}$$

The solution y to (5.12) is the imaginary part of z, or

$$y(t) = -\frac{1}{6}[\cos 3t + 2\sin 3t],$$

which agrees with (5.10). ◆

It is up to the individual to decide which of these methods is preferable. If you are comfortable with complex arithmetic, then the complex method is probably quicker. The complex method is preferred by some engineers and physicists because of the insight the answers provide.

Polynomial forcing terms

The derivative of a polynomial is another polynomial of lower degree. Consequently, a polynomial forcing term

$$f(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$$

has a form which is replicated under differentiation. We can find a particular solution by the method of undetermined coefficients.

EXAMPLE 5.16 ◆ Find a particular solution to the differential equation

$$y'' + 2y' - 3y = 3t + 4. \tag{5.17}$$

The right-hand side is a polynomial of degree 1, so we look for a particular solution of the same form. In this case that means a polynomial of the same degree, or

$$y = at + b,$$

where a and b are constants to be determined. The derivatives of y are

$$y' = a \quad \text{and} \quad y'' = 0.$$

Inserting these into our differential equation gives

$$y'' + 2y' - 3y = 0 + 2a - 3(at + b) = -3at + (2a - 3b). \tag{5.18}$$

Equating the coefficient of t and the constant term in (5.17) and (5.18) gives two equations for a and b

$$\begin{aligned} -3a &= 3, \\ 2a - 3b &= 4. \end{aligned}$$

The solutions are $a = -1$ and $b = -2$, and our particular solution is

$$y = -t - 2.$$

We encourage the reader to check that y is indeed a solution. ◆

Exceptional cases

The method of undetermined coefficients looks straightforward. There are, however, some exceptional cases to look out for. Suppose we change the forcing term in equation (5.6) in Example 5.5 to $3e^{-t}$, getting the equation

$$y'' - y' - 2y = 3e^{-t}.$$

If we look for a solution of the indicated form, $y(t) = ae^{-t}$, we run into trouble. To see this, let's insert y into the left-hand side of the differential equation. We get

$$y'' - y' - 2y = ae^{-t} + ae^{-t} - 2ae^{-t} = 0.$$

There is no choice of the constant a which will make this equal to $3e^{-t}$.

The problem arises because the forcing term, and hence the proposed solution, is a solution to the associated homogeneous equation. This is the source of the difficulty in all exceptional cases. However, even if the solution to the homogeneous equation is only a part of the forcing term, we can have an exceptional case. As we will see, exceptional cases do arise in applications, and they are often the most interesting cases.

What do we do? In this case we look for a solution of the form $y(t) = ate^{-t}$. We multiply the usual general form by t. This is the way to find a solution whenever the usual form does not work. If this method does not work, multiply by t once more and try again.

(AMPLE 5.19 ◆ Find a particular solution to the equation

$$y'' - y' - 2y = 3e^{-t}. \tag{5.20}$$

We know from the preceding discussion that the forcing term is a solution to the homogeneous equation, so we look for a solution of the form $y(t) = ate^{-t}$. Substituting into the left-hand side of the equation, we get

$$y'' - y' - 2y = a(t-2)e^{-t} - a(1-t)e^{-t} - 2ate^{-t} = -3ae^{-t}.$$

This will give a solution to (5.20) provided that

$$-3ae^{-t} = 3e^{-t}.$$

Therefore, we need $a = -1$, and our particular solution is $y(t) = -te^{-t}$. ◆

Typically, we do not notice that the forcing function is exceptional. The first indication that arises is when we are presented with equations for the undetermined coefficients that are inconsistent. At this point apply the remedy—multiply the trial solution by t and try again. If that still leads to inconsistent equations, multiply by t^2.

Combination forcing terms

The method we just used can be used whenever the forcing term is a linear combination of expressions of the forms we have already handled.

XAMPLE 5.21 ◆ Find a particular solution to the equation

$$y'' - y' - 2y = e^{-2t} - 3e^{-t}. \tag{5.22}$$

The forcing term is a sum, $f(t) = f_1(t) + f_2(t)$. where $f_1(t) = e^{-2t}$, and $f_2(t) = -3e^{-t}$. Suppose we break up the equation and solve the equations separately for each part of the forcing term. This means that we find functions y_1 and y_2 such that

$$\begin{aligned} y_1'' - y_1' - 2y_1 &= e^{-2t}, \quad \text{and} \\ y_2'' - y_2' - 2y_2 &= -3e^{-t}. \end{aligned} \tag{5.23}$$

Adding these two equations, we get

$$(y_1 + y_2)'' - (y_1 + y_2)' - 2(y_1 + y_2) = e^{-2t} - 3e^{-t},$$

so $y = y_1 + y_2$ is a solution to (5.22).

It remains to solve the two equations in (5.23). The first is solved using the method in Example 5.5. We find that $y_1(t) = e^{-2t}/4$. In the second equation we have an exceptional case, so we use the method of Example 5.19. The solution is $y_2(t) = te^{-t}$. Hence the solution to (5.22) is

$$y(t) = y_1(t) + y_2(t) = \frac{1}{4}e^{-2t} + te^{-t}.$$ ◆

As the example illustrates, when the forcing term is a linear combination of the terms, we need only solve the equation for each individual term and take the linear combination of the solutions.

Let's look at one more example.

EXAMPLE 5.24 ◆ Find a particular solution to the equation

$$y'' + 4y = \cos 2t - 2\sin 2t. \tag{5.25}$$

We will use the complex method. It is easiest to treat the two summands in the forcing term separately. This means that we look for solutions y_1 and y_2 to the equations

$$\begin{aligned} y_1'' + 4y_1 &= \cos 2t \quad \text{and} \\ y_2'' + 4y_2 &= -2\sin 2t. \end{aligned} \tag{5.26}$$

Then it is easily seen that $y = y_1 + y_2$ is a solution to (5.25).

To solve the first equation in (5.26), we notice that $\cos 2t = \text{Re}(e^{2it})$, so we solve the complex equation

$$z'' + 4z = e^{2it}. \tag{5.27}$$

The forcing term is a solution to the homogeneous equation, so we look for a solution of the form $z(t) = ate^{2it}$. In the usual way we find that the solution is

$$z(t) = -\frac{it}{4}e^{2it}.$$

The solution to the first equation in (5.26) is

$$y_1(t) = \text{Re}\, z(t) = \text{Re}\left(-\frac{it}{4}e^{2it}\right) = \frac{t\sin 2t}{4}.$$

Notice that the imaginary part of z,

$$u(t) = \text{Im}\, z(t) = \text{Im}\left(-\frac{it}{4}e^{2it}\right) = -\frac{t\cos 2t}{4},$$

is a solution to the equation

$$u'' + 4u = \text{Im}\, e^{2it} = \sin 2t.$$

The similarity between this equation and the second equation in (5.26) allows us to conclude that

$$y_2(t) = -2u(t) = \frac{t\cos 2t}{2}.$$

The solution to (5.25) is

$$y(t) = y_1(t) + y_2(t) = \frac{t\sin 2t}{4} + \frac{t\cos 2t}{2}.$$ ◆

More complicated forcing terms

In addition to the three cases we have considered so far, we could have forcing terms which are products of two of these, or of all three. The most general situation would be a forcing term of the form

$$f(t) = e^{rt} P(t) \cos \omega t + e^{rt} Q(t) \sin \omega t,$$

where $P(t)$ and $Q(t)$ are polynomials. Again $f'(t)$ has the same form, so we can use undetermined coefficients.

The method of undetermined coefficients is summarized in Table 1, which shows the allowed forcing functions and the type of particular solution to be used. Even for the most general case the method works in the way we have indicated in our examples.

Table 1 The method of undetermined coefficients

Forcing function $f(t)$	Trial solution $y_p(t)$	Comments
e^{rt}	ae^{rt}	
$\cos \omega t$ or $\sin \omega t$	$a \cos \omega t + b \sin \omega t$	
$P(t)$	$p(t)$	P is a polynomial; p is a polynomial of the same degree
$P(t) \cos \omega t$ or $P(t) \sin \omega t$	$p(t) \cos \omega t +$ $q(t) \sin \omega t$	P is a polynomial; p & q are polynomials of the same degree
$e^{rt} \cos \omega t$ or $e^{rt} \sin \omega t$	$e^{rt}[a \cos \omega t +$ $b \sin \omega t]$	
$e^{rt} P(t) \cos \omega t$ or $e^{rt} P(t) \sin \omega t$	$e^{rt}[p(t) \cos \omega t +$ $q(t) \sin \omega t]$	P is a polynomial; p & q are polynomials of the same degree

EXERCISES

In Exercises 1–4, use the technique demonstrated in Example 5.5 to find a particular solution for the given differential equation.

1. $y'' + 3y' + 2y = 4e^{-3t}$

2. $y'' + 6y' + 8y = -3e^{-t}$

3. $y'' + 2y' + 5y = 12e^{-t}$

4. $y'' + 3y' - 18y = 18e^{2t}$

In Exercises 5–8, use the form $y_p = a \cos \omega t + b \sin \omega t$, as in Example 5.7, to help find a particular solution for the given differential equation.

5. $y'' + 4y = \cos 3t$

6. $y'' + 9y = \sin 2t$

7. $y'' + 7y' + 6y = 3 \sin 2t$

8. $y'' + 7y' + 10y = -4 \sin 3t$

9. Suppose that $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{i\omega t}. \tag{5.28}$$

Substitute $z(t)$ into equation (5.28), then compare the real and imaginary parts of each side of the resulting equation to prove two facts:

$$x'' + px' + qx = A\cos\omega t,$$
$$y'' + py' + qy = A\sin\omega t.$$

Write a short paragraph summarizing the significance of this result.

In Exercises 10–13, use the complex method, as in Example 5.11, to find a particular solution for the differential equation.

10. Exercise 5

11. Exercise 6

12. Exercise 7

13. Exercise 8

In Exercises 14–17, use the technique shown in Example 5.16 to find a particular solution for the given differential equation.

14. $y'' + 5y' + 4y = 2 + 3t$

15. $y'' + 6y' + 8y = 2t - 3$

16. $y'' + 5y' + 6y = 4 - t^2$

17. $y'' + 3y' + 4y = t^3$

In Exercises 18–23, use the technique of Section 4.3 to find a solution of the associated homogeneous equation, then use the technique of this section to find a particular solution. Use Theorem 5.2 to form the general solution. Then find the solution satisfying the given initial conditions.

18. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0$

19. $y'' - 4y' - 5y = 4e^{-2t}, \quad y(0) = 0, \quad y'(0) = -1$

20. $y'' + 2y' + 2y = 2\cos 2t, \quad y(0) = -2, \quad y'(0) = 0$

21. $y'' - 2y' + 5y = 3\cos t, \quad y(0) = 0, \quad y'(0) = -2$

22. $y'' + 4y' + 4y = 4 - t, \quad y(0) = -1, \quad y'(0) = 0$

23. $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0$

In Exercises 24–29, the forcing term is also a solution of the associated homogeneous solution. Use the technique of Example 5.19 to find a particular solution.

24. $y'' - 3y' - 10y = 3e^{-2t}$

25. $y'' - y' - 2y = 2e^{-t}$

26. $y'' + 4y = 4\cos 2t$

27. $y'' + 9y = \sin 3t$

28. $y'' + 4y' + 4y = 2e^{-2t}$

29. $y'' + 6y' + 9y = 5e^{-3t}$

30. If $y_f(t)$ is a solution of

$$y'' + py' + qy = f(t)$$

and $y_g(t)$ is a solution of

$$y'' + py' + qy = g(t),$$

show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of

$$y'' + py' + qy = \alpha f(t) + \beta g(t),$$

where α and β are any real numbers.

Use the technique suggested by Examples 5.21 and 5.24, as well as Exercise 30, to help find particular solutions for the differential equations in Exercises 31–38.

31. $y'' + 2y' + 2y = 2 + \cos 2t$

32. $y'' - y = t - e^{-t}$

33. $y'' + 25y = 2 + 3t + \cos 5t$

34. $y'' + 2y' + y = 3 - e^{-t}$

35. $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

36. $y'' + 2y' + 2y = 3\cos t - \sin t$

37. $y'' + 4y' + 4y = e^{-2t} + \sin 2t$

38. $y'' + 16y = e^{-4t} + 3\sin 4t$

39. Use the form $y_p(t) = (at + b)e^{-4t}$ in an attempt to find a particular solution of the equation $y'' + 3y' + 2y = te^{-4t}$.

Use an approach similar to that in Exercise 39 to find particular solutions of the equations in Exercises 40–43.

40. $y'' - 3y' + 2y = te^{-3t}$

41. $y'' + 2y' + y = t^2e^{-2t}$

42. $y'' + 5y' + 4y = te^{-t}$

43. $y'' + 3y' + 2y = t^2e^{-2t}$

44. Use the form $y_p = e^{-2t}(a\cos t + b\sin t)$ in an attempt to find a particular solution of $y'' + 2y' + 2y = e^{-2t}\sin t$.

45. If $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{(a+bi)t},$$

show that $x(t)$ and $y(t)$ are solutions of

$$x'' + px' + qx = Ae^{at}\cos bt \quad \text{and} \quad y'' + py' + qy = Ae^{at}\sin bt,$$

respectively.

46. Use the technique suggested by Exercise 45 to find a particular solution of the equation in Exercise 44.

47. Prove that the imaginary part of the solution of $z'' + z' + z = te^{it}$ is a solution of $y'' + y' + y = t\sin t$. Use this idea to find a particular solution of $y'' + y' + y = t\sin t$.

Variation of Parameters

In this section we introduce a technique called ***variation of parameters***. This technique is used to find a particular solution to more general higher-order equations, provided we know a fundamental set of solutions to the associated homogeneous equation. As we did in the previous section, we will illustrate the method for second-order equations. The method also works for higher-order equations, but it is usually more efficient to solve the associated first-order system using variation of parameters. This will be discussed in a later chapter.

We are interested in solving the equation

$$y'' + p(t)y' + q(t)y = g(t). \tag{6.1}$$

Notice that we are allowing the coefficients $p(t)$ and $q(t)$ to be functions of t. In particular, we are not restricting them to be constants. This might seem to be a great increase in generality, but there is a rather strong constraint. We will have to assume that we have computed a fundamental set of solutions y_1 and y_2 to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{6.2}$$

Then the general solution to the homogeneous equation is

$$y_h = C_1 y_1 + C_2 y_2, \tag{6.3}$$

where C_1 and C_2 are arbitrary constants.

The idea behind variation of parameters is to replace the constants C_1 and C_2 in (6.3) by unknown functions $v_1(t)$ and $v_2(t)$ and look for a particular solution to the inhomogeneous equation (6.1) of the form

$$y_p = v_1 y_1 + v_2 y_2. \tag{6.4}$$

You will notice the similarity with the method of variation of parameters as it was used to solve first-order linear equations in Chapter 2.

EXAMPLE 6.5 ◆ Find a particular solution to the equation

$$y'' + y = \tan t. \tag{6.6}$$

The right side, $\tan t$, is not one of the forms that can be handled with undetermined coefficients. We will find a solution by the method of variation of parameters. First, note that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions to the associated homogeneous equation $y'' + y = 0$. So we look for a particular solution of the form

$$y_p = v_1 \cos t + v_2 \sin t. \tag{6.7}$$

We start by computing the first derivative of y_p.

$$\begin{aligned} y_p' &= v_1' \cos t - v_1 \sin t + v_2' \sin t + v_2 \cos t \\ &= (v_1' \cos t + v_2' \sin t) - v_1 \sin t + v_2 \cos t \end{aligned} \tag{6.8}$$

The differential equation (6.6) puts only one constraint on the two functions v_1 and v_2. We can impose one more constraint. Notice that if we set the term in parentheses on the right in (6.8) equal to zero, then the expression simplifies, and we will eliminate the first derivatives of v_1 and v_2. Then when we compute y_p'' no second derivatives of v_1 and v_2 will appear. Hence we set

$$v_1' \cos t + v_2' \sin t = 0. \tag{6.9}$$

Now (6.8) simplifies to

$$y_p' = -v_1 \sin t + v_2 \cos t.$$

Differentiating this equation gives

$$\begin{aligned} y_p'' &= -v_1' \sin t - v_1 \cos t + v_2' \cos t - v_2 \sin t \\ &= -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t. \end{aligned}$$

Inserting these expressions for y_p and y_p'' into the left-hand side of our differential equation, we obtain

$$\begin{aligned} y_p'' + y_p &= \left(-v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t\right) + (v_1 \cos t + v_2 \sin t) \\ &= -v_1' \sin t + v_2' \cos t. \end{aligned}$$

Comparing this with (6.6), we see that y_p is a solution to (6.6) provided $-v_1' \sin t + v_2' \cos t = \tan t$. This is our second equation for v_1' and v_2'. Let's restate it along with (6.9).

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0 \\ -v_1' \sin t + v_2' \cos t &= \tan t \end{aligned}$$

This is a system of two linear equations for the unknowns v_1' and v_2'. The coefficient matrix is

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The system can be solved provided the determinant of this matrix is nonzero. Since the determinant is $\cos^2 t + \sin^2 t = 1$, we can solve. Let's solve them by elimination. Multiply the first by $\sin t$ and the second by $\cos t$ to get

$$\begin{aligned} \sin t(v_1' \cos t + v_2' \sin t) &= 0 \\ \cos t(-v_1' \sin t + v_2' \cos t) &= \sin t. \end{aligned}$$

When added together, the term involving v_1' is eliminated:

$$v_2'(\sin^2 t + \cos^2 t) = \sin t \quad \text{or} \quad v_2' = \sin t.$$

Similarly, if we multiply the first equation by $\cos t$ and the second by $\sin t$ and then subtract the second from the first, we get

$$v_1'(\cos^2 t + \sin^2 t) = v_1' = -\sin t \tan t \quad \text{or} \quad v_1' = -\frac{\sin^2 t}{\cos t}.$$

When we integrate the equations for v_1' and v_2', we get

$$v_1(t) = -\ln|\sec t + \tan t| + \sin t \quad \text{and} \quad v_2(t) = -\cos t. \qquad (6.10)$$

Notice that we have left off the constants of integration. Since we are after one particular solution, we can set the constants equal to zero.

Substituting the formulas for v_1 and v_2 into 6.7 gives our particular solution

$$\begin{aligned} y_p(t) &= (-\ln|\sec t + \tan t| + \sin t)\cos t - \cos t \sin t \\ &= -(\cos t)\ln|\sec t + \tan t|. \end{aligned}$$

◆

The general case

Does this procedure always work? Let's examine the general case and see what happens. To find a particular solution to

$$y'' + p(t)y' + q(t)y = g(t), \tag{6.11}$$

we look for a solution of the form

$$y_p = v_1 y_1 + v_2 y_2, \tag{6.12}$$

where v_1 and v_2 are functions that are yet to be determined, and y_1 and y_2 are a fundamental set of solutions to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{6.13}$$

We compute the derivative of y_p,

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' = [v_1' y_1 + v_2' y_2] + v_1 y_1' + v_2 y_2'.$$

As in Example 6.5, we set the expression in the square brackets equal to zero, or

$$v_1' y_1 + v_2' y_2 = 0. \tag{6.14}$$

The first derivative then simplifies to

$$y_p' = v_1 y_1' + v_2 y_2'.$$

Differentiating again, we obtain

$$y_p'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''.$$

Inserting y_p, y_p', and y_p'' into the left-hand side of equation (6.11), we get

$$\begin{aligned} y_p'' + p(t)y_p' + q(t)y_p &= \left(v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''\right) \\ &\quad + p(t)\left(v_1 y_1' + v_2 y_2'\right) + q(t)(v_1 y_1 + v_2 y_2) \\ &= v_1(y_1'' + p(t)y_1' + q(t)y_1) \\ &\quad + v_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &\quad + v_1' y_1' + v_2' y_2'. \end{aligned}$$

Since y_1 and y_2 are solutions to the homogeneous equation (6.13), this simplifies to

$$y_p'' + p(t)y_p' + q(t)y_p = v_1' y_1' + v_2' y_2'.$$

Now we see that y_p is a solution to (6.11) provided that

$$v_1' y_1' + v_2' y_2' = g(t).$$

This is our second equation for v_1 and v_2. We restate it along with Equation (6.14):

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= g(t). \end{aligned} \tag{6.15}$$

This is a system of two linear equations for v_1' and v_2'. The coefficient matrix is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

The system can be solved provided that the determinant of this matrix is nonzero. The determinant will be recognized as the Wronskian of y_1 and y_2,

$$W = y_1y_2' - y_2y_1'.$$

Since y_1 and y_2 form a fundamental set of solutions to the homogeneous equation, they are linearly independent and we know by Proposition 1.26 that the Wronskian $W(t) \neq 0$.

These equations can be solved by elimination (multiply the first by y_1' and the second by y_1 and then subtract) to obtain

$$v_2' = \frac{y_1 g}{y_1y_2' - y_1'y_2} \quad \text{and} \quad v_1' = \frac{-y_2 g}{y_1y_2' - y_1'y_2}.$$

Notice that the denominator in the above expressions is the Wronskian of y_1 and y_2 and is always nonzero.

We can integrate these equations:

$$\begin{aligned} v_1(t) &= \int \frac{-y_2(t)g(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}, \quad \text{and} \\ v_2(t) &= \int \frac{y_1(t)g(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}. \end{aligned} \tag{6.16}$$

When they are inserted into (6.12), we get a particular solution.

Summary

To find a particular solution to the inhomogeneous equation $y'' + py' + qy = g$, follow these steps.

1. Find a fundamental set of solutions y_1, y_2 to the associated homogeneous equation $y'' + py' + qy = 0$.
2. Form $y_p = v_1y_1 + v_2y_2$ where v_1 and v_2 are functions to be determined.
3. Find v_1 and v_2. Here there are two possible ways to proceed. The first way is to use the formulas in (6.16). This requires that you remember these formulas or that you have the use of a book that contains the formulas.

   The second way is to follow the procedure of the method of variation of parameters. The procedure has the following steps:
   (a) Differentiate y_p:

   $$y_p' = (v_1'y_1 + v_2'y_2) + v_1y_1' + v_2y_2'$$

   and set the first term on the right equal to zero.

   $$v_1'y_1 + v_2'y_2 = 0 \tag{6.17}$$

(b) Take the second derivative of y_p and insert y_p, y'_p, and y''_p into the differential equation. After simplifying, a second equation will appear:

$$v'_1 y'_1 + v'_2 y'_2 = g(t). \tag{6.18}$$

(c) Solve (6.17) and (6.18) for v'_1 and v'_2 by elimination.

(d) Integrate to find v_1 and v_2.

4. Substitute v_1 and v_2 into $y_p = v_1 y_1 + v_2 y_2$.

It is up to the reader to decide which of the methods to use in step 3.

Although variation of parameters always works theoretically, its success requires a fundamental set of solutions to the homogeneous equation and the ability to compute the integrals in (6.16). The method is of significant theoretical importance, however.

EXERCISES

For Exercises 1–12, find a particular solution to the given second-order differential equation.

1. $y'' + 9y = \tan(3t)$

2. $y'' + 4y = \sec(2t)$

3. $y'' - y = t + 3$

4. $x'' - 2x' - 3x = 4e^{3t}$

5. $y'' - 2y' + y = e^t$

6. $x'' - 4x' + 4x = e^{2t}$

7. $x'' + x = \tan^2 t$

8. $x'' + x = \sec^2 t$

9. $x'' + x = \tan^3 t$

10. $x'' + x = \sec^3 t$

11. $y'' + y = \tan t + \sin t + 1$

12. $y'' + y = \sec t + \cos t - 1$

13. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2 y''(t) + 3ty'(t) - 3y(t) = 0.$$

Use variation of parameters to find the general solution to

$$t^2 y''(t) + 3ty'(t) - 3y(t) = \frac{1}{t}.$$

14. Verify that $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln t$ are solutions to the homogeneous equation

$$t^2 y''(t) + 3ty'(t) + y(t) = 0, \quad \text{for } t > 0.$$

Use variation of parameters to find the general solution to

$$t^2 y''(t) + 3ty'(t) + y(t) = \frac{1}{t}, \quad \text{for } t > 0.$$

orced Harmonic Motion

In this section, we apply the technique of undetermined coefficients to analyze harmonic motion with an external sinusoidal forcing term. We derived the model in Section 4.4, and with a sinusoidal forcing term, equation (4.4) becomes

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t. \tag{7.1}$$

The constant A is the amplitude of the driving force, and ω is the ***driving frequency***.

To focus our thinking, we may suppose that we have an iron mass m suspended on a spring, with the top of the spring attached to a motor that moves the top of the spring up and down. We can also consider an RLC circuit in which the source voltage is sinusoidal.

We will first treat the case with no damping.

Forced undamped harmonic motion

The equation we will deal with here comes from (7.1) with $c = 0$ or

$$x'' + \omega_0^2 x = A\cos\omega t. \tag{7.2}$$

The associated homogeneous equation is

$$x'' + \omega_0^2 x = 0, \tag{7.3}$$

with general solution

$$x_h = C_1\cos\omega_0 t + C_2\sin\omega_0 t.$$

We have to consider separately the cases when the driving frequency ω is equal to the natural frequency and when it is not.

Case 1: $\omega \neq \omega_0$. If the driving frequency is not equal to the natural frequency, we look for a particular solution of the form

$$x_p = a\cos\omega t + b\sin\omega t, \tag{7.4}$$

where a and b are undetermined constants. Substituting x_p into the left-hand side of the inhomogeneous differential equation (7.2) gives

$$x_p'' + \omega_0^2 x_p = a(\omega_0^2 - \omega^2)\cos\omega t + b(\omega_0^2 - \omega^2)\sin\omega t.$$

Comparing this with (7.2), we see that we have a solution provided that

$$a = \frac{A}{\omega_0^2 - \omega^2} \quad \text{and} \quad b = 0.$$

So our particular solution is

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

Notice that x_p is oscillatory, with the same frequency as the driving force. The amplitude of this oscillation also depends on the driving frequency and gets larger as the driving frequency approaches the natural frequency of the spring. This is our first indication of resonance. We will have more to say about that shortly.

The general solution to the inhomogeneous equation is

$$x(t) = x_h(t) + x_p(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

Let's look at the solution where the motion starts at equilibrium. This means the initial conditions are $x(0) = x'(0) = 0$. It is easily seen that $C_1 = -A/(\omega_0^2 - \omega^2)$, and $C_2 = 0$. Hence the solution is

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t). \tag{7.5}$$

This is a superposition of two oscillations with different frequencies. The result is interesting.

EXAMPLE 7.6 ◆ Suppose $A = 23$, $\omega_0 = 11$, and $\omega = 12$. With these values of the parameters the solution (7.5) becomes

$$x(t) = \cos 11t - \cos 12t.$$

The graph is shown in Figure 1. This figure shows the phenomenon called ***beats***. It occurs whenever two frequencies that are almost equal interfere with each other as we see in (7.5). ◆

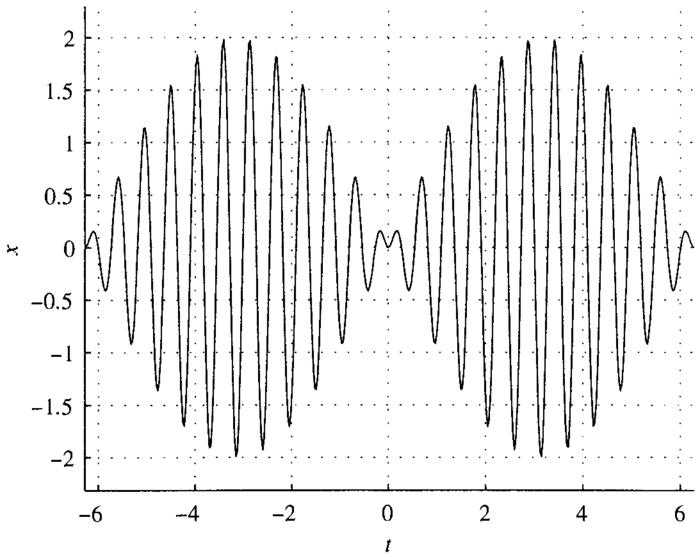
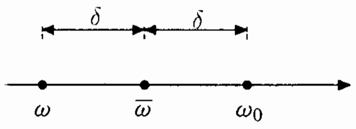


Figure 1 Beats in forced, undamped, harmonic motion.

We can understand the phenomenon of beats better after a little algebra. We introduce $\overline{\omega} = (\omega_0 + \omega)/2$, which is the ***mean frequency***, and $\delta = (\omega_0 - \omega)/2$, the ***half difference***. Then

$$\omega = \overline{\omega} - \delta \quad \text{and} \quad \omega_0 = \overline{\omega} + \delta.$$



2 The relationship of ω ω_0 to $\overline{\omega}$ and δ.

The relationship between these variables is shown in Figure 2. If we substitute these equations into (7.5) and use the addition law for the cosine, we get

$$x(t) = \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t. \tag{7.7}$$

In Example 7.6, $\delta = -1/2$ and $\overline{\omega} = 23/2$. In that case the factor $\sin \delta t = \sin(-t/2)$ oscillates very slowly, especially in comparison with the faster oscillation with frequency $\overline{\omega} = 23/2$. Thus the solution in (7.7) can be seen to be a fast oscillation with a frequency $\overline{\omega}$ and an amplitude

$$\left| \frac{A \sin \delta t}{2\overline{\omega}\delta} \right|, \tag{7.8}$$

which oscillates much more slowly. If we were to superimpose a graph of this slow oscillating amplitude on the graph in Figure 1, it would appear as a curve through the maxima of the faster oscillation. This is called the ***envelope*** of the faster oscillation. (See Exercise 2.)

The phenomenon of beats illustrated in Figure 1 occurs in many situations. For example, it is used by a piano tuner to be sure that particular piano strings are properly tuned. The tuner strikes a tuning fork, which vibrates at the correct frequency. He next hits the poorly tuned piano key, which vibrates at a slightly different frequency. The two sounds interfere in a way modeled by (7.5), and therefore by (7.7). The tuner hears the mean frequency with an amplitude that is modulated by the difference frequency, as shown in Figure 1. This modulation gives rise to beats in the tone that are readily audible. When the string is properly tuned, the beats go away.

Case 2: $\omega = \omega_0$. In this case, the particular solution given in (7.4) is a solution to the homogeneous equation. Therefore, we look for a particular solution of the form

$$x_p = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

Inserting x_p into the left-hand side of the differential equation in 7.2, we get

$$\begin{aligned} x_p'' + \omega_0^2 x_p &= \left[2\omega_0(-a \sin \omega_0 t + b \cos \omega_0 t) + t\omega_0^2(-a \cos \omega_0 t - b \sin \omega_0 t)\right] \\ &\quad + \omega_0^2 t(a \cos \omega_0 t + b \sin \omega_0 t) \\ &= 2\omega_0(-a \sin \omega_0 t + b \cos \omega_0 t). \end{aligned}$$

Setting the right side equal to $A \cos \omega_0 t$, we see that we have a solution provided that

$$b = \frac{A}{2\omega_0} \quad \text{and} \quad a = 0.$$

The particular solution is

$$x_p = \frac{A}{2\omega_0} t \sin \omega_0 t,$$

and the general solution to the inhomogeneous equation is

$$x(t) = x_h(t) + x_p(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t.$$

In the special case where $x(0) = C_1 = 0$, $x'(0) = C_2 = 0$, $A = 8$, and $\omega_0 = 4$, this reduces to

$$x(t) = t \sin 4t.$$

The graph of this function is shown in Figure 3. Notice how the solution grows with time. This growth is due to the fact that the frequency of the forcing term equals the natural vibrating frequency of the spring. The force pulls and pushes at a frequency equal to the natural frequency of the spring. Thus the amplitude of the oscillatory motion of the mass increases. This type of behavior is called ***resonance***, and engineers try to eliminate it in the design of mechanical systems, since oscillations that grow in amplitude can eventually cause the system to break.

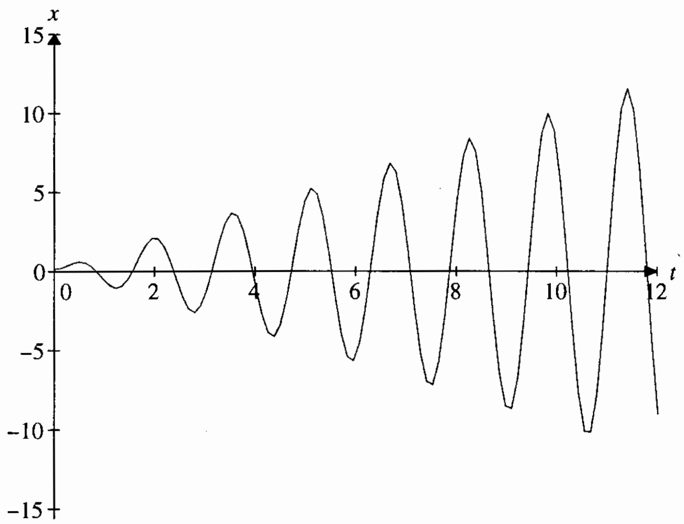


Figure 3 Forced, undamped, harmonic motion where the driving frequency equals the natural frequency.

In reality, mechanical systems always have some damping (friction) that will keep the solution from getting infinitely large. However, resonance can still cause mechanical motions to become large enough to cause major structural damage. A common example of the destructiveness of resonance is the shattering of a glass by the voice of a singer. The singer has to sing a note at a frequency that is very close to the natural frequency of the glass in order to cause it to resonate and eventually shatter. Another example involves the behavior of a troop of soldiers marching over a bridge. If they were to continue to march in perfect step, they might be marching at the natural frequency of some component of the bridge and cause it to resonate and eventually break. Consequently, since time immemorial troops of soldiers have broken ranks when marching over a bridge.

On the other hand, resonance is exploited in electrical systems that are governed by the same model as (7.2) in order to tune radios, for example.

Forced damped harmonic motion

If we add a damping term to the system, interesting things happen. Now we are dealing with the equation in (7.1),

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t. \tag{7.9}$$

The associated homogeneous equation is

$$x'' + 2cx' + \omega_0^2 x = 0. \tag{7.10}$$

The characteristic roots are

$$\lambda = -c \pm \sqrt{c^2 - \omega_0^2}.$$

Let's suppose that we are in the underdamped case, where $c < \omega_0$. We found in Section 4.4 that the general solution is

$$x_h = e^{-ct}\left(C_1\cos(\eta t) + C_2\sin(\eta t)\right),$$

where

$$\eta = \sqrt{\omega_0^2 - c^2}.$$

To find a particular solution to the inhomogeneous equation, we use undetermined coefficients, but this time it is easier to use the complex method. This means we look for a solution $z(t) = ae^{i\omega t}$ to the equation

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t} \tag{7.11}$$

and then set $x_p = \mathrm{Re}(z)$. Let's substitute $z(t)$ into the left side of (7.11). We get

$$z'' + 2cz' + \omega_0^2 z = \left[(i\omega)^2 + 2c(i\omega) + \omega_0^2\right]ae^{i\omega t} = P(i\omega)ae^{i\omega t},$$

where the polynomial

$$P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$$

is the characteristic polynomial of the differential equation in (7.11) or (7.10). Thus equation (7.11) becomes

$$P(i\omega)ae^{i\omega t} = Ae^{i\omega t}.$$

Solving for a we get $a = A/P(i\omega)$. Hence

$$z(t) = ae^{i\omega t} = \frac{A}{P(i\omega)}e^{i\omega t} = H(i\omega)Ae^{i\omega t}, \tag{7.12}$$

where we have set

$$H(i\omega) = \frac{1}{P(i\omega)}.$$

The function $H(i\omega)$ is called the ***transfer function***. To see what it is like, we look at its reciprocal

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2 = (\omega_0^2 - \omega^2) + 2ic\omega.$$

We want to write this complex number in polar form, which means that we want to find a positive number R and an angle ϕ such that

$$P(i\omega) = Re^{i\phi} = R[\cos\phi + i\sin\phi].$$

We need

$$R\cos\phi = \omega_0^2 - \omega^2 \quad \text{and} \quad R\sin\phi = 2c\omega.$$

This is just polar coordinates, so

$$R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2},$$

while the angle ϕ is defined by the pair of equations

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}} \quad \text{and} \quad \sin\phi = \frac{2c\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

(See the discussion of amplitude and phase in Section 4.4.) Notice that $2c\omega > 0$, so $\sin\phi > 0$. This means that the phase always satisfies $0 < \phi < \pi$. Furthermore, we have

$$\cot\phi = \frac{\omega_0^2 - \omega^2}{2c\omega} \quad \text{or} \quad \phi = \phi(\omega) = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right). \tag{7.13}$$

The last equation defines ϕ uniquely because $0 < \phi < \pi$.

Now we see that the transfer function can be written as

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R}e^{-i\phi}.$$

We will define the ***gain*** G by

$$G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}, \tag{7.14}$$

and then the transfer function is

$$H(i\omega) = G(\omega)e^{-i\phi(\omega)}. \tag{7.15}$$

From (7.12) we see that the solution to (7.11) is

$$z(t) = H(i\omega)Ae^{i\omega t} = G(\omega)Ae^{i(\omega t - \phi)}.$$

Finally, the solution to (7.9) is

$$x_p(t) = \operatorname{Re} z(t) = G(\omega)A\cos(\omega t - \phi). \tag{7.16}$$

This form makes it clear that x_p is sinusoidal with the same frequency as the driving force. It also shows that the amplitude of x_p is the product of the gain $G(\omega)$ times the amplitude of the driving force. In addition, x_p is out of phase with the driving force by the amount $\phi = \phi(\omega)$, given in (7.13).

The general solution to the inhomogeneous equation is

$$\begin{aligned} x &= x_h + x_p \\ &= e^{-ct}\left(C_1\cos(\eta t) + C_2\sin(\eta t)\right) + G(\omega)A\cos(\omega t - \phi). \end{aligned} \tag{7.17}$$

Notice that x_h has the factor e^{-ct}, which quickly decays to 0 as $t \to \infty$. For this reason, this term is called the ***transient*** term. The rate of the decay of the transient term is governed by the exponential factor e^{-ct}. After the passage of time equal to $T_c = 1/c$, this decreases to $e^{-1} \approx 0.3679$, and the amplitude of the transient term has decreased to e^{-1} times its original value. T_c is called the ***time constant***, and it is defined by this property. Notice that after the passage of time equal to $4T_c$ the amplitude of the transient is reduced to $e^{-4} \approx 0.0183$ times its original value, after which the effect of the transient term is usually negligible.

The particular solution x_p in (7.16) does not decay. It is therefore called the ***steady-state*** term. It is this term that is driven by the external force $A\cos\omega t$. After the passage of time equal to a few time constants, it is the only term that is important.

(AMPLE 7.18 ◆ Consider a forced vibrating spring where $m = 5$kg, $\mu = 7$ kg/s, and $k = 3$ kg/s^2, with a forcing term $2\cos 4t$. Suppose the initial conditions are such that the constants in (7.17) are $C_1 = 0$ and $C_2 = 1$. Find the amplitude and the phase of the steady-state solution. Plot the displacement of the resulting oscillation versus time. Add a plot of the steady-state oscillation.

The equation for the vibrating spring with the given parameters is

$$5x'' + 7x' + 3x = 2\cos 4t.$$

Dividing by the mass 5, we get

$$x'' + \frac{7}{5}x' + \frac{3}{5}x = \frac{2}{5}\cos 4t.$$

This is the form of the equation for the forced harmonic oscillator. The natural frequency is $\omega_0 = \sqrt{3/5} \approx 0.7746$. In addition, $c = 7/10$. With the assumptions of the example, the transient solution is

$$x_h(t) = e^{-ct}\sin\eta t,$$

where $\eta = \sqrt{\omega_0^2 - c^2} \approx 0.3317$. The gain is computed from (7.14) to be $G(4) \approx 0.0610$. Hence the amplitude of the steady-state solution is $G(4) \times (2/5) \approx 0.0244$. Finally, we compute the phase from (7.13) and get $\phi = \phi(4) = 2.7928$. A plot of the solution together with its steady-state term is given in Figure 4. Note how the solution x converges to the steady-state solution as t gets large. ◆

Transient solutions can be quite large in comparison to the steady-state solution, as we have seen in Example 7.18, as shown in Figure 4. Such large transient currents in electrical circuits can be destructive. It is almost certain that you have experienced this. Transients arise whenever an electrical circuit is turned on or off. How often have you turned on a light and one of the bulbs burned out immediately? This is caused by a large transient current flowing through an already weakened bulb. How

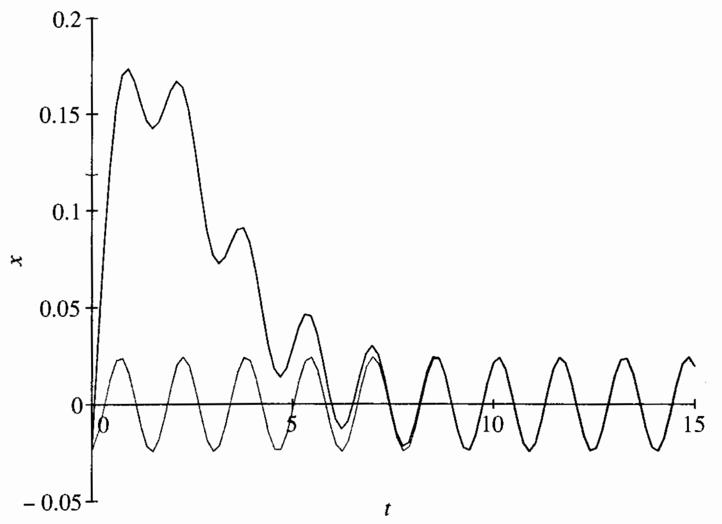


Figure 4 The motion of a forced spring. The steady-state solution is in blue.

often have you turned on a light and one of the bulbs was burned out, although it had been working fine the last time you used the light? This time the bulb most likely burned out when the lights were turned out, and again it was a large transient caused by the shutdown that did the job.

The destructive effect of transients is the reason why it is often a good idea to leave electrical equipment running even when it is not being used. For example, it is highly recommended that computers be always left on, unless it is known that they will not be used for a couple of days at the minimum. Transients are particularly harmful to hard disks.

We now examine the steady-state term (7.16) more closely. We will concentrate on the amplitude. We will examine how the gain G depends on the driving frequency ω. There are too many terms in the formula (7.14) for G to easily understand what will happen for all possible values of the parameters. We remedy this by ***lumping parameters***. First, we will want to see how G changes as the driving frequency changes relative to the natural frequency, so we will introduce $s = \omega/\omega_0$ by setting $\omega = s\omega_0$ and consider G as a function of s. Then we introduce $D = 2c/\omega_0$, by substituting $c = D\omega_0/2$. The new constant D measures the effect of the damping force. When these changes are introduced, the gain takes the simpler form

$$G = \frac{1}{\omega_0^2\sqrt{(1-s^2)^2 + D^2s^2}}, \quad \text{where} \quad s = \frac{\omega}{\omega_0} \quad \text{and} \quad D = \frac{2c}{\omega_0}.$$

While still somewhat complicated, this expression for the gain is much easier to understand than (7.14). It allows us to easily see how the gain varies as the quotient $s = \omega/\omega_0$ varies. Because the natural frequency is fixed, let's look at the quantity

$$\omega_0^2 G = \frac{1}{\sqrt{(1-s^2)^2 + D^2s^2}}.$$

The behavior of $\omega_0^2 G$ as a function of s and D is shown in Figure 5. Since the natural frequency ω_0 is fixed, $D = 2c/\omega_0$ is proportional to the damping constant. Notice that for small damping constants, the gain has a significant maximum for a driving frequency that is close to the natural frequency (when $s = \omega/\omega_0$ is near 1). The maximum gets larger as the damping constant decreases. This is another example of resonance. As we see in Figure 5, the resonance increases as the damping constant decreases.

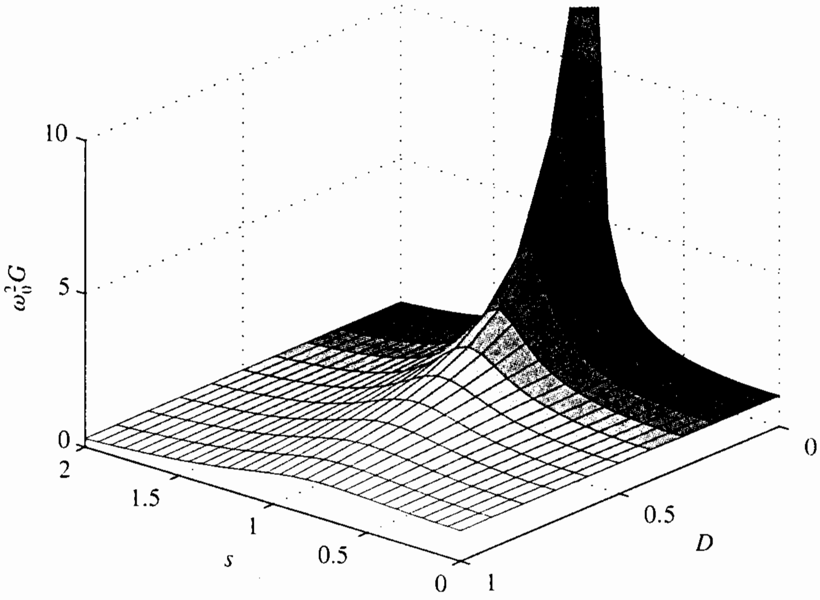


Figure 5 The gain of a forced harmonic oscillator with damping. $s = \omega/\omega_0$ and $D = 2c/\omega_0$.

Notice from Figure 5 that the maximum gain is at a frequency that is slightly smaller than the natural frequency. We will leave it as an exercise to determine the precise frequency at which the maximum gain occurs.

EXERCISES

1. In the narrative (Case 1), the substitution $x_p = a\cos\omega t + b\sin\omega t$ produced

$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t$$

as a particular solution of $x'' + \omega_0^2 x = A\cos\omega t$, when $\omega \neq \omega_0$.

(a) Use the substitution $x_p = a\cos\omega t$ to produce the same result.

(b) Use the complex method to produce the same result.

2. The function $x = \cos 6t - \cos 7t$ has mean frequency $\overline{\omega} = 13/2$ and half difference $\delta = 1/2$. Thus,

$$\begin{aligned}\cos 6t - \cos 7t &= \cos\left(\frac{13}{2} - \frac{1}{2}\right)t - \cos\left(\frac{13}{2} + \frac{1}{2}\right)t,\\ &= 2\sin\frac{1}{2}t\sin\frac{13}{2}t.\end{aligned}$$

Plot the graph of x, and superimpose the "envelope" of the beats, which is the slow frequency oscillation $y(t) = \pm 2\sin(1/2)t$. Use different line styles or colors to differentiate the curves.

In Exercises 3–6, plot the given graph on on an appropriate time interval. Use the technique of Exercise 2 to superimpose the plot of the envelope of the beats in a different linestyle and/or color.

3. $\cos 10t - \cos 11t$

4. $\cos 9t - \cos 10t$

5. $\sin 12t - \sin 11t$

6. $\sin 11t - \sin 10t$

7. Use a computer to plot the graph of the solution

$$x(t) = \cos 11t - \cos \omega t \tag{7.19}$$

for $\omega = 9$, 10, 10.5, 10.9, and 10.99 on the time interval $[0, 24]$. Is the last case ($\omega = 10.99$) an example of resonance? *Hint*: Use the suggestions in the narrative to place equation (7.19) in the form $x(t) = A\sin\delta t \sin\overline{\omega}t$. Use this result to justify your conclusion.

8. If the system doesn't start from equilibrium, the beats might not be as pronounced, but they are still there. Use a numerical solver to plot the solution of

$$y'' + 144y = \cos 11t, \quad y(0) = y_0, \quad y'(0) = 0,$$

for each $y_0 = 0, 0.1, 0.2, 0.3, 0.4$, and 0.5. Plot the solutions on the same time interval $[0, 4\pi]$ and compare the plots. Are the "slow" and "fast" frequencies still present?

9. A 1 kg mass is attached to a spring ($k = 4$ kg/s^2) and the system is allowed to come to rest. The spring mass system is attached to a machine that supplies an external driving force $f(t) = 4\cos\omega t$ Newtons. The system is started from equilibrium, the mass having no initial displacement nor velocity. Ignore any damping forces.

(a) Find the position of the mass as a function of time.

(b) Place your answer in the form $x(t) = A\sin\delta t \sin\overline{\omega}t$. Select an ω near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot that shows the "beats" and include the envelope of the beating motion in your plot (see Exercise 2).

10. An undamped spring-mass system with external driving force is modeled with

$$x'' + 25x = 4\cos 5t.$$

The parameters of this equation are "tuned" so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates, rather than showing the "beating" character of previous exercises?

(b) Sketch the solution found in part (a).

11. An inductor (1 H) and a capacitor (0.25 F) are connected in series with a signal generator that provides an emf $E(t) = 12\cos\omega t$. Assume that the system is started from equilibrium (no initial charge on the capacitor, no initial current) and ignore any damping effects.

(a) Find the current in the system as a function of time, assuming that the signal generator provides a driving force at near resonant frequency. Plot a sample solution, including the "envelope" of the beats.

(b) Find the current in the system as a function of time, this time assuming that the signal generator provides a driving force at resonant frequency. Plot your solution.

In Exercises 12–15, place the transfer function in the form

$$H(iw) = \frac{1}{R}e^{-i\phi}.$$

Use this result to find the steady-state solution of the given equation.

12. $x'' + x' + 4x = 3\cos 2t$

13. $x'' + 2x' + 2x = 3\sin 4t$

14. $x'' + 2x' + 4x = 2\sin 2\pi t$

15. $x'' + 4x' + 8x = 3\cos 2\pi t$

In Exercises 16–23, find a particular solution to the differential equation using undetermined coefficients as in Examples 5.7 or 5.11. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady-state solution. Use different line styles or colors to differentiate the curves.

16. $x'' + 5x' + 4x = 2\sin 2t, \quad x(0) = 1, \quad x'(0) = 0$

17. $x'' + 7x' + 10x = 3\cos 3t, \quad x(0) = -1, \quad x'(0) = 0$

18. $x'' + 2x' + 2x = \cos 2t, \quad x(0) = 0, \quad x'(0) = 2$

19. $x'' + 4x' + 5x = 3\sin t, \quad x(0) = 0, \quad x'(0) = -3$

For each equation in Exercises 20–23, calculate the time constant T_c. Plot the transient response over $[0, 4T_c]$, showing that this response dies out in $4T_c$, as advertised in the narrative.

20. The equation in Exercise 16

21. The equation in Exercise 17

22. The equation in Exercise 18

23. The equation in Exercise 19

An underdamped, driven, spring-mass system is modeled with the equations in Exercises 24–25.

(i) Find the steady-state solution and place your answer in the form $x_p(t) = A\cos(\omega t - \phi)$. Use a computer to plot this solution.

(ii) Use a numerical solver to plot solutions of the model with initial conditions $(x(0), x'(0)) = (-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(2, 0)$. Select a time interval that shows all of these solutions approaching the steady-state solution as $t \to \infty$.

24. $x'' + 2x' + 4x = 3\cos t$

25. $x'' + 2x' + 5x = 2\cos 3t$

Consider the equations in Exercises 26–27. Find the transfer function, the gain, the phase, and the steady-state response. Plot the driving function $\cos t$ and the steady-state response on the same graph. Explain how one can read the gain and phase from this graph.

26. $x'' + 0.4x' + x = \cos t$

27. $x'' + 0.4x' + 2x = \cos t$

For the equations in Exercises 28 and 29 use a numerical solver to plot the solution with the initial conditions $x(0) = 0$ and $x'(0) = 0$. Being mindful of the time constant, select a time interval where the transient response has died out and superimpose the graph of the forcing function on this interval. Estimate the gain and phase from the resulting graph.

28. $x'' + 0.2x' + 1.44x = \cos t$

29. $x'' + 0.4x' + 1.69x = \cos t$

30. In Figure 6, the solution of $x'' + 2cx' + \omega_0^2 = \cos t$ is drawn in blue. The driving force, $\cos t$, is drawn in black. The initial conditions are unimportant, as you have seen that all solutions eventually approach the steady-state response when $c^2 < \omega_0^2$, which you may assume is the case here. Estimate the gain and the phase, then use equations (7.14) and (7.13) to calculate the values of c and ω_0^2. Use these computed values of c and ω and your numerical solver to reproduce the image in Figure 6 (use $x(0) = 3$ and $x'(0) = 0$).

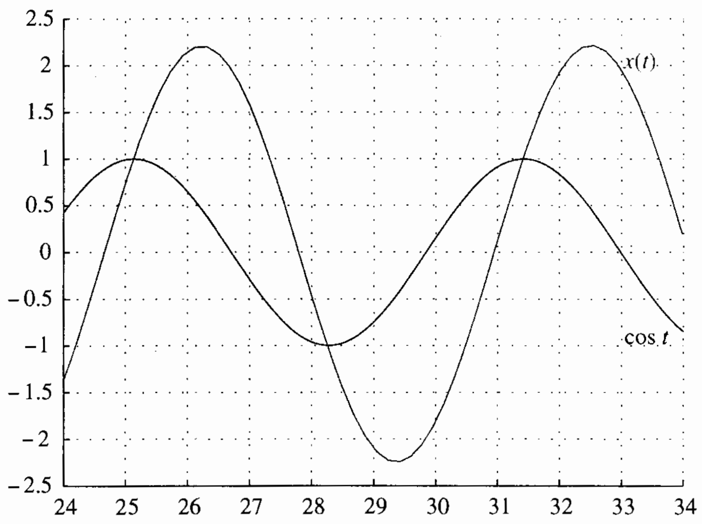


Figure 6 The response to cos t.

Find the gain for each equation in Exercises 31–34 as a function of ω. Plot the graph of the gain versus the driving frequency ω and use the graph to estimate the maximum gain and the frequency at which it occurs.

31. $y'' + 0.01y' + 49y = A\cos\omega t$

32. $y'' + 0.5y' + 4y = A\sin\omega t$

33. $y'' + 0.05y' + 25y = A\sin\omega t$

34. $y'' + 0.25y' + y = A\cos\omega t$

Find the gain of

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

as a function of ω. If $\omega_0^2 > 2c^2$, show that the maximum gain occurs at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2c^2} \tag{7.20}$$

known as the ***resonant frequency*** of the driven oscillator. In Exercises 35–38, use formula (7.20) to calculate the resonant frequency and compare with an estimate found from the plot of the steady-state solution.

35. Exercise 31

36. Exercise 32

37. Exercise 33

38. Exercise 34

In Exercises 39–42, plot the gain versus the frequency ω and estimate the resonant frequency from the graph. Use equation (7.20) to verify this estimate. Use a numerical solver to plot solutions of the given oscillator for selections of the driving frequency ω both near and far from the resonant frequency. Write a short paragraph describing what you learned from this exercise.

39. $y'' + 0.1y' + 25y = \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0$

40. $y'' + 0.2y' + 4y = \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0$

41. $y'' + 0.2y' + 49y = \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0$

42. $y'' + 0.2y' + 9y = \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0$

43. A driven *LRC* circuit is modeled by the equation

$$LI' + RI + \frac{1}{C}Q = A\cos\omega t.$$

Assume the underdamped case.

(a) Show that the charge on the capacitor (once transients have died) will achieve a maximum when the driving frequency is

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}.$$

(b) Show that the current will achieve a maximum (again, after transients have disappeared) when the driving frequency is

$$\omega = \frac{1}{\sqrt{LC}}.$$

44. An inductor (0.1 H), resistor (100 Ω), and capacitor (10^{-3} F) are connected in series with an emf $E(t) = 100\sin 120\pi t$ volts. At time $t = 0$, there is no charge on the capacitor nor any current in the system. Find the current in the system as a function of time.

45. A 50 g mass (1000 g $=$ 1 kg) stretches a spring 10 cm (100 cm $=$ 1 m). As the system moves through the air, a resistive force is supplied that is proportional to, but opposite the velocity, with magnitude $0.01v$. The system is hooked to a machine that applies a driving force to the mass that is equal to $F(t) = 5\cos 4.4t$ Newtons. If the system is started from equilibrium (no displacement, no velocity), find the position of the mass as a function of time. *Hint*: Remember that 1000 g $=$ 1 kg and 100 cm $=$ 1 m.

Project 4.8 Nonlinear Oscillators

Hooke's law (see equation (1.6) in Section 4.1) says that the magnitude of the restoring force of a spring is proportional to the displacement. While this is often valid, it is not always so, and even when it is, it is only valid for small values of the displacement. Here we will discuss springs that do not obey Hooke's law.

To be precise, we will study oscillators that are modeled by the equation

$$x'' + cx' + kx + lx^3 = f(t). \tag{8.1}$$

The implication of this equation is that the restoring force of the oscillator has the form

$$R(x) = -kx - lx^3,$$

where x is the displacement from the equilibrium of the oscillator. Once we give up Hooke's law, there is a wide variety of possible restoring forces. This is just one of many possible choices. Notice that Hooke's law is being assumed if $l = 0$. If $l > 0$ the oscillator is said to be *hard* and if $l < 0$ it is *soft*. With $l \neq 0$, equation (8.1) is called ***Duffing's equation***.

If an oscillator is hard or soft, it is said to be nonlinear, simply because in these cases the differential equation (8.1) is nonlinear. The purpose of this project is to discover differences between the behaviors of nonlinear and linear oscillators. The nonlinear equation (8.1) cannot be solved explicitly, so we will have to rely on numerical solutions.

The period of undamped unforced oscillation

For a linear oscillator, modelled by equation (8.1) with $l = 0$, the natural frequency is $\omega_0 = \sqrt{k}$. This is the frequency of undamped, unforced oscillations. The period of the oscillation is $T = 2\pi/\omega_0$. Notice that the period is completely independent of the amplitude of the oscillation. Is the same true for a hard oscillator?

1. Use the parameters $k = 3$ and $l = 1$. Since the motion is undamped, we have $c = 0$, and since it is unforced, $f(t) = 0$. For several values of the amplitude A between 0.1 and 10, compute the solution to (8.1) with initial conditions $x(0) = A$ and $x'(0) = 0$. In each case, discover the period of the oscillation. Make a graph of the period T versus the amplitude A.
2. Write a short paragraph summarizing your results, emphasizing how the motion of the nonlinear oscillator differs from that of the linear oscillator. In particular describe what is happening to the period as the amplitude approaches 0, and as it gets very large.

Some comments are in order about how to perform this numerical experiment. Perhaps the easiest way to determine the period is to estimate it from the graph of the solution. You can use the graph of the displacement to do this, but then you will have to estimate where on the graph of the displacement a maximum occurs. If you use the velocity $v = x'$ instead, then it is necessary to estimate the time of a zero crossing, since $v = x' = 0$ at a maximum. It is much easier to be precise about the location of a zero crossing than that of a maximum. However, $x' = v = 0$ at a minimum as well, so care will have to be taken to insure that the measurement is done at a maximum.

If the period is determined by graphical estimation, there is a way to decrease the measurement error. Instead of measuring one period, measure the time required for a number of periods, say 3 or 4. Then divide by the number of periods to find the period itself.

Depending on what software you are using, it might be possible to have the computer compute the period. If your computer has a routine that computes the zeros of functions, you might be able to define a function, using your differential equation solver, that returns the velocity $v(t)$ as a function of t for the oscillation. Of course, this function has to be defined in a format that is compatible with the zero finding routine, but this is possible in many mathematical software systems.

Frequency response of a hard oscillator

We examined the frequency response of a linear harmonic oscillator in Section 4.7. We discovered that the total response was the sum of a transient response and a steady-state response. The steady-state response was affected only by the driving force, and in no way depended on the initial conditions. (See equation (7.17) and the discussion that follows it.) We discovered in particular that the amplitude of the steady-state response has the form $G(\omega)A$, where A is the amplitude of the driving force and $G(\omega)$ is the gain given in (7.14). It is the graph of the gain $G(\omega)$ versus ω that we call the frequency response. Here we will see if the frequency response in the nonlinear case has behavior similar to the linear case.

1. Use the parameters $c = 1$, $k = 1$, and $l = 3$ in (8.1). We will use the sinusoidal forcing term $f(t) = 20\cos\omega t$. Experiment by solving the differential equation numerically with a number of values of ω and various initial conditions until you are convinced that the motion settles into a steady-state solution after some time. It is important for what follows that you take notice of about how long it takes to reach steady-state.[1] Submit the plot of one solution that shows the emergence of the steady-state solution.
2. Use the initial conditions $x(0) = 0$ and $v(0) = x'(0) = 0$ for several different driving frequencies ω between 2 and 6. Be sure that $\omega = 4$ is one of those values. In each case find the amplitude of the steady-state solution. Plot the amplitude versus the frequency.
3. Do the same for the initial conditions $x(0) = 6$ and $v(0) = x'(0) = 0$.
4. Plot the frequency response curves from steps 2 and 3 together on one figure, using different line types or colors to distinguish them. The phenomenon that you see illustrated is called ***Duffing's hysteresis***.
5. Write a short paragraph describing what you have discovered. Emphasize how the frequency response for the nonlinear oscillator is different from that of a linear oscillator.

[1] Although the steady-state solution is periodic, it will be more complicated than a simple trigonometric function. In general it is not possible to compute the steady-state solution exactly.

The amplitude of the steady-state solution can be estimated from a plot of the solution, or by examining the numerical data. However, in many computing circumstances, it will be possible to have the computer do this for you. This can be done by removing all values of the computed displacement that might still have a significant transient content, and then finding the remaining value with the largest magnitude.