IV. Autonomous Systems

Throughout this chapter we will consider an *autonomous* system of differential equations

$$x' = f(x), \tag{4.1}$$

where $f : D \to \mathbb{R}^n$ is a continuous mapping defined on an open, connected subset $D \subset \mathbb{R}^n$. As indicated in Remark 1.2, such equations describe physical systems which are isolated from external influences. We will always assume that f is such that every initial value problem $x' = f(x)$, $x(t_0) = x^0$ for $x^0 \in D$ has a unique solution; for example, it suffices that f be locally Lipschitz in D. Finally, we make the convention that, unless otherwise specified, when we refer to a *solution* of (4.1) we always mean a solution which is defined on a maximal interval. With this convention, uniqueness implies that two solutions of the IVP have the same interval of definition as well as the same values there.

4.1 Orbits and asymptotic limit sets

The following simple lemma is basic to the understanding of what makes autonomous systems so special.

Lemma 4.1: *Suppose that $x(t)$ is a solution of (4.1) defined on an interval I, and that $h \in \mathbb{R}$. Then the function $y(t) \equiv x(t+h)$, defined on the interval $J = I - h \equiv \{t \mid t+h \in I\}$, is also a solution.*

Proof: For $t \in J$, $y'(t) = x'(t+h) = f(x(t+h)) = f(y(t))$. Moreover, J must be maximal, for an extension of y to $\tilde{y}$ would yield an extension $\tilde{x} = \tilde{y}(t-h)$ of x. ■

In the notation of Chapter II, the lemma says that

$$\hat{x}(t; t_0, x^0) = \hat{x}(t+h; t_0+h, x^0). \tag{4.2}$$

Note also that while we have assumed above that the initial value problem given by (4.1) and the condition $x(t_0) = x^0$ has a unique solution for all (t_0, x^0), the argument of the lemma implies that it would suffice to assume this for any one t_0 and all $x^0 \in D$.

We next introduce some terminology associated with autonomous systems.

Definition 4.1: An *orbit* or *trajectory* of the system (4.1) is a set $C \subset D$ of the form $\{x(t) \mid t \in I\}$, where $x(t)$ is a solution of (4.1) defined on I.

Definition 4.2: An *equilibrium point* of the system (4.1) (also called a *critical* or *singular* point) is a point $x^0 \in D$ such that $f(x^0) = 0$.

Theorem 4.2: *(a) Every point of D belongs to precisely one orbit of the system (4.1).*
(b) If x^0 is an equilibrium point of the system then $\{x^0\}$ is an orbit.

Proof: If $x^0 \in D$ then the IVP $x' = f(x)$, $x(0) = x^0$ has a solution, and x^0 lies on the corresponding orbit. On the other hand, if x^0 belongs to the orbits C and $\tilde{C}$ then there must exist solutions $x(t)$, $\tilde{x}(t)$ defined on I, $\tilde{I}$ satisfying $C = \{x(t)\}$, $\tilde{C} = \{\tilde{x}(t)\}$, and $x(t_0) = \tilde{x}(\tilde{t}_0) = x^0$ for some $t_0, \tilde{t}_0 \in \mathbb{R}$. Then by Lemma 4.1, $y(t) \equiv x(t + t_0 - \tilde{t}_0)$ is a solution defined on $J = \{t \mid t + t_0 - \tilde{t}_0 \in I\}$, and since $y(\tilde{t}_0) = \tilde{x}(\tilde{t}_0) = x^0$, y and $\tilde{x}$ are identical, by uniqueness. Hence

$$C = \{x(t) \mid t \in I\} = \{y(t) \mid t \in J\} = \{\tilde{x}(t) \mid t \in \tilde{I}\} = \tilde{C}. \quad \blacksquare$$

(b) This is an immediate consequence of the fact that the constant function $x(t) \equiv x^0$ is a solution of (4.1): $x'(t) = 0 = f(x^0) = f(x(t))$.

Remark 4.1: (a) The function $f : D \to \mathbb{R}^n$ is sometimes called a *vector field* on D. We may visualize $f(x)$ as a vector with its tail at x; then the orbits C are tangent to these vectors, and a solution $x(t)$ has speed given by the length of the vector.

(b) Suppose that f and D are such that every solution of (4.1) is defined on all of $\mathbb{R}$. Then for each $t \in \mathbb{R}$ we define a mapping $\Phi_t : D \to D$ by $\Phi_t(x) = \hat{x}(t_0 + t; t_0, x)$; Φ, considered as defined on $\mathbb{R} \times D$ (that is, as a function of both t and x) is called the *flow* generated by the system (4.1). By Lemma 4.1 the flow is independent of the value of t_0 used to define it. Equation (2.16) becomes

$$\Phi_{t+s}(x) = \Phi_t\big(\Phi_s(x)\big) \qquad \text{or} \qquad \Phi_{t+s} = \Phi_t \circ \Phi_s, \tag{4.3}$$

a formula which is summarized by saying that the maps Φ_t form a *one parameter group.* For each fixed t, Theorem 2.11 implies that Φ_t is continuous and hence is a homeomorphism (a continuous map with continuous inverse Φ_{-t}) from D to D. If $f \in C^1(D)$ then, by Theorem 2.15, Φ_t and $(\Phi_t)^{-1} = \Phi_{-t}$ are also C^1, i.e., are *diffeomorphisms* of D.

(c) Even if the special hypothesis of (b) is abandoned—if some solutions of (4.1) have domain a proper subset of $\mathbb{R}$—we may define $\Phi_t(x) = \hat{x}(t_0 + t; t_0, x)$; as a function of t and x, Φ will now have domain $\{(t, x) \mid t_0 + t \in I_{(t_0,x)}\}$. The first formula in (4.3) will now hold whenever the right hand side is defined.

Before stating our next result, we recall a standard definition: a *simple closed curve* in $\mathbb{R}^n$ is the image of a continuous mapping $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(s) = \gamma(t)$ with $s < t$ if and only if $s = a$ and $t = b$.

Theorem 4.3: *If $x(t)$ defined on I is a solution of (4.1) which is not constant, and if there exist distinct $t_1, t_2 \in I$ with $x(t_1) = x(t_2)$, then*

(a) $I = \mathbb{R}$;

(b) x is periodic;

(c) the set of strictly positive periods of x contains a minimal element h;

(d) two points $t, t' \in \mathbb{R}$ satisfy $x(t) = x(t')$ if and only if $t - t' = kh$ for some $k \in \mathbb{Z}$; in particular, the set of all periods is $h\mathbb{Z} \equiv \{kh \mid k \in \mathbb{Z}\}$;

(e) the orbit corresponding to $x(t)$, called a periodic orbit, is a simple closed curve.

Proof: Let Γ denote the set of all $\tau \in \mathbb{R}$ such that $x(t_0 + \tau) = x(t_0)$ for some $t_0 \in \mathbb{R}$. Of course, $0 \in \Gamma$, but by hypothesis also $t_2 - t_1 \in \Gamma$ and $t_2 - t_1 \neq 0$. Choose $\tau \in \Gamma$ and let $y(t) \equiv x(t + \tau)$; y is defined on $J = \{t \mid t + \tau \in I\}$. Since y is a solution of (4.1) and $y(t_0) = x(t_0)$ for the t_0 whose existence is guaranteed by $\tau \in \Gamma$, x and y are identical by uniqueness; in particular, $I = J$.

Now if τ in the above argument is chosen nonzero, then since $I = J = I - \tau$ the interval I is invariant under a non-zero translation; this is possible only if $I = \mathbb{R}$, proving (a). Moreover, since $\tau \in \Gamma$ implies $y(t) \equiv x(t+\tau) = x(t)$ for all t, $x(t)$ is periodic, proving (b) and implying that Γ is precisely the set of periods of x. From this property it follows easily that Γ is a group under addition.

Now Γ cannot contain arbitrarily small positive numbers, since if $\{\tau_n\}$ were a sequence of points of Γ with $\tau_n \searrow 0$, then

$$f(x(0)) = x'(0) = \lim_{n\to\infty} \tau_n^{-1}[x(\tau_n) - x(0)] = 0,$$

so that $x(0)$ would be an equilibrium point and $x(t)$ a constant solution, contradicting the hypothesis. The infimum h of the positive elements of Γ must lie in Γ, since if $\tau_n \searrow h$ with $\tau_n \in \Gamma$, then by continuity of x, $x(t+h) = \lim_{n\to\infty} x(t+\tau_n) = \lim_{n\to\infty} x(t) = x(t)$ for all $t \in \mathbb{R}$; this verifies (c). Further, $\Gamma = h\mathbb{Z}$ (which is all that remains to prove in (d)) by the group property of Γ: clearly $\Gamma \supset h\mathbb{Z}$ and if $\tau \in \Gamma$ with $kh < \tau < (k+1)h$ then $\tau - kh \in \Gamma$ is a positive element smaller than h. Finally, the orbit is $x([0,h])$ and for $s < t \in [0,h]$, $x(s) = x(t)$ if and only if $s = 0$ and $t = h$; this verifies that C is a simple closed curve. ∎

We now turn to the question of the limiting behavior of a solution $x(t)$ as t approaches infinity. The simplest possibility is that $\lim_{t\to\infty} x(t)$ actually exists, but there are more general cases in which we want to say that $x(t)$ has a well defined limiting behavior. For example, one typical behavior of solutions in the case $n = 2$ is shown in Figure 4.1; a solution may spiral outward and approach a closed curve C which is itself a periodic orbit.

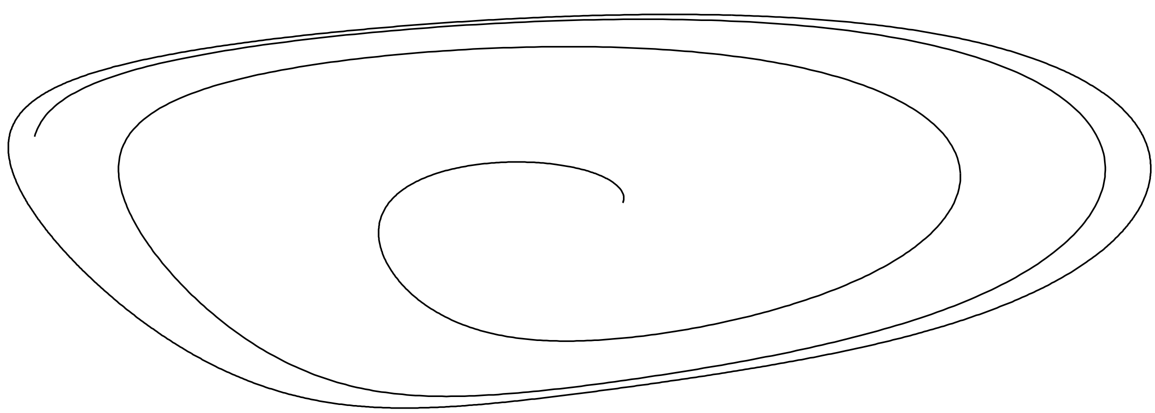


Figure 4.1

Definition 4.3: Let $x(t)$ be a solution of (4.1) whose interval of definition contains some interval $[a,\infty)$. A point $x^0 \in D$ is an *ω-limit point* of this solution if there is a sequence $\{t_n\}$ of real numbers with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n) = x^0$. The set of ω-limit points of x is denoted $\Omega[x]$. *α-limit points* are defined similarly, but with a sequence of times $\{t_n\}$ such that $t_n \to -\infty$.

If $\lim_{t\to\infty} x(t) \equiv x^0$ exists then $\Omega[x] = \{x^0\}$. In the case shown in Figure 4.1, $\Omega[x]$ is the entire limiting curve C.

The basic properties of the set of ω-limit points are contained in the next theorem. To state it, we need one more definition: we call a set $E \subset D$ *invariant* for the equation (4.1) if for every $x^0 \in E$ the solution of the IVP $x' = f(x)$, $x(t_0) = x^0$ satisfies $x(t) \in E$ for all t; equivalently, if every orbit which intersects E is contained in E.

Theorem 4.4: *Let $x(t)$ be a solution of (4.1) which is defined for all $t \geq a$. Then $\Omega \equiv \Omega[x]$ is relatively closed in D and invariant. If also there is a compact subset $K \subset D$ with $x(t) \in K$ for $t \in [a, \infty)$, then Ω is nonempty, compact, and connected; moreover, every solution $y(t)$ whose orbit intersects Ω (and hence is contained therein) is defined on all of $\mathbb{R}$.*

Proof: To see that Ω is relatively closed, suppose that $\{x^k\}$ is a sequence of points of Ω with $\lim_{k\to\infty} x^k = x^0 \in \Omega$. Then there exists a t_k with $|x(t_k) - x^k| < 1/k$; we may choose the t_k inductively so that $t_k \to \infty$ and hence $x^0 = \lim_{k\to\infty} x(t_k) \in \Omega$. To check invariance, suppose that $y(t)$ is a solution with $y(t_0) \in \Omega$, and let t_1 be any number in the domain of y. There exists $\{s_k\}$ with $s_k \to \infty$ and $x(s_k) \to y(t_0)$; then by (4.2),

$$x(s_k + t_1 - t_0) = \hat{x}(s_k + t_1 - t_0; s_k, x(s_k)) = \hat{x}(t_1; t_0, x(s_k)) \underset{k\to\infty}{\longrightarrow} \hat{x}(t_1; t_0, y(t_0)) = y(t_1),$$

where in taking the limit we have used the continuity of $\hat{x}$ in the initial conditions. Thus $y(t_1) \in \Omega$.

Now suppose that $K \subset D$ is compact and that $x(t) \in K$ for $t \in [a, \infty)$. Then $\Omega \subset K$ and hence, since K is closed, every limit point of Ω belongs to K and hence to D; thus Ω is closed, not just relatively closed, and since K is compact, so is Ω. The sequence of points $\{x(k)\}_{k=1}^{\infty}$ lies in K for $k \geq a$ and hence contains a convergent subsequence, say $\{x(k_j)\}_{j=1}^{\infty}$; $\lim_{j\to\infty} x(k_j) \in \Omega$ and therefore Ω is non-empty. Moreover, since Ω is compact, our extension theorem Theorem 2.19 immediately implies that the domain of any solution contained in Ω is $\mathbb{R}$. Finally, to verify that Ω is connected, suppose the contrary; by definition this means that we may write $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 closed (and hence compact) and disjoint. Let $d = d(\Omega_1, \Omega_2) > 0$ be the distance between these two subsets. For $i = 1, 2$ choose $x^i \in \Omega_i$ and a sequence $\{t_{ik}\}$ with $t_{ik} \to \infty$ and $x(t_{ik}) \to x^i$; we may clearly suppose that $t_{1k} < t_{2k}$ and, by discarding an initial segment of these sequences if necessary, that $|x(t_{ik}) - x^i| < d/4$ for all k. Consider now the continuous function $g(t) = d(x(t), \Omega_1)$, which satisfies $g(t_{1k}) < d/4$ and $g(t_{2k}) > 3d/4$. By the intermediate value theorem, $g(s_k) = d/2$ for some $s_k \in (t_{1k}, t_{2k})$. Because $x(t)$ lies in the compact set K for sufficiently large t we may choose a convergent subsequence of $\{x(s_k)\}$; the limit x^0 of this subsequence lies in $\Omega = \Omega_1 \cup \Omega_2$ but satisfies $d(x^0, \Omega_1) = d/2$ and $d(x^0, \Omega_2) \geq d/2$, which is clearly impossible.

Remark 4.2: The hypothesis $x([a, \infty)) \subset K$ in the second part of the theorem is certainly necessary. For example, in the system $x' = v$, where v is some fixed element of $\mathbb{R}^n$, all orbits are straight lines parallel to v and no solution has any ω-limit points. Figure 4.2 shows one possible behavior of a system (with $n = 2$) which leads to a disconnected and non-compact $\Omega[x]$.

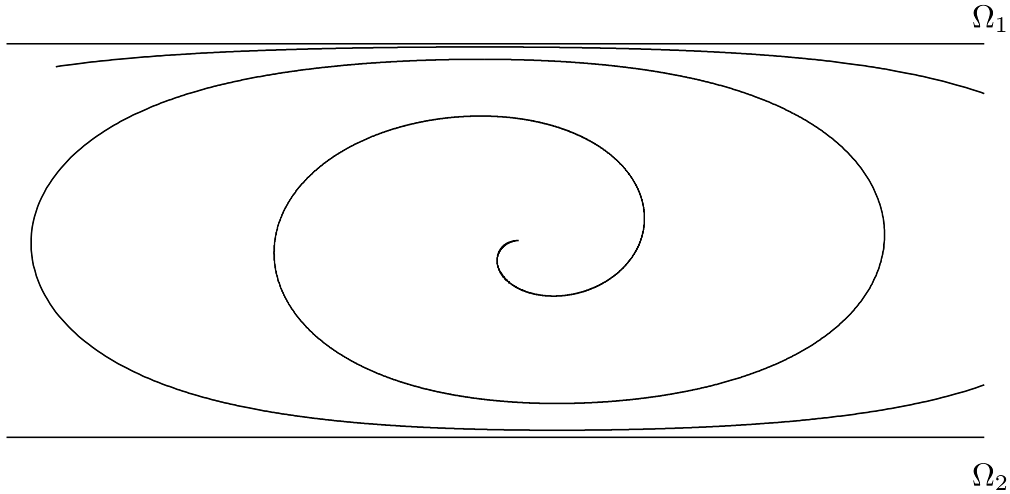


Figure 4.2

4.2 Two dimensional autonomous systems and linearization

In this section we discuss some examples of two dimensional autonomous systems, as well as the general technique of *linearization* for obtaining information about the behavior of non-linear systems. The Euclidean space $\mathbb{R}^2$ in which the the variable x lies is usually called the *phase plane*, and one reason that the qualitative behavior of specific two dimensional systems is relatively easy to understand is that we may draw typical orbits in this phase plane, and thus visualize the flow generated by the system of o.d.e.'s. In such drawings, we usually put an arrow on the trajectories to indicate the direction of increasing t.

Example 4.1: Two-dimensional constant coefficient homogeneous linear systems. It should be noted first that every constant coefficient homogeneous linear system is autonomous, so that the concepts of orbit, equilibrium point, ω-limit point, etc. make sense for the systems studied in Section 3.2. The origin is always an equilibrium point, and other equilibrium points exist if and only if zero is an eigenvalue of the coefficient matrix. Here we specialize to the two dimensional case: $x' = Ax$ with $x \in \mathbb{R}^2$, A a constant 2×2 matrix. The qualitative picture of the phase plane depends only on the Jordan form of A.

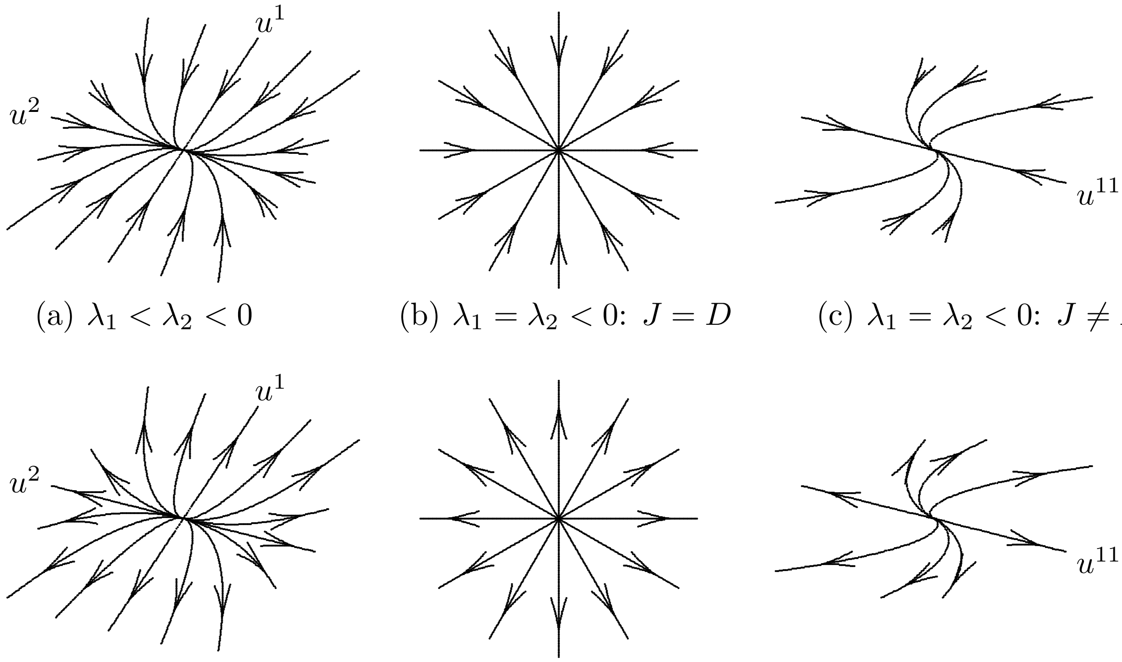
If both eigenvalues λ_1, λ_2 of A are real, then from (3.17) the general solution has the form

$$x(t) = c_1 e^{t\lambda_1} u^1 + c_2 e^{t\lambda_2} u^2, \tag{4.4a}$$

when $\lambda_1 \neq \lambda_2$ and when $\lambda_1 = \lambda_2$ but the Jordan form is diagonal, or

$$x(t) = (c_1 + tc_2) e^{t\lambda} u^{11} + c_2 e^{t\lambda} u^{12}, \tag{4.4b}$$

when $\lambda_1 = \lambda_2 = \lambda$ and the Jordan form is one 2×2 block. To say more it is necessary to consider various special cases. The resulting phase planes are shown in Figure 4.3a and 4.3b; in these figures we write $J = D$ to indicate the case of a diagonal Jordan form, $J \neq D$ to indicate a 2×2 block.



(a) $\lambda_1 < \lambda_2 < 0$ (b) $\lambda_1 = \lambda_2 < 0$: $J = D$ (c) $\lambda_1 = \lambda_2 < 0$: $J \neq D$

(d) $\lambda_1 > \lambda_2 > 0$ (e) $\lambda_1 = \lambda_2 > 0$: $J = D$ (f) $\lambda_1 = \lambda_2 > 0$: $J \neq D$

Case 1: λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$: stable and unstable nodes

Figure 4.3a

Case 1: λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$: We number the eigenvalues so that $|\lambda_2| \leq |\lambda_1|$. Suppose first that both eigenvalues are negative: $\lambda_1 \leq \lambda_2 < 0$.

(a) If $\lambda_1 \neq \lambda_2$, then solutions will have the form (4.4a); as $t \to -\infty$ typical solutions will approach infinity parallel to u^1, while as $t \to \infty$ they will approach the origin parallel to u^2. There are also two special solutions which lie along the vector u^1 (corresponding to $c_2 = 0$ and two possible signs of c_1), two along u^2, and the zero solution.

(b) If $\lambda_1 = \lambda_2$ and J is diagonal then the solution (4.4a) is always parallel to the vector $c_1u^1 + c_2u^2$; orbits (other than the origin itself) are rays directed inward to the origin.

(c) If $\lambda_1 = \lambda_2$ and solutions have the form (4.4b) then for $t \to \pm\infty$ solutions are parallel to u^{11}, but in opposite directions. There are still two special solutions along u^{11} and the zero solution.

In all these cases, the flow of every solution is into the origin; the origin is called a *stable node*. If both eigenvalues are positive, the same considerations apply (leading to cases (d)–(f)), but solutions now flow from the origin to infinity; the origin is now called an *unstable node*.

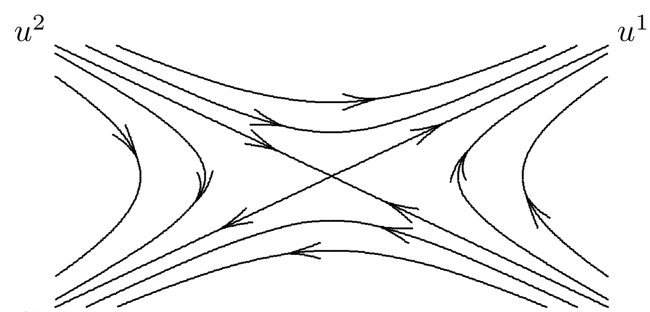
Case 2: λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$: Suppose that $\lambda_1 > 0 > \lambda_2$. Solutions necessarily have the form (4.4a), typical solutions approach infinity parallel to u^2 as $t \to -\infty$, and parallel to u^1 as $t \to \infty$. There are also two special solutions along each of u^1 (oriented out of the origin) and u^2 (oriented into the origin), and as usual the origin is itself an orbit. In this case the origin is called a *saddle point*.

Case 3: λ_1, λ_2 real, $\lambda_1\lambda_2 = 0$: This special case is less important than the others because the origin is not an *isolated* equilibrium point, that is, there are other equilibrium points arbitrarily close to it, and this makes the linear analysis difficult to apply to nonlinear problems. Nevertheless, we include it for completeness.

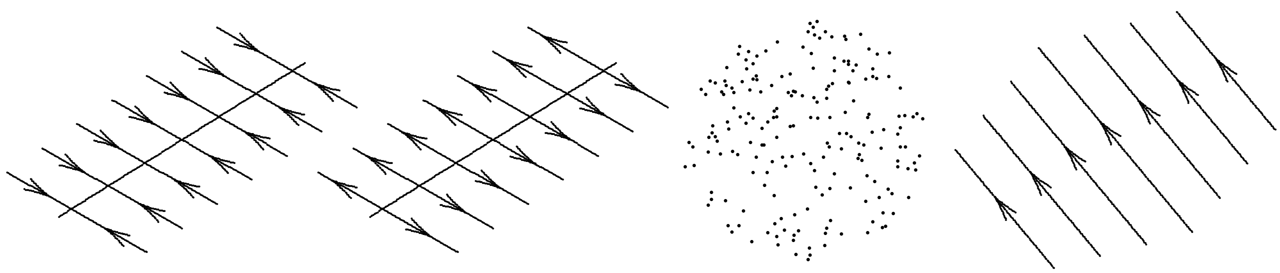
(a,b) If $\lambda_2 = 0$ but $\lambda_1 \neq 0$ then (from (4.4a)) all points of the form cu^1 are equilibrium points, and other trajectories lie on straight lines parallel to u^2, (a) approaching the equilibria as $t \to \infty$ if $\lambda_1 < 0$, or (b) approaching infinity as $t \to \infty$ if $\lambda_1 > 0$.

(c) If $\lambda_1 = \lambda_2 = 0$ and A is diagonal, then $A = 0$ and every point of the plane is an equilibrium point.

(d) If $\lambda_1 = \lambda_2 = 0$ and A is not diagonal, then from (4.4b) the orbits are on straight lines parallel to u^{11}; here each straight line is a single orbit (rather than three orbits as in (a,b)).



Case 2: λ_1, λ_2 real, $\lambda_1 > 0 > \lambda_2$: saddle point



(a) $\lambda_1 < 0$, $\lambda_2 = 0$ (b) $\lambda_1 > 0$, $\lambda_2 = 0$ (c) $\lambda_1 = \lambda_2 = 0$, $J = D$ (d) $\lambda_1 = \lambda_2 = 0$, $J \neq D$

Case 3: λ_1, λ_2 real, $\lambda_2 = 0$

Figure 4.3b

We now turn to

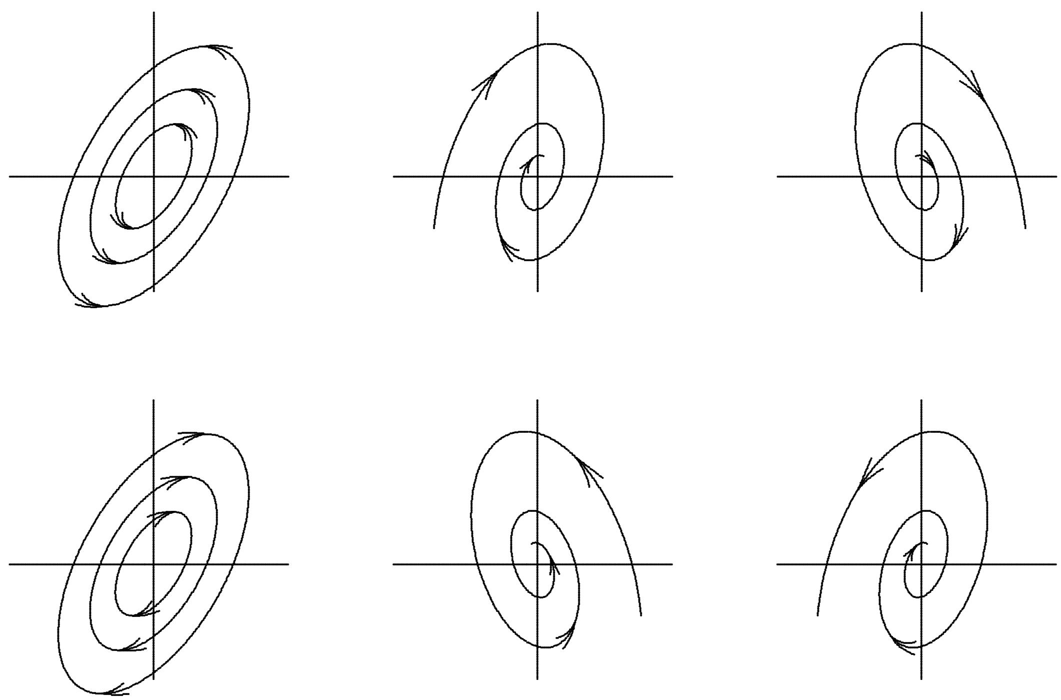
Case 4: λ_1, λ_2 complex: We now write the eigenvalues as $\lambda = \alpha \pm i\beta$, where we take $\beta > 0$ by convention. By (3.18) every solution has the form

$$\begin{aligned} x(t) &= e^{t\alpha}[(c_1 \cos t\beta + c_2 \sin t\beta)v - (c_1 \sin t\beta - c_2 \cos t\beta)w] \\ &= ce^{t\alpha}[\cos(t\beta - \delta)v - \sin(t\beta - \delta)w], \end{aligned} \tag{4.5}$$

where in to obtain the second line we have set $c = (c_1^2 + c_2^2)^{1/2}$ and $c_1 = c\cos\delta$, $c_2 = c\sin\delta$. Again we consider various cases, shown in Figure 4.4.

(a) If $\alpha = 0$ then the solution is periodic, with period $2\pi/\beta$; inspection of (4.5) makes it clear that orbits are closed curves encircling the origin. In this case the origin is called a *center*. In fact, these curves are ellipses, and they are traced clockwise if $\det B > 0$ and counter-clockwise if $\det B < 0$, where $B = [v\ w]$ is the 2×2 matrix with columns v, w. To see this, let $\tilde{x}(t)$ denote the curve given by (4.5) in the case $v = e^1$ and $w = e^2$; it is clear that $\tilde{x}$ is a circle traced clockwise; in fact, $\tilde{x}(t)^T\tilde{x}(t) \equiv c^2$. But $x(t) = B\tilde{x}(t)$ and thus $x(t)^T Cx(t) \equiv c^2$, where C is the symmetric matrix $C = (BB^T)^{-1}$; this is the equation of an ellipse. Moreover, B preserves the orientation in going from the basis $[e^1, e^2]$ to the basis $[v, w]$ if $\det B > 0$, and reverses it otherwise, and this orientation is equivalent to the direction of rotation.

(b,c) If $\alpha \neq 0$ then a growth or decay in $x(t)$ is superimposed on the periodic behavior of (a), leading to solutions which (b) spiral inward if $\alpha < 0$, or (c) spiral outward if $\alpha > 0$. The origin is then called a stable or unstable *spiral point*, respectively. The direction of rotation is determined as in (a).



(a) $\alpha = 0$: center (b) $\alpha < 0$: stable spiral (c) $\alpha > 0$: unstable spiral

Figure 4.4: Complex eigenvalues $\lambda = \alpha \pm \beta$.

Having analyzed the phase plane for these linear systems, we now want to show how the analysis can be applied to gain insight into non-linear systems. We first turn from the two-dimensional case to consider the general idea of using *linearization* to study a non-linear autonomous system. Suppose that $f \in C^1(D)$, and that $x^0 \in D$ is an equilibrium point of the system (4.1). For x near x^0 we have a natural approximation $f(x) \simeq Df_{x^0}(x - x^0)$, and setting $u = x - x^0$ we are led to approximate (4.1) by a constant coefficient linear system

$$u' = Au, \qquad A = Df_{x^0}. \tag{4.6}$$

Such an approximation should be useful when u is small ($x \simeq x^0$).

Remark 4.3: We are interested not in numerical approximation but rather in gaining some qualitative understanding of the behavior of the solutions of the non-linear problem. Linearization is an important tool in this endeavor. What success can we hope for from its application?

(a) If $A = Df_{x^0}$ has any eigenvalues with real part zero then the qualitative behavior of (4.1) and (4.6) may be quite different. For example, the system $x' = x|x|^2 = x \sum x_i^2$ has an isolated equilibrium point at the origin, but the linearization $x' = 0$ has every point of $\mathbb{R}^n$ as an equilibrium point. Similarly, it is easy to find systems for which the linearization has the origin as a center but for which the higher order terms cause solutions to spiral either inward or outward. For this reason we confine our attention to equilibrium points at which A has no eigenvalues with real part zero; such equilibrium points are called *hyperbolic.*

(b) The ideal situation for a hyperbolic equilibrium point would be that solutions of (4.6) give a qualitatively exact picture of the solutions of (4.1) near x^0—in fact, that the two equations differ by a change of variable. Thus we would seek an invertible map $F : U \to V$, with U and open neighborhood of x^0 and V an open neighborhood of 0, such that $x(t)$ is a solution of (4.1) if and only if $F(x(t))$ is a solution of (4.6). It can be shown (see Hartman) that a continuous F with this property always exists. In most problems, however, f has certain differentiability properties, say $f \in C^1$, and we would like F also to be in C^1. It is not always possible to find such an F; certain "obstructions" may occur. Again, we refer to Hartman.

(c) A lesser goal, which we will discuss in some detail later, is to prove that the true system near a hyperbolic equilibrium x^0 inherits some of the properties of the linear system. For example, we will show that if all eigenvalues of A have strictly negative real part, so that every solution $u(t)$ of (4.6) satisfies $\lim_{t\to\infty} u(t) = 0$, then every solution $x(t)$ of (4.1) with $x(t_0)$ sufficiently near x^0 satisfies $\lim_{t\to\infty} x(t) = x^0$.

In the next example we ignore the difficulties discussed above and assume that the linearization gives a good qualitative picture of the behavior of solutions near a hyperbolic critical point. This assumption can in fact be justified because the dimension of of the phase space is only two. We will thus refer to an equilibrium as a stable node if the corresponding linearized problem has two negative eigenvalues, and similarly for unstable nodes, saddle points, and stable and unstable spiral points.

Example 4.2: Competing species. We consider a simple ecological model of *competing species*, a model somewhat similar to those considered in Example 1.1. (This model is also discussed in Hirsch and Smale.) Let x_1 and x_2 denote the populations of two species inhabiting the same environment and competing for the same resources. We suppose that

x_1 and x_2 satisfy the autonomous system

$$\begin{aligned} x_1' &= (r_1 - \alpha_1 x_1 - \beta_1 x_2)x_1, \\ x_2' &= (r_2 - \alpha_2 x_2 - \beta_2 x_1)x_2. \end{aligned} \tag{4.7}$$

For the interpretation of these equations we refer to the discussion of Example 1.1. Briefly, r_i is the growth rate of species i at very low population levels, and α_i and β_i represent the effect respectively of intra- and of inter-species competition on the growth of this species. All these coefficients are positive, and we suppose that $\alpha_1\alpha_2 \neq \beta_1\beta_2$ to avoid consideration of special cases.

There are four equilibrium points in the model:

$$\begin{aligned} E_0 &= (0,0), & E_2 &= (0, r_2/\alpha_2), \\ E_1 &= (r_1/\alpha_1, 0), & E_3 &= (\alpha_1\alpha_2 - \beta_1\beta_2)^{-1}(\alpha_2 r_1 - \beta_1 r_2, \alpha_1 r_2 - \beta_2 r_1). \end{aligned}$$

The ecological interpretation of these equilibria is simple: at E_0 no individuals of either species exist; at E_1 and E_2 one species has died out and the second is existing at the carrying capacity of its environment. Since populations are non-negative, E_3 has an ecological interpretation only if both its components are non-negative; in this case, it represents a situation in which the two species are coexisting.

Further discussion of the model depends on the relative sizes of the coefficients. For simplicity we analyze only one of several possible cases: we assume that

$$\alpha_2 r_1 < \beta_1 r_2 \qquad \text{and} \qquad \alpha_1 r_2 < \beta_2 r_1. \tag{4.8}$$

Note that (4.8) implies that

$$\alpha_1\alpha_2 < \beta_1\beta_2, \tag{4.9}$$

and hence E_3 lies in the first quadrant; (4.9) may be interpreted as a statement that intraspecies competition is weaker than interspecies competition. Linearization at the equilibria shows that E_0 is an unstable node, E_1 and E_2 are stable nodes, and E_3 is a saddle point. To illustrate the method we discuss E_1. Linearized equations are most easily obtained by substituting $x_1 = u_1 + r_1/\alpha_1$ and $x_2 = u_2$ into (4.7) and dropping quadratic terms in the u_1, u_2; the resulting linearized equations are

$$\begin{aligned} u_1' &= -r_1 u_1 - (r_1\beta_1/\alpha_1)u_2, \\ u_2' &= -[(\beta_2 r_1 - \alpha_1 r_2)/\alpha_1]u_2. \end{aligned}$$

The eigenvalues here are $\lambda_1 = -r_1$ and $\lambda_2 = -(\beta_2 r_1 - \alpha_1 r_2)/\alpha_1$, both negative. The first eigenvector is $u^1 = (1,0)$, the trajectory directly along u^1 in the linearized problem persists in the non-linear problem, since the x-axis is invariant.

We are now ready to sketch the first quadrant of the phase plane. Preliminary sketches are shown in Figure 4.5. The first summarizes the information from the local behavior at the equilibria; the second introduces a second simple technique frequently of use in such problems—indication of the various regions and curves in the plane corresponding to

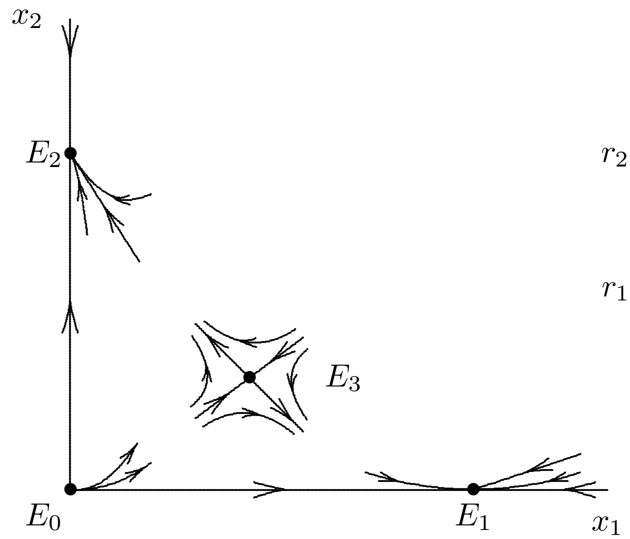


Figure 4.5.a

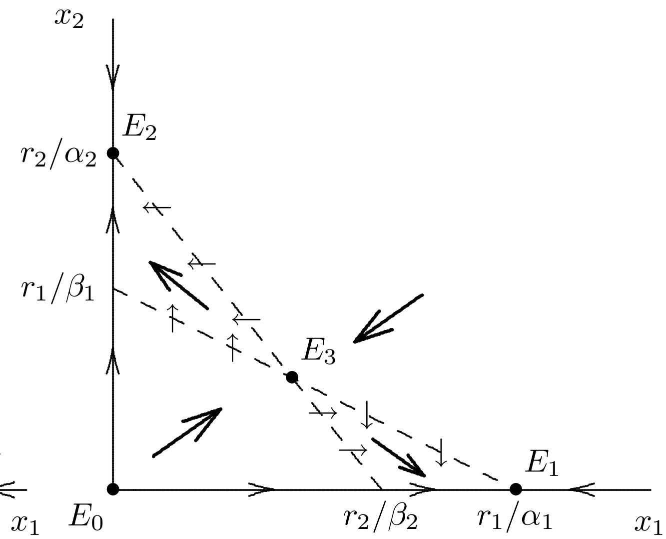


Figure 4.5.b

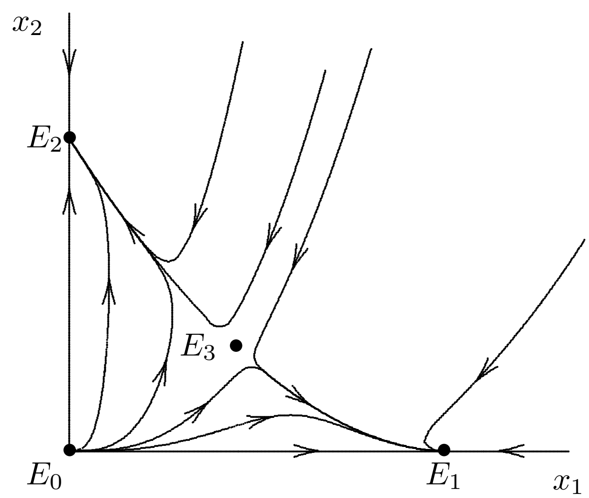


Figure 4.6

known signs (positive, negative, or zero) of the derivatives of the components of x. The final sketch is shown in Figure 4.6. Note that the final state of the system may correspond to the survival of either species, depending on the initial conditions; the survival of both is a practical impossibility because the equilibrium E_3 is not stable.

Sketches of the phase plane may be prepared similarly for other choices of parameters. When the inequalities (4.8) are reversed, so that intraspecies competition is stronger than interspecies competition, the equilibrium for joint coexistence becomes stable. The resulting phase plane is sketched in Figure 4.7.

Example 4.3: The Van der Pol oscillator. We study the autonomous system

$$\begin{aligned} x' &= f_1(x,y) \equiv -y, \\ y' &= f_2(x,y) \equiv x + y - y^3, \end{aligned} \tag{4.10}$$

and will let $z \in \mathbb{R}^2$ denote the pair (x, y), so that (4.10) is $z' = f(z)$. Our analysis will be closely based on that of Hirsch and Smale, to whom we also refer for a discussion of how

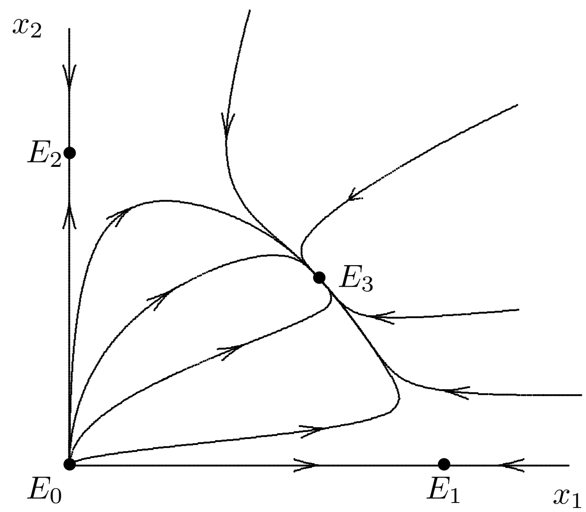


Figure 4.7

(4.10) may be viewed as the equations of an electrical circuit with a non-linear resistor. We make two preliminary observations. First, the origin is the unique equilibrium point of the system, and linearization there yields an unstable spiral point in the linearized system $u' = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} u$. Second, the system is invariant under the reflection (through the origin) $R : \mathbb{R}^2 \to \mathbb{R}^2$ given by $Rz = -z$; specifically, if $z(t)$ is a solution, then so is $Rz(t)$.

Our goal is to prove

Theorem 4.5: *The system (4.10) has a unique periodic orbit C_p with the origin in its interior. Every solution $z(t)$ is defined on some interval $[a, \infty)$ and if z is not the identically zero solution then $\Omega[z] = C_p$.*

A *limit cycle* of an autonomous system is a periodic orbit which is contained in the set of ω-limit points (or α-limit points) of some solution $x(t)$ but which is not identical with the orbit of $x(t)$. The theorem implies that C_p is a limit cycle for the Van der Pol oscillator. We will see that in fact every solution starting outside C_p spirals inward to C_p, and every non-zero solution starting inside C_p spirals outward to C_p. The behavior of typical solutions is shown in Figure 4.8; note that following the inward spiraling solutions backwards does not lead to an infinite outward spiral; in fact, we can show that for such a solution $x(t)$ there exists a $T > -\infty$ such that $\lim_{t\to T+} |x(t)| = \infty$, and that for t sufficiently close to T the solution remains in one of the two regions A_2 and A_4 defined below.

Proof: We present the proof in a series of steps. It may appear at times that we are appealing to geometric intuition; in fact, however, it is fairly easy to convert these appeals into rigorous proofs, and we try to alert the reader to check our work in such cases. Some verifications are also left as explicit exercises for the reader.

(a) We first partition $\mathbb{R}^2$ into (nine) sets corresponding to the signs (positive, negative, or zero) of the functions $f_1(x, y) = -y$ and $f_2(x, y) = x + y - y^3$, or equivalently the signs of the derivatives x' and y'. The origin $\{0\}$ corresponds to $f_1 = f_2 = 0$. There are four

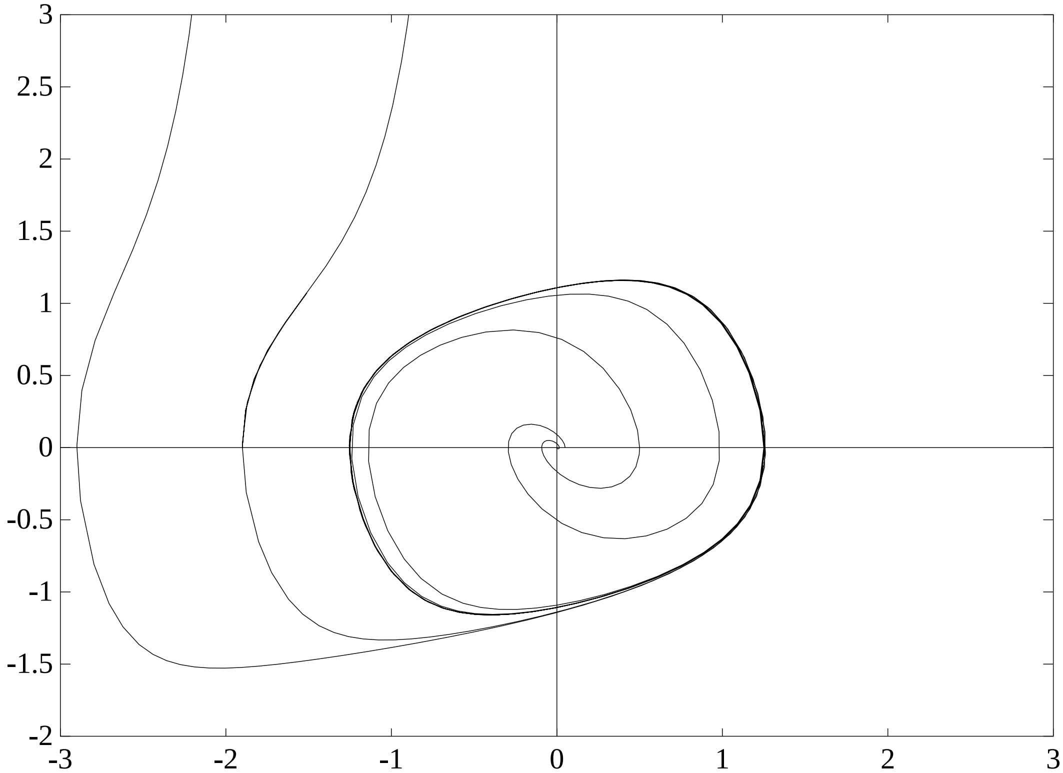


Figure 4.8

sets where precisely one of f_1, f_2 vanishes: the positive x-axis, a_1, the negative x-axis, a_3, and the branches a_2, a_4 of the curve $x = y^3 - y$ lying respectively in the upper and lower half planes. Finally, there are the four connected open sets obtained by removing $\{0\}$ and $a_1, \ldots, a_4$ from $\mathbb{R}^2$, on which both f_1 and f_2 are non-zero and have fixed sign: $A_1 = \{z \mid x > y^3 - y,\ y > 0\}$, $A_2 = \{z \mid x < y^3 - y,\ y > 0\}$, $A_3 = R(A_1)$, and $A_4 = R(A_2)$. See Figure 4.9.

(b) Suppose now that $z(t)$ is a solution and t_0 a point in its interval of definition. We claim that (i) if $z(t_0) \in a_i$, then $z(t_0 + s) \in A_i$ for all sufficiently small $s > 0$; (ii) if $z(t_0) \in A_i$, then there exists a $T > t_0$ such that $z(t) \in A_i$ for $t_0 \le t < T$, and $z(T) \in a_{i+1}$ (where if $i = 4$ then a_{i+1} is interpreted as a_1). To verify (i) we simply inspect the signs in (4.10); for example, when $z(t_0) = (x^0, 0) \in a_1$, $x'(t_0) = 0$ and $y'(t_0) > 0$, so that, for small s, $x'(t_0 + s) = -y(t_0 + s) < 0$ and $y'(t_0 + s) > 0$, i.e., $z(t_0 + s) \in A_1$. To check (ii), suppose first that $z(t_0) = (x^0, y^0) \in A_1$; then while $z(t) \in A_1$ we have $y'(t) > 0$ and hence $x'(t) = -y(t) < -y^0$ and $x(t) < x^0 - ty^0$. This clearly implies that the solution intersects a_2 in finite time. The argument for $z(t_0) \in A_2$ is similar but slightly more complicated; we leave it as an exercise. The cases $z(t) \in A_3, A_4$ follow from these by application of the symmetry R.

(c) Now suppose $c > 0$ and consider the solution $z_c(t) = (x_c(t), y_c(t))$ which satisfies $z(0) = (c, 0) \in a_1$. From (b) it follows that $z_c(t)$ must return to a_1 in finite time after a trip around the origin; let $\tau(c)$ be the smallest positive number such that $z_c(\tau(c)) \equiv (\phi(c), 0) \in a_1$. The map $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is the key to our further analysis. It is an exercise

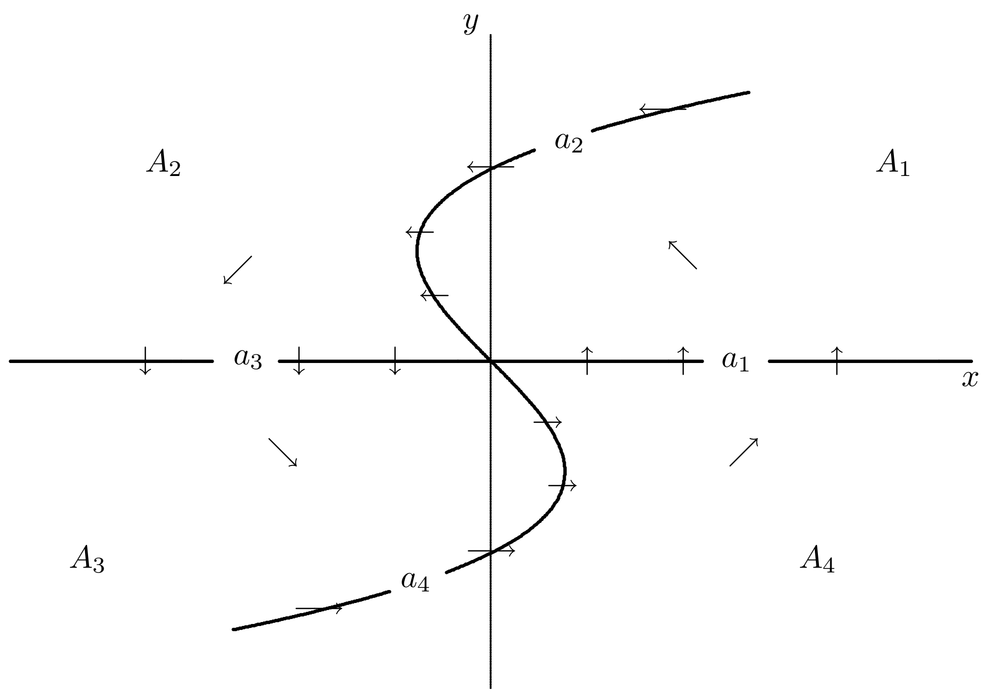


Figure 4.9

to check that ϕ is continuous and remains so when extended by $\phi(0) = 0$ (this follows from continuous dependence on initial conditions, but a little work is needed). ϕ must be monotonic increasing since if $c > d$ but $\phi(c) \leq \phi(d)$ then z_c and z_d would intersect but not coincide (check!). If $\phi(c) > c$ then the solution is spiraling outwards, if $\phi(c) < c$ then the solution is spiraling inwards, and if c is a fixed point of ϕ, that is, if $c = \phi(c)$, then the solution corresponds to a periodic orbit of the system. We will prove the

Claim: There exists a number c_0 such that $\phi(c_0) = c_0$, $\phi(c) > c$ if $c < c_0$, and $\phi(c) < c$ if $c > c_0$.

The claim immediately implies the statement of the theorem and most of the qualitative properties of the orbits indicated in the paragraph following its statement (the exception is the $t \to -\infty$ behavior of the solutions outside C_p). C_p is just the orbit of z_{c_0}. Any solution through a point outside C_p must coincide (up to a time shift) with z_c for some $c > c_0$; then $c, \phi(c), \phi^2(c), \ldots$ is a decreasing sequence which must decrease to c_0, since any limit point of this sequence is a fixed point of ϕ by continuity. This implies that $C_p = \Omega[z_c]$. Solutions beginning inside C_p are treated similarly.

(d) Now consider again the solution $z_c(t)$, and introduce two new notations. First, let $\sigma(c)$ be the smallest positive number such that $z_c(\sigma(c)) \equiv (-\psi(c), 0) \in a_3$. We will prove the claim with ϕ replaced by ψ; since $\phi = \psi \circ \psi$, the claim as stated follows immediately. Second, let c_1 be the unique point in $\mathbb{R}_+$ such that when $c = c_1$ the solution first intersects a_2 at the point $(0, 1)$; such a point must exist because the solution satisfying $z(t_1) = (0, 1)$ may be shown by methods similar to those of (b) to intersect a_1 at some finite time $t_2 < t_1$ (check!). For $c \in \mathbb{R}_+$ let

$$F(c) = c^2 - \psi(c)^2;$$

we will show that $F(c) < 0$ for $c < c_1$ but that $F(c)$ monotonically increases for $c > c_1$ and eventually attains a positive value; this will establish the claim for ψ, with c_0 the unique zero of F.

(e) Define $G(x, y) = x^2 + y^2$, so that

$$\begin{aligned} F(c) &= -\int_0^{\sigma(c)} \frac{d}{dt} G(x_c(t), y_c(t))\, dt \\ &= -2 \int_0^{\sigma(c)} [y_c(t)^2 - y_c(t)^4]\, dt. \end{aligned} \tag{4.11}$$

If $c < c_1$ then $0 < y(t) < 1$ on $(0, \sigma(c))$ (since $y(t)$ must reach its maximum at the intersection of $z_c(t)$ with a_2, and this intersection point must lie between the origin and $(0,1)$, by choice of c_1) and hence (4.11) implies that $F(c) < 0$ (check!—see Figure 4.10a).

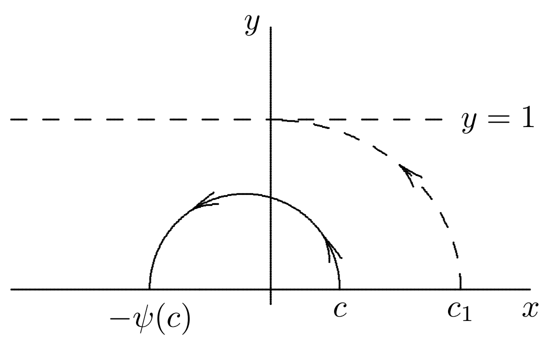


Figure 4.10a

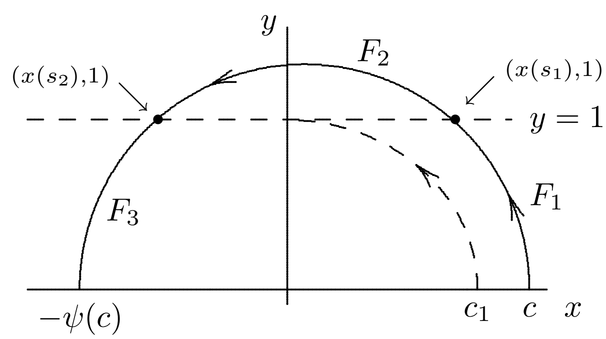


Figure 4.10b

If $c > c_1$ then $z_c(t)$ must intersect the line $y = 1$ for precisely two values s_1, s_2 in the interval $0 \le t \le \sigma(c)$, because $y_c(t)$ increases monotonically until $z_c(t)$ intersects a_2, then decreases monotonically until $t = \sigma$ (check!—see Figure 4.10b). We split the integral in (4.11) into the sum of three terms F_1, F_2, and F_3, given by integrating over the intervals $[0, s_1]$, $[s_1, s_2]$, and $[s_2, \sigma]$, respectively. On the first interval $y_c(t)$ is a monotonically increasing function and hence $y = y_c(t)$ may be introduced as a variable of integration:

$$F_1(c) = -2 \int_0^1 \frac{y^2 - y^4}{\tilde{x}_c(y) + y - y^3}\, dy,$$

where we have used $dy/dt = x + y - y^3$ and have written $\tilde{x}_c(y) \equiv x_c(y_c^{-1}(y))$ in this reparameterization. It is clear that $\tilde{x}_c(y)$ is an increasing function of c (check!) and is easy to verify that in fact $\tilde{x}_c(y) \to \infty$ uniformly as $c \to \infty$ (check!), so that $F_1(c)$ increases to 0. The same argument shows that $F_3(c)$ increases to 0 as $c \to \infty$ (check!). Finally, the integral for $F_2(c)$ may be parameterized by x:

$$F_2(c) = 2 \int_{x_c(s_2)}^{x_c(s_1)} [\tilde{y}_c(x)^3 - \tilde{y}_c(x)]\, dx,$$

where as above $\tilde{y}_c(x) \equiv y_c(x_c^{-1}(x))$. Since $\tilde{y}_c(x) > 1$ for $x_c(s_2) < x < x_c(s_1)$, the integrand is positive; moreover, as c increases the function $\tilde{y}_c(x)$ increases, $x_c(s_2)$ decreases, and $x_c(s_1)$ increases, so that F_2 increases (check! check! check!). We conclude that $F(c)$ is increasing and is eventually positive. In fact, one may verify that $\lim_{c\to\infty} F_2(c) = \lim_{c\to\infty} F(c) = \infty$.

4.3 The Poincaré-Bendixson Theorem

Throughout this section we take $f(x)$ as a continuous vector field defined on an open, connected set $D \subset \mathbb{R}^2$, and such that the autonomous system (4.1), $x' = f(x)$, has unique solutions. Our main goal is to prove the Poincare-Bendixson Theorem:

Theorem 4.6: *Suppose that $x(t)$ is a solution of (4.1) which is defined on some interval $[a, \infty)$, and satisfies $x([a,\infty)) \subset K$ for some compact set $K \subset D$. Suppose further that $\Omega[x]$ contains no equilibrium point of f. Then $\Omega[x]$ is a periodic orbit of (4.1).*

Note that Theorem 4.4 implies that $\Omega[x]$ is nonempty and connected, and that $\Omega[x]$ and $x(t)$ may be related in either of two ways: $x(t)$ may be a periodic solution, in which case $\Omega[x]$ is identical with the orbit of $x(t)$, or $\Omega[x]$ may be a limit cycle disjoint from this orbit but contained in its closure.

We will try to give a careful proof which does not rely on geometric intuition. To do so we must introduce a certain amount of machinery.

Definition 4.4: A *transversal* to the vector field f is a finite, open straight line segment $L = \{u + \tau v \mid u, v \in \mathbb{R}^2, v \neq 0;\ \tau \in (a,b)\}$ such that the closure $\overline{L}$ of L is contained in D and such that, if $\xi \in \overline{L}$, then $f(\xi)$ is not parallel to L, that is, $f(\xi) \neq av$ for any $a \in \mathbb{R}$. (We denote points of transversals by ξ or ζ.) Points of L have a natural order, up to an overall reversal: if $\xi^i = u + \tau_i v$ for $i = 1, 2$ we write $\xi^1 \prec \xi^2$ iff $\tau_1 < \tau_2$. Finally, we let n be the unit vector normal to L ($n \cdot v = 0$) oriented so that $f(\xi) \cdot n$ is positive for $\xi \in L$; then $L \subset \tilde{L} \equiv \{x \in \mathbb{R}^2 \mid x \cdot n = c\}$ for some constant c.

Remark 4.4: It is clear that if $x \in D$ is not an equilibrium point, i.e., if $f(x) \neq 0$, then there exists a transversal L with $x \in L$.

Lemma 4.7: *Let L be a transversal to f. Then there exists an $\epsilon > 0$ such that, if $W \equiv (-\epsilon, \epsilon) \times L$, then:*

(i) $\phi : W \to D$ with $\phi(t, \xi) = \Phi_t(\xi)$ is well defined, i.e., $t \in I_{(0,\xi)}$ for $\xi \in L$ and $|t| < \epsilon$;

(ii) ϕ is injective on W;

(iii) $\phi(W)$ is open and ϕ^{-1} is continuous there.

Remark 4.5: The mapping ϕ of this lemma is called a *flow box*; it may be thought of as a non-linear change of coordinates which converts the flow in D into a simple flow along parallel straight lines in W (see Figure 4.11). The lemma extends immediately to the construction of a flow box in any dimension, if the transversal L is replaced by an open subset of a hypersurface which is transverse to f. The proof is considerably simplified if

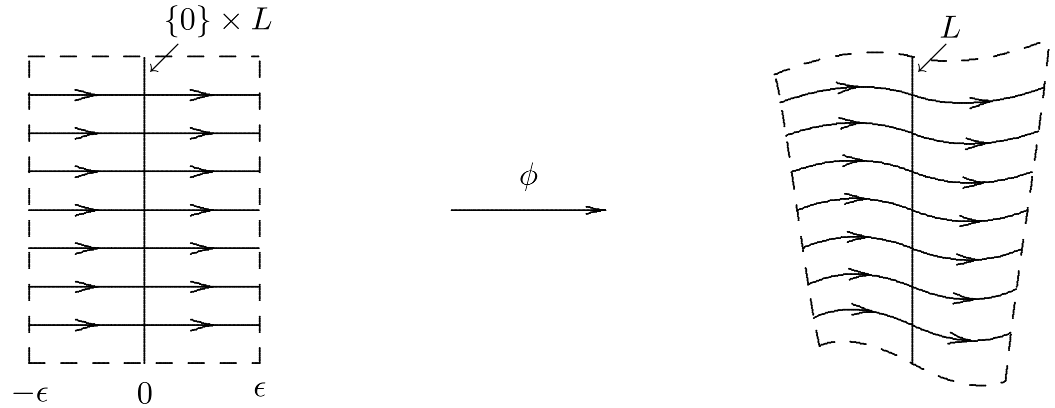


Figure 4.11

we assume that f is a C^1 vector field, in which case (iii) is an immediate consequence of the inverse function theorem. See Hirsch and Smale.

Proof: It follows from the proof of uniform existence, Corollary 2.8, taking the compact set K of that corollary to be L, that for any open set V satisfying $\overline{L} \subset V \subset \overline{V} \subset D$ there exists an $\epsilon > 0$ such that $\Phi_t(\xi)$ is defined and lies in $\overline{V}$ for $|t| < \epsilon$ and $\xi \in L$; this verifies (i). We may choose V small enough so that $\alpha \equiv \inf_{x\in\overline{V}} f(x) \cdot n > 0$. Then for $(t,\xi) \in W$,

$$\phi(t,\xi) \cdot n = \xi \cdot n + \int_0^t f(\phi(s,\xi)) \cdot n\, ds = c + \tilde{\alpha} t, \tag{4.12}$$

with $\tilde{\alpha} \equiv t^{-1} \int_0^t f(\phi(s,\xi)) \cdot n\, ds \ge \alpha$, so that

$$\operatorname{sgn}\bigl(\phi(t,\xi) \cdot n - c\bigr) = \operatorname{sgn}(t), \tag{4.13}$$

where the signum function sgn is 0 or ± 1 according to the sign of its argument. Since $x \cdot n = c$ is the equation of the infinite extension $\tilde{L}$ of L, (4.13) says that all points $\phi(t,\xi)$ with the same sign of t lie on the same side of $\tilde{L}$. Thus if $\phi(t,\xi) = \phi(s,\zeta)$ then s and t have the same sign—say $0 < s < t$—so that $\zeta = \Phi_{t-s}(\xi) = \phi(t-s,\xi)$. Again, since $\xi, \zeta \in L$, (4.13) implies that $t = s$ and hence $\xi = \zeta$. This verifies (ii).

Now suppose that $y^0 = \phi(t_0, \xi^0) \in \phi(W)$; we show that there is a neighborhood N of y^0 such that if $y^1 \in N$, then $y^1 \in \phi(W)$, and that $|\phi^{-1}(y^1) - \phi^{-1}(y^0)|$ may be made small by choosing N sufficiently small. Since $\Phi_{-t_0}(y^0) = \xi^0$ is well defined, Theorem 2.11 implies that $\Phi_\tau(y)$ is well defined for (τ, y) close to $(-t_0, y^0)$; specifically, there exists an $\eta > 0$ and a neighborhood U of y^0 with compact closure such that $\Phi_\tau(y)$ is defined on the compact set specified by $y \in \overline{U}$, $|\tau + t_0| \le \eta$. By the same theorem, $\Phi_\tau(y)$ is continuous and hence uniformly continuous on this compact set; thus for any $\delta > 0$ there exists a neighborhood $N(= N_{\delta,\eta}) \subset U$ of y^0 such that, for $y^1 \in N$ and $|s| \le \eta$, $\Phi_{-t_0+s}(y^1)$ is defined and satisfies

$$|\Phi_{-t_0+s}(y^1) - \Phi_{-t_0+s}(y^0)| = |\Phi_{-t_0+s}(y^1) - \phi(s,\xi^0)| < \delta. \tag{4.14}$$

Fix η; then for δ sufficiently small, (4.13) and (4.14) imply

$$\operatorname{sgn}\bigl(\Phi_{-t_0\pm\eta}(y^1) \cdot n - c\bigr) = \operatorname{sgn}\bigl(\phi(\pm\epsilon', \xi^0) \cdot n - c\bigr) = \pm 1,$$

i.e., that the two points $\Phi_{-t_0\pm\eta}(y^1)$ lie on opposite sides of the line $\tilde{L}$, so that $\Phi_{-t_0+s_1}(y^1) \equiv \xi^1 \in \tilde{L}$ for some s_1, $|s_1| \le \eta$ (see Figure 4.12). Finally, if we choose δ less than the distance from $\{\phi(s,\xi^0) \mid |s| \le \eta\}$ to $\tilde{L} \setminus L$, we see that $\xi^1 \in L$. By choosing $\eta < \epsilon - |t_0|$ we may guarantee that $|t_0 - s_1| < \epsilon$; thus $y^1 = \phi(t_1,\xi^1) \in \phi(W)$ for $t_1 = t_0 - s_1$. To verify continuity of ϕ^{-1}, we set $M = \sup_{\overline{V}} |f|$ and observe that

$$\begin{aligned}
|(t_1,\xi^1) - (t_0,\xi^0)| &\le |s_1| + |\xi^1 - \xi^0| \\
&\le |s_1| + |\xi^1 - \Phi_{-s_1}(\xi^1)| + |\Phi_{-s_1}(\xi^1) - \xi^0| \\
&\le |s_1| + M|s_1| + |\Phi_{-t_0}(y^1) - \Phi_{-t_0}(y^0)| \\
&\le \eta(1+M) + \delta,
\end{aligned}$$

which may be made arbitrarily small by choice of η and then δ. ■

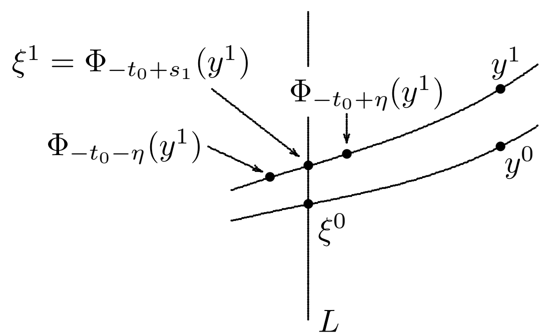


Figure 4.12

To prove the next lemma we need a classical topological result, which we quote without proof.

Jordan Curve Theorem: *Let $\Gamma \subset \mathbb{R}^2$ be a simple closed curve. Then $\mathbb{R}^2 \setminus \Gamma$ is the union of two disjoint nonempty open sets O_1 and O_2 such that $\partial O_1 = \partial O_2 = \Gamma$. One of O_1, O_2 is bounded and is called the interior of Γ; the other is unbounded and is called the exterior of Γ.*

Lemma 4.8: *Suppose that L is a transversal to f, that $x(t)$ is a solution to (4.1) in D, and that $x(t) \in L$ for values $t_1 < t_2 < \ldots$ and no other values of t greater than t_1. Then the points $x(t_i)$ are arranged monotonically in the natural order on L.*

Proof: It suffices to consider three successive crossings of L at times t_1, t_2, and t_3; we write $\xi^i = x(t_i)$ for $i = 1,2,3$. If $\xi_1 = \xi_2$ then $x(t)$ is periodic and the result is immediate; hence we suppose that $\xi^1 \prec \xi^2$. Let Γ be the curve obtained by following $x(t)$ from ξ^1 to ξ^2, then following L back to ξ^1; since $x(t)$ does not intersect L for $t_1 < t < t_2$, Γ is a simple closed curve. Let $\phi : W \to D$ with $W = (-\epsilon,\epsilon) \times L$ be a flow box for L; $\phi^{-1}(\Gamma)$ certainly contains the three line segments $\{(0,\xi) \mid \xi^1 \preceq \xi \preceq \xi^2\}$, $\{(t,\xi^1) \mid 0 \le t < \epsilon\}$, and $\{(t,\xi^2) \mid -\epsilon < t \le 0\}$, and in fact must equal their union, for if $\phi(s,\xi) = x(t)$ for

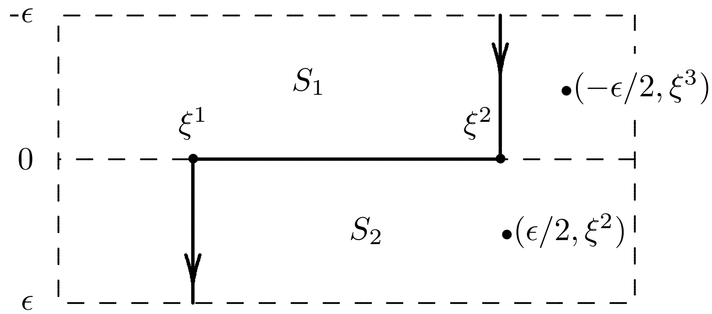


Figure 4.13. The set W; the heavy line is $\phi^{-1}(\Gamma)$.

$t_1 < t < t_2$ then $\xi = \Phi_{-s}(\phi(s,\xi)) = \Phi_{-s}(x(t)) = x(t-s) \in L$ but $x(t-s') \notin L$ for s' between s and 0; so that either $s > 0$ and $\xi = \xi^1$ or $s < 0$ and $\xi = \xi^2$. Thus $W \setminus \phi^{-1}(\Gamma)$ consists of two connected components

$$S_1 = \{(t,\xi) \mid \xi \prec \xi^1, t \geq 0 \text{ or } \xi \prec \xi^2, t < 0\}$$
$$S_2 = \{(t,\xi) \mid \xi \succ \xi^2, t \leq 0 \text{ or } \xi \succ \xi^1, t > 0\}$$

(see Figure 4.13). The Jordan curve theorem implies that $\phi(S_1)$ and $\phi(S_2)$ must be contained in different components O_1, O_2 of $\mathbb{R}^2 \setminus \Gamma$, since if both belonged to, say, O_1, then the interior of the segment of Γ coincident with L would not lie in ∂O_2. Now the points $x(t_2 + \epsilon/2)$ and $x(t_3 - \epsilon/2)$ are connected by a path ($x(t)$ itself) which does not intersect Γ, and hence these points must belong to the same component of $\mathbb{R}^2 \setminus \Gamma$; since $\phi^{-1}\big(x(t_2+\epsilon/2)\big) = (\epsilon/2, \xi^2) \in S_2$, $\phi^{-1}\big(x(t_3-\epsilon/2)\big) = (-\epsilon/2, \xi^3) \in S_2$ also, i.e., $\xi^3 \succ \xi^2$. ■

Lemma 4.9: *Let $x(t)$ be a solution of (4.1) satisfying the hypotheses of Theorem 4.6, and suppose that $x^0 \in \Omega[x]$ and that L is a transversal to f which contains x^0. Then*

(a) there exists a sequence $\{t_k\}$ with $t_k \to \infty$ such that $x(t_k) \in L$ and $\lim_{k\to\infty} x(t_k) = x^0$;

(b) L intersects $\Omega[x]$ in the single point x^0.

Proof: Since $x^0 \in \Omega[x] \cap L$ there exists a sequence of times $t'_k \to \infty$ such that $x(t'_k) \to x^0$ and for sufficiently large k we must have $x(t'_k) \in \phi(W)$, where ϕ is a flow box for L. But then $x(t'_k) = \phi(s_k, \xi^k)$, and the points $\xi^k = x(t_k)$ for $t_k = t'_k - s_k$ all lie in L; moreover, $\xi^k \to x^0$ by the continuity of ϕ^{-1}. If y^0 were a point of $\Omega[x] \cap L$ distinct from x^0 we could generate a similar sequence $\zeta_k \to y^0$, but the existence of two such sequences would certainly violate the monotonicity required by Lemma 4.8. ■

Proof of the Poincaré-Bendixson Theorem. By Theorem 4.4, $\Omega[x]$ is not empty. Choose $y^0 \in \Omega[x]$; by Theorem 4.4 the solution $y(t)$ of (4.1) which satisfies $y(0) = y^0$ is defined for all $t \in \mathbb{R}$ and lies in $\Omega[x]$. Again, $\Omega[y] \neq \emptyset$; choose $y^1 \in \Omega[y]$. Since $\Omega[y] \subset \Omega[x]$, the hypotheses of the theorem imply that y^1 is not an equilibrium point of f; hence (Remark 4.4) we may choose a transversal L to f which contains y^1. Now application of Lemma 4.9 to $y(t)$ implies both that $y(t)$ must intersect L infinitely often and, since $y(t) \in \Omega[x]$, that the orbit of $y(t)$ can intersect L in at most one point. We conclude that $y(t)$ must be a periodic orbit and that $\Omega[y]$ is identical with this orbit.

Now we have found a periodic orbit $C_p \equiv \Omega[y]$ contained in $\Omega[x]$, and it remains to be shown that $\Omega[x]$ is identical with C_p. Suppose it is not. We may write $\Omega[x] = C_p \cup (\Omega[x] \setminus C_p)$; since by Theorem 4.4, $\Omega[x]$ is connected and $C_p = \Omega[y]$ is closed, and since a connected set cannot be the union of nonempty disjoint closed sets, $C_p \cup (\Omega[x] \setminus C_p$ is not closed. But $\Omega[x]$ is closed, so there must exist a point $z^0 \in C_p$ and a sequence of points $\{z^i\}$ with $z^i \in \Omega[x] \setminus C_p$ and $z^0 = \lim_{i\to\infty} z^i$. As above, we may choose a transversal, called again L, through z^0, and construct a corresponding flow box ϕ. Choose i so large that $z^i = \phi(s, \zeta^i) \in \phi(W)$; then $\Phi_{-s}(z^i) \equiv \zeta^i \in L$. But because $\Omega[x]$ is invariant we also have $\zeta^i \in \Omega[x]$, and because z^0 lies on the periodic orbit C_p and z^i does not, $\zeta^i \neq z^0$. Thus L intersects $\Omega[x]$ in the two points ζ^i and z^0, contradicting Lemma 4.9. ■

Remark 4.6: The Poincaré-Bendixson Theorem may be used to establish the existence of periodic orbits in plane autonomous systems. To do so, we would typically seek an open set U with $\overline{U} \subset D$ such that (i) a solution $x(t)$ with $x(t_0) \in U$ satisfies $x(t) \in U$ for $t > t_0$ (U is then called *positively invariant*), and (ii) no equilibrium point x^0 of U can lie in any ω-limit set other than $\Omega[x^0]$. From these properties it follows that U must contain a periodic orbit.

(a) One possibility for verifying (i) is an energy conservation argument as sketched in the discussion of Theorem 2.22; we will consider similar, more general techniques when we discuss Lyapunov functions in Chapter V. Another, related approach is the following: U will be positively invariant if the boundary ∂U is a smooth curve such that, if $x^0 \in \partial U$, then $f(x^0)$ is transverse to ∂U at x^0 and is oriented inward to U. (To see this, suppose that $x(t_0) \in U$, let $t_1 = \inf\{t > t_0 \mid x(t) \notin U\}$, and consider the flow near $x(t_1)$.) There are also simple generalizations of this criteria when ∂U is not smooth.

(b) The simplest way to verify (ii) is to show that U contains no equilibrium points. It is also easy to verify that no equilibrium at which linearization yields an unstable node or an unstable spiral point can be an ω-limit point; see Exercise 6.1 of Cronin.