Part I

Classical Geometry

γῆ, earth (including land and sea) ...
μετρέω, measure out ...
(Liddel and Scott, *Greek-English Lexicon*, Oxford)

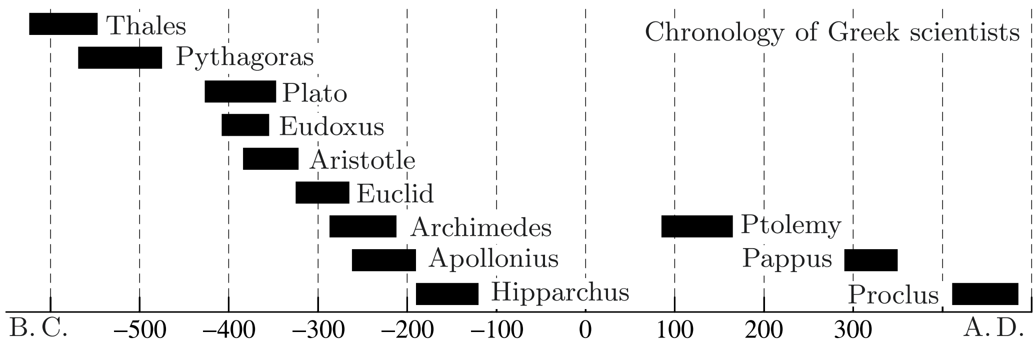
"In the tomb of Khaemhet at Thebes we see a number of men equipped with ropes and writing material measuring a field, ..."
(T.E. Peet, *Rhind mathematical papyrus*, 1923, p. 32)

"The Mathematick Lecturer to read first some easy & usefull practical things, then Euclid, Sphericks, the Projections of the Sphere, the Construction of Mapps, Trigonometry, Astronomy, Opticks, Musick, Algebra, &c." (I. Newton, *Of Educating Youth in the Universities*, MS. Add. 4005, fol. 14–15, Cambridge 1690)

"Development of Western science is based on two great achievements: the invention of the formal logical system (in Euclidean geometry) by the Greek philosophers, and the discovery of the possibility to find out causal relationships by systematic experiment (during the Renaissance)."
(A. Einstein in a letter to J.S. Switzer, 23 Apr. 1953)

"Quoique la Géométrie soit par elle-même abstraite, il faut avoüer cependant que les difficultés qu'éprouvent ceux qui commencent à s'y appliquer, viennent le plus souvent de la maniére dont elle est enseignée dans les Elémens ordinaires. On y débute toûjours par un grand nombre de définitions, de demandes, d'axiomes, & de principes préliminaires, qui semblent ne promettre rien que de sec au lecteur". (A.-C. Clairaut, *Elémens de Géométrie*, 1741)

We see in the chronology below that Euclid, who lived around 300 B.C., was not the first great geometer, despite the fact that his famous *Elements* "with all its definitions, postulates, axioms & preliminary principles, which seem to



promise nothing but arid reading" (see the above quotation from Clairaut) usually serve as a model for the beginning of a course on geometry. But mathematical results had already been obtained in the preceding centuries, in order to measure (μετρέω) land (γῆ), to survey fields after the regular floods of the Nile, to compute the quantity of corn in a cylindrical container, and to construct spectacular temples and pyramids. We therefore start in Chap. 1 with "some easy & usefull practical things", the theorems of Thales and Pythagoras, which are the oldest theorems of humanity and fundamental tools for geometry. They allow one to deal with most practical applications.

A first flaw in this paradise was revealed by the discovery of irrational numbers, which showed that the concept and the proof of Thales' theorem were not as simple as had been thought. In parallel with this were the efforts, influenced by the Greek philosophers, from the Pythagoreans to Plato, to separate geometry from its practical applications, to raise it to an abstract science studying unchangeable objects and to lift the soul towards eternal truth. The nails, ropes and walls used by the temple builders were replaced by mathematical points, lines, rectangles etc., objects of pure reasoning, which require a list of definitions, axioms and postulates (see Chap. 2). This is the origin of the style of nearly all mathematical thought and exposition since then.

In Chaps. 3 and 4 we describe the achievements of the post-Euclidean period, the new curves and theorems invented by Apollonius, Nicomedes, Archimedes and Pappus, often in order to solve one of the three great problems of Greek geometry: squaring the circle, trisecting any angle or duplicating the cube. Chap. 4 also contains many more recent beautiful results, which the Greeks *could* have found with their methods.

Chap. 5 is devoted to the last great creation of the Greek period, plane and spherical trigonometry by Hipparchus and Ptolemy and their application to one of the dreams of mankind, understanding the movements of the heavenly bodies. This gave rise to modern astronomy and the physical sciences.

1

Thales and Pythagoras

> "... la théorie des lignes proportionnelles et la proposition de Pythagore, qui sont les bases de la Géométrie ... [the theory of proportional lines and the theorem of Pythagoras, which form the basis of geometry]" (J.-V. Poncelet, 1822, p. xxix)

> "... the original works of the forerunners of Euclid, Archimedes and Apollonius are lost, having probably been discarded and forgotten almost immediately after the appearance of the masterpieces of that great trio." (T.L. Heath, 1926, vol. I, p. 29)

The most beautiful discoveries of this period concern relations between *lengths* (Thales' intercept theorem), *angles* (the central angle theorem or Eucl. III.20) and *areas* (the Pythagorean theorem). A quick look at the index shows that these three theorems are by far the most basic and frequently used results of geometry.

The only original documents which have survived from the pre-Euclidean period are some cuneiform Babylonian tablets (from approximately 1900 B.C.), the Egyptian *Rhind papyrus* and the *Moscow papyrus* from approximately the same period. The achievements of Thales, Pythagoras and his pupils the Pythagoreans are only documented in commentaries, often contradictory, written many centuries later.

1.1 Thales' Theorem

> "I tried (unsuccessfully) to get each high school in which my children were enrolled to go outside during geometry and find out how tall the oak in the yard really is."
> (D. Mumford, President IMU; Preface in H. M. Enzensberger, *Zugbrücke außer Betrieb [Drawbridge Up]*, 1999)

Thales was born in Miletus (Asia Minor, nowadays Turkey). He travelled to Babylon and to Egypt, calculated the height of the pyramids by measuring

A. Ostermann and G. Wanner, *Geometry by Its History*,
Undergraduate Texts in Mathematics, DOI: 10.1007/978-3-642-29163-0_1,

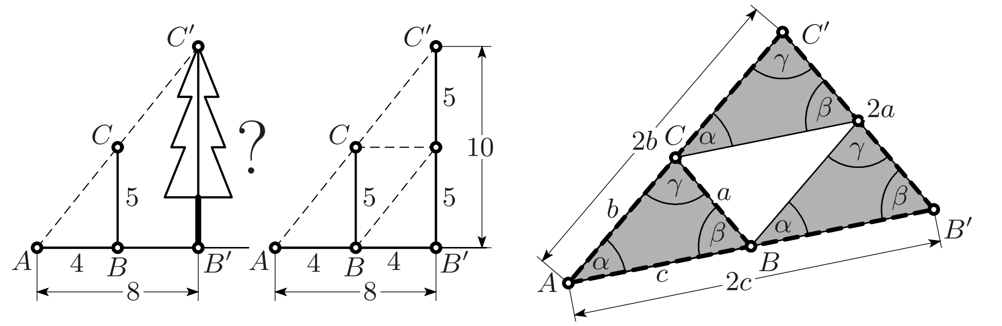


Fig. 1.1. The "oak" in Mumford's school yard and Thales' theorem for ratio 2

the length of their shadow, calculated the distance of ships from the shore, and predicted a solar eclipse in 585 B.C.

Thales is certainly the man to tell us how to measure the height of a tree $B'C'$, without having to climb it (see Fig. 1.1, left). Let AB' be the shadow of the tree; we erect a vertical stick BC in such a manner that AB is the shadow of the stick.[1] We then measure the distance AB, say 4 metres, the distance AB', say 8 metres, and the stick BC, say 5 metres. By parallel displacements of the triangle ABC we see that, since AB' measures twice AB, the height $B'C'$ will measure twice BC (see the middle picture), hence $B'C' = 2 \cdot 5 = 10$

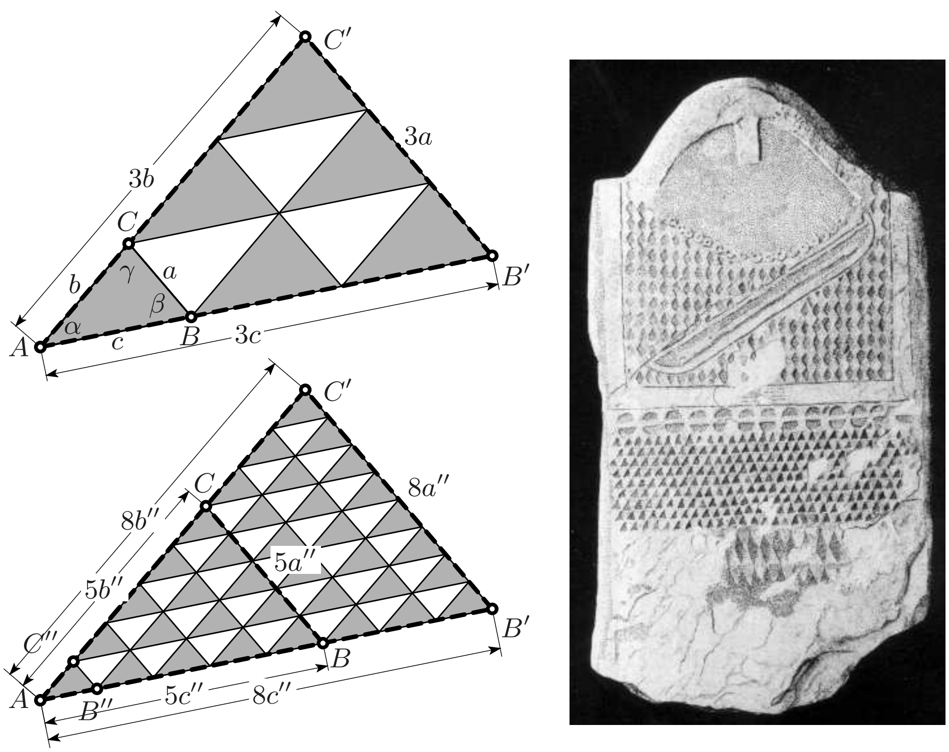


Fig. 1.2. The proof of Thales' theorem; right: Neolithic stele, Sion 2500 B.C. (courtesy Prof. A. Gallay)

[1]As recorded by Plutarch; see Heath (1921, p. 129)

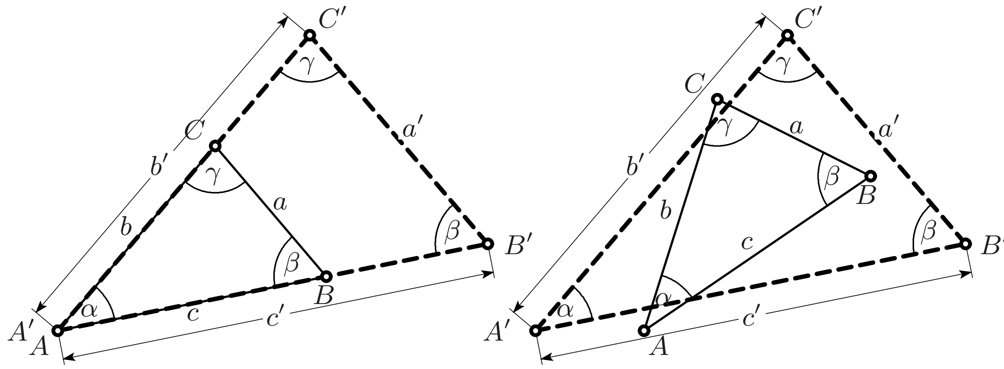


Fig. 1.3. Thales' intercept theorem

metres. The same argument can be applied to translations of any triangle ABC (see Fig. 1.1, right). We see that if a side of a triangle is doubled and the angles are preserved, then the other two sides are also doubled.

If our tree were still taller, we might have to displace our triangle *three* times and would arrive at the situation of Fig. 1.2 (upper left) where the sides of $AB'C'$ are three times as long as the sides of ABC.

By using still finer subdivisions, we arrive at the lower left picture of Fig. 1.2 where the ratios of these lengths are 8:5. We have thus discovered that the following theorem is valid for any rational fraction. We call this proof, which could have been inspired by the Neolithic stele from 2500 B.C., and which will be severely criticised later, the *Stone Age proof*.

Theorem 1.1 (Thales' intercept theorem). *Consider an arbitrary triangle ABC (see Fig. 1.3, left) and let AC be extended to C' and AB to B', so that $B'C'$ is parallel to BC. Then the lengths of the sides satisfy the relations*

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} \qquad \textit{and hence} \qquad \frac{a'}{c'} = \frac{a}{c}, \qquad \frac{c'}{b'} = \frac{c}{b}, \qquad \frac{b'}{a'} = \frac{b}{a}.$$

These proportions are also preserved when the triangle is displaced and rotated, see Fig. 1.3 (right). As a consequence we get the following result. *If corresponding angles of two triangles are equal, then corresponding sides are proportional*. Triangles having these properties are called *similar*.

1.2 Similar Figures

A more general view of Thales' theorem appeared in the works of Clavius, Viète and others: figures are said to be *similar with similarity centre O* when corresponding points A_i, B_i, C_i lie on lines through O, and the corresponding lines A_iA_j, B_iB_j, C_iC_j are parallel (see Fig. 1.4). Applying Thales' theorem to selected pairs of triangles with a vertex in O shows that all corresponding lengths of similar figures are proportional. Such similar figures were an important source of inspiration for many of the great masters (see Fig. 1.5).

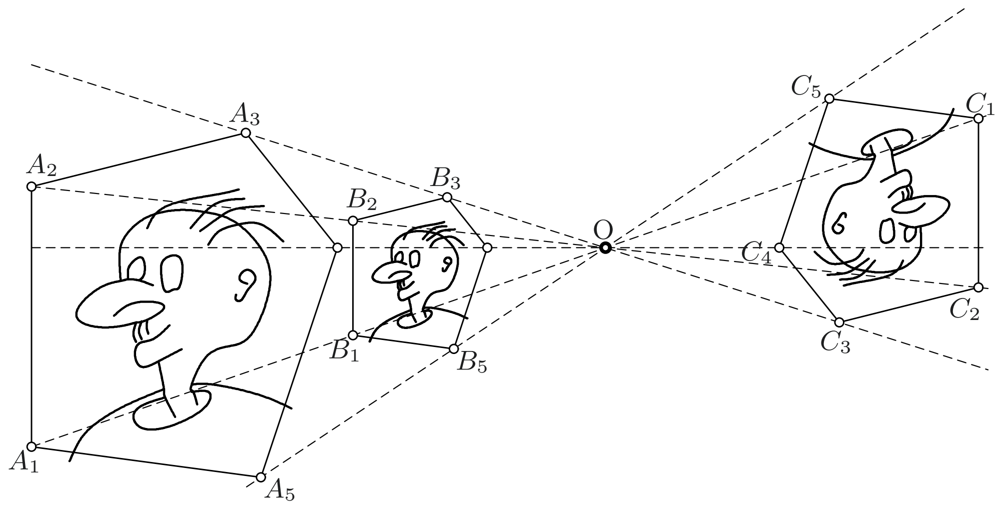
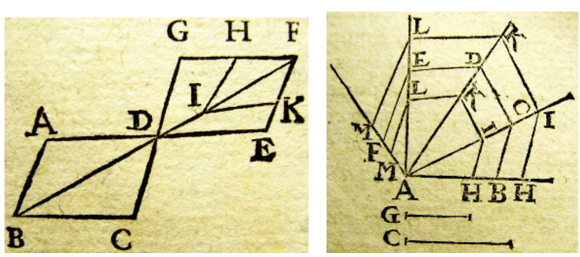
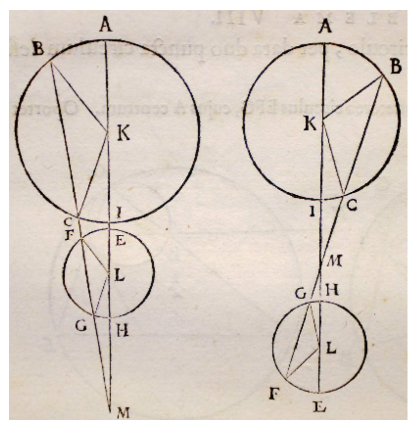


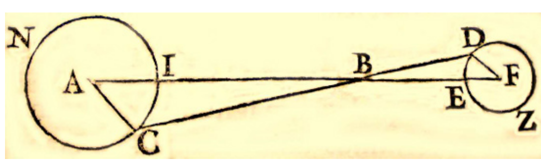
Fig. 1.4. Similar figures: illustration inspired by Clavius and Viète, improved by modern computer technology



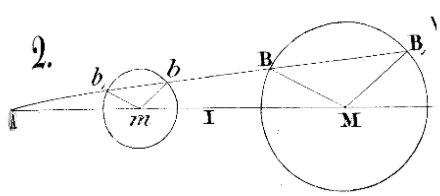
(Clavius' ed. of Euclid 1574)



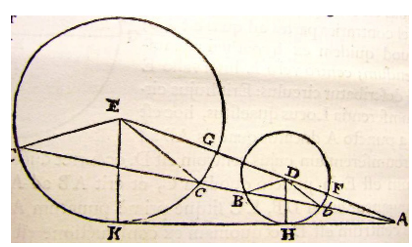
(Viète, 1600)



(Fermat, 1629a)



(Steiner, 1826a)



(Apollonius; Simson, 1749)

Fig. 1.5. Similar figures in the publications of several masters

Constructing rational lengths. Consider two distinct points 0 and 1 on a line. We call the length of the segment joining these two points the *unit length*. By carrying this unit forward on the line, we easily construct the integer points 2, 3, etc. But how can we construct points corresponding to rational values? For this we draw an arbitrary ray, not parallel to the line, through the point 0. We then carry forward several times (five times, say)

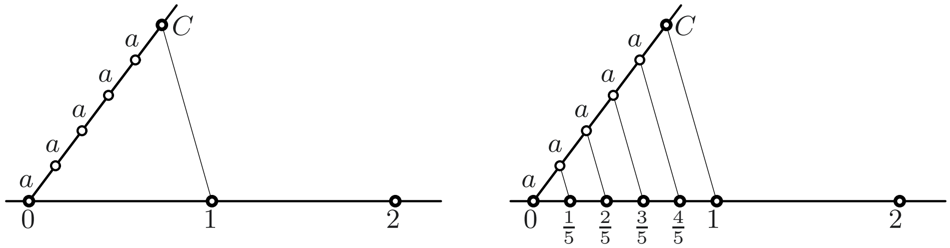


Fig. 1.6. Constructing rational lengths

an arbitrary length a. This construction yields a point C, see Fig. 1.6 (left). If we now draw the corresponding parallels to the line joining C with 1 (see Fig. 1.6, right), we obtain by Thales' theorem the required points $\frac{1}{5}$, $\frac{2}{5}$, etc. (this procedure will later be called Eucl. VI.9).

1.3 Properties of Angles

> Emil Artin (1898–1962) was famous for the extremely clear and extraordinarily well presented lectures that he always gave without any notes. One day, midway in a proof, he suddenly hesitated and said: "this conclusion is trivial". After a few seconds, he repeated: "it is trivial, but I no longer know why". He then thought about the question for another minute and said: "I *know* that it is trivial, but I no longer understand it". He reflected on it a few moments more and finally said: "excuse me, I have to look at my lecture notes". He then left the room and came back ten minutes later saying: "it *really* is trivial".
>
> (Witnessed by Prof. Josef Schmid, Fribourg)

> "I still remember a guy sitting on the couch, thinking very hard, and another guy standing in front of him, saying, 'And therefore such-and-such is true.' 'Why is that?' the guy on the couch asks. 'It's trivial! It's trivial!' the standing guy says ..."
>
> (R.P. Feynman, souvenir from the math-physics common lounge at Princeton; quoted from *Surely You're Joking, Mr. Feynman*, 1985, p. 69)

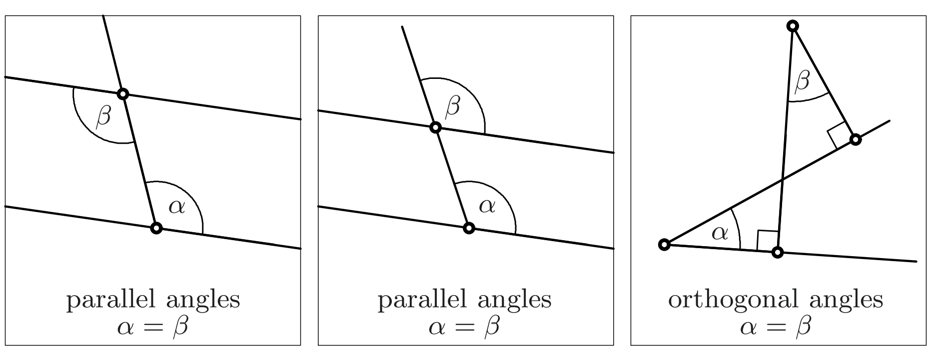
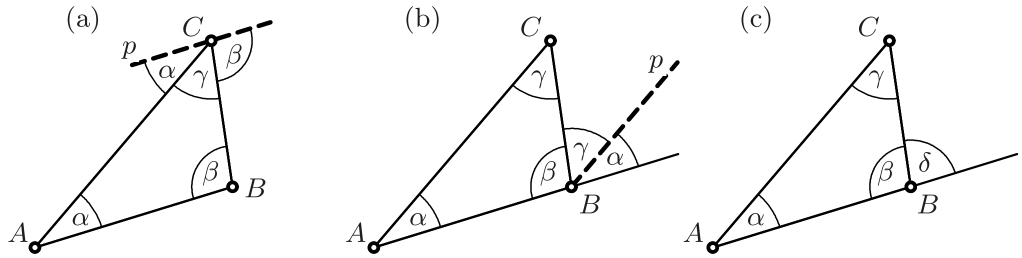


Fig. 1.7. Parallel and orthogonal angles



Pythagorean proof Euclid's proof Exterior angle $\delta = \alpha + \gamma$

Fig. 1.8. Angles in a triangle (Eucl. I.32)

Angles in a triangle. Some basic equality properties of angles (parallel and orthogonal angles) are shown in Fig. 1.7. As in pre-Euclidean times, we consider (for the moment) these properties to be "trivial". A more thorough treatment will follow in Chap. 2.

> "Die Winkelsumme im Dreieck kann nicht nach den Bedürfnissen der Kurie abgeändert werden. [The sum of the angles of a triangle can not be modified according to the requirements of the Curia.]"
> (B. Brecht, *Leben des Galilei*, 1939, scene 8)

Theorem 1.2 (Eucl. I.32). *The sum of the three angles of an arbitrary triangle ABC is equal to two right angles:*[2]

$$\alpha + \beta + \gamma = 2\,⊾ = 180^\circ. \tag{1.1}$$

For its *proof*, the Pythagoreans draw a line p through C parallel to the opposite side AB, see Fig. 1.8 (a). Euclid extends the side AB, draws a parallel to AC through B (Fig. 1.8 (b)) and uses the parallel angles α and γ.

Euclid's method yields the following corollary.

Corollary 1.3. *Each exterior angle is the sum of the non-adjacent interior angles, see Fig. 1.8 (c):*

$$\delta = \alpha + \gamma\,. \tag{1.2}$$

Angles in a circle. On a circle with centre O and diameter AB, we choose an arbitrary point C (other than A or B) and join it to A and to O, see Fig. 1.9 (a). Since the triangle AOC is isosceles, we have the same angle β at A as at C (see Eucl. I.5 in Sect. 2.1). Hence, by (1.2), the angle BOC is twice the angle BAC. We shall call BOC the *central angle on the arc* BC, and BAC an *inscribed angle* on this arc. More generally, in Fig. 1.9 (b), we call CAD an inscribed angle on the arc CD and COD the central angle on this arc. We next choose an arbitrary point D on the circle, such that C and D are on opposite sides of the diameter AB, see Fig. 1.9 (b). Deleting this diameter, we obtain in Fig. 1.9 (c) an important relation for $\alpha = \beta + \gamma$:

[2]Inspired by Euclid (cf. Euclid's Postulate 4 in Sect. 2.1), we use a specific symbol ⊾ for a right angle; similarly, Steiner (1826c) used the symbol R, and Miquel (1838a) the symbol d (*angle droit*), so we are in good company.

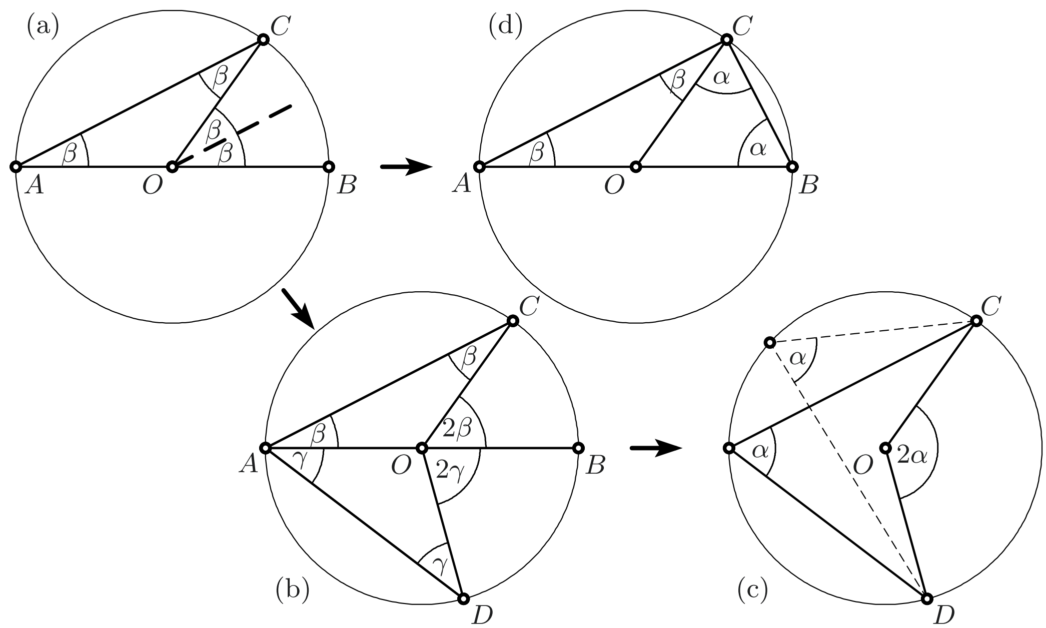


Fig. 1.9. Central angle and inscribed angle

Theorem 1.4 (Eucl. III.20). *A central angle of a circle is twice any inscribed angle on the same arc, see Fig. 1.9 (c).*

Theorem 1.5 (Eucl. III.31). *If AB is a diameter and C a point (other than A or B) on the circle, then ACB is a right angle, see Fig. 1.9 (d).*

Proof. This follows from the equality of the two angles denoted by α and β and Eucl. I.32, because $2\alpha+2\beta = 2\,\llcorner$ implies $\alpha+\beta = \llcorner$. It can also be considered as a special case of Eucl. III.20 by taking $2\alpha = 2\,\llcorner$ in Fig. 1.9 (c). ◻

The Thales circle. The circle with a given segment AB as diameter is called the *Thales circle* of the segment, see also Fig. 2.1, Def. 21. Any triangle ABC with C on this circle is right-angled. For the converse to Theorem 1.5, see Exercise 4 of Chap. 2, page 54.

1.4 The Regular Pentagon

Regular polygons have fascinated geometers since the dawn of science. The Babylonians had understood the equilateral triangle and the square (see Sect. 1.6 below), therefore the Greeks directed their attention to the regular pentagon, a polygon with five vertices.

Length of the diagonal. By drawing all the diagonals of a regular pentagon, we obtain a star as shown in Fig. 1.10 (b).

This beautiful star was for the Pythagoreans a symbol of recognition between members, a tradition which has survived until today in revolutionary movements and luxury hotels.

We will determine the length, say Φ, of the diagonal of a regular pentagon of side length 1, see Fig. 1.10 (a). Since the central angles on the arcs AB, BC, etc. are $72°$ by construction, the inscribed angles on these arcs are $\alpha = 36°$ (Eucl. III.20). We consider the triangle ACD, see Fig. 1.10 (c). It contains the smaller triangle CDF which is similar to ACD. Hence, we get

$$\text{Thales:} \quad s = \frac{1}{\Phi} \qquad \text{isosceles:} \quad \Phi = 1 + s \tag{1.3}$$

which leads to

$$\Phi^2 = \Phi + 1 \qquad \text{and} \qquad s^2 + s = 1\,. \tag{1.4}$$

A geometrical construction for these values, showing that $\Phi = \frac{\sqrt{5}}{2} + \frac{1}{2}$ and $s = \frac{\sqrt{5}}{2} - \frac{1}{2}$, was probably known to the Pythagoreans, and is numbered II.11 in Euclid's *Elements* (see Exercise 15 on page 57).

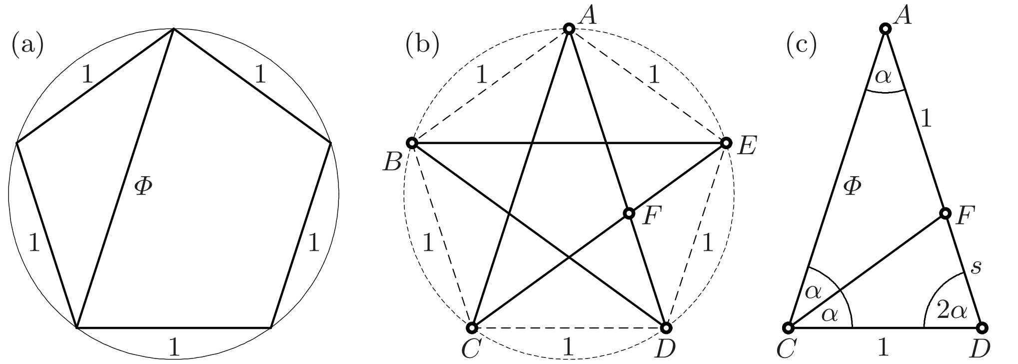


Fig. 1.10. The regular pentagon

The number Φ is called *the golden ratio*. The fact that many beautiful ancient buildings, in particular the *Parthenon* on the Acropolis, fit so perfectly into a "golden rectangle" (a rectangle with sides length 1 and Φ) led to the notation Φ in honour of Φειδίας, its architect.

The discovery of irrational numbers. *All is number*, claimed Pythagoras, who apparently had only *rational numbers* in mind. However, it was soon discovered that $\sqrt{2}$ and Φ are not rational.

To give a proof for Φ, we assume that the rational number $\frac{m}{n}$ is a solution of (1.3). We further assume that this fraction is reduced, i.e. that m and n are relatively prime. Hence, by (1.3), we have

$$\frac{m}{n} = 1 + \frac{1}{\frac{m}{n}} = 1 + \frac{n}{m} = \frac{m+n}{m}\ . \tag{1.5}$$

But if m and n are relatively prime, so are m and $m+n$ (for more details, see Eucl. VII.2 in Sect. 2.4, in particular Fig. 2.19). Hence, the fractions m/n and $(m+n)/m$ cannot be equal and Φ *cannot be a rational number.*

The fact that the regular pentagon, considered holy by the Pythagoreans, has a non-measurable diagonal was a real shock. A legend says that Hippasus, having discovered this fact and talked too much, was drowned at sea.

This discovery was also a major upset to the theory: the proof given above for Thales' theorem is not valid for irrational proportions. This considerably complicated Euclid's *Elements*, see Chap. 2.

1.5 The Computation of Areas

> A study for the Department of Education ... found nearly one in three adults (29%) in England could not calculate the floor area of a room in feet or metres — with or without calculators or paper and pens.
>
> (BBC News Online [Education], Sunday, May 5, 2002)

The calculation of areas will lead us to the Pythagorean theorem, the third pillar of this chapter, after Thales' theorem and Eucl. III.20. We start with the *area of a rectangle*, which is $a \cdot b$. This is the number of wine bottles (28) that can be stored in a bin holding 4 layers, each of 7 bottles, see Fig. 1.11, left.

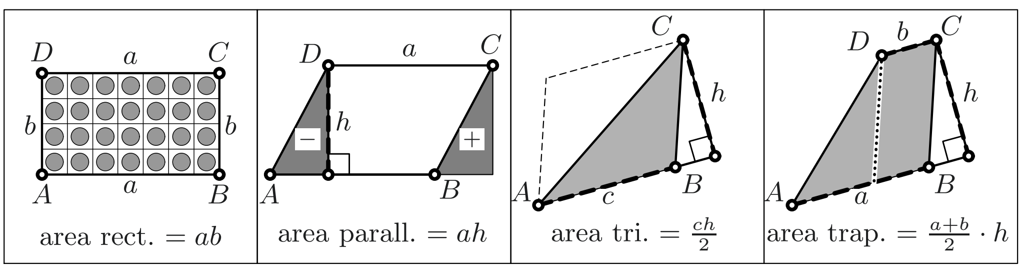


Fig. 1.11. Areas of rectangle, parallelogram, triangle and trapezium

The area of a parallelogram is $a \cdot h$, where h is the *altitude* of the parallelogram (Eucl. I.35). There are two ways to see this: (a) We cut off the triangle on the left and add it on the right to obtain a rectangle (Euclid's proof, see the second figure in Fig. 1.11); (b) We cut the parallelogram parallel to AB into a large number of very slim rectangles ("method of exhaustion" of Eudoxus and Archimedes, in this form in the commentaries of Legendre (1794); see also Fig. 2.34, right).

The *area of a triangle* is half the area of the parallelogram,

$$\mathcal{A} = \textit{ area of triangle } = \text{ base} \times \text{altitude divided by 2 } = \frac{c \cdot h}{2} \qquad (1.6)$$

(Eucl. I.41), see third picture of Fig. 1.11.

Finally the *area of a trapezium* (see Fig. 1.11, right) is found by cutting the trapezium into a parallelogram and a triangle, which gives by combining the two previous results $\mathcal{A} = bh + \frac{a-b}{2} \cdot h = \frac{a+b}{2} \cdot h$.

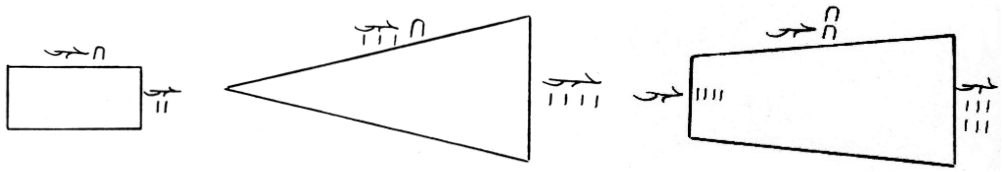


Fig. 1.12. Area calculations in Rhind papyrus; rectangle $10 \cdot 2 = 20$ (No. 49, left); triangle $\frac{4 \cdot 10}{2} = 20$ (No. 51, middle); trapezium $\frac{4+6}{2} \cdot 20 = 100$ (No. 52, right); reproductions of transcriptions by Peet (1923)

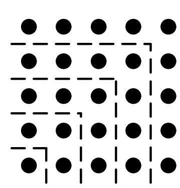
The Rhind papyrus. In 1858, the Scottish egyptologist A.H. Rhind bought in a market place at Luxor two pieces of a papyrus roll (now papyri 10057 and 10058 of the British Museum) which was written around 1650 B.C. and claimed to be a copy of a still older document from the 19th century B.C. The first extensive analysis and translation was made by A. Eisenlohr (1877), who numbered the examples of the papyrus from 1 to 84. In 1898 a facsimile was published by the British Museum. A very careful treatment with transcriptions directly from the papyrus was given by T.E. Peet (1923). The Egyptians noted numbers in a decimal system, using the symbols 1 = 𓏺, 10 = 𓎆, 100 = 𓍢, 1000 = 𓆼, so that, for example, the number

4678 = 𓆼𓆼𓆼𓆼𓍢𓍢𓍢𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺

requires a great deal of writing. In the Rhind papyrus, the area of a rectangle is treated in No. 49 (see Fig. 1.12, left). The result for the rectangle of sides 𓏺𓏺 and 𓎆 *khet*, which should be 𓎆𓎆 *setat*, is unfortunately buried under "scribe's errors of the worst description" (Peet 1923, p. 90). However, in No. 51 (Fig. 1.12, middle) the area of a triangle of base 𓏺𓏺𓏺𓏺 and altitude 𓎆 is correctly computed as 𓎆𓎆, but the discussion of whether the Egyptian scribe correctly understood the meaning of the altitude, fills four pages in Peet (1923), pp. 91–94). In No. 52 we find the correct computation of the area 𓍢 of a trapezium with sides 𓏺𓏺𓏺𓏺𓏺𓏺 and 𓏺𓏺𓏺𓏺 and altitude 𓎆𓎆. Again the meaning of the altitude is not completely clear (Fig. 1.12, right).

Areas of similar triangles. Take the triangle ABC of Fig. 1.2 (Stone Age proof of Thales' theorem) with sides that are 5 times longer than those of the triangle $AB''C''$. It is composed of

$$1 + 3 + 5 + 7 + 9 = \quad = 5^2 \qquad (1.7)$$



copies of the small triangle (this was one of the favourite arguments of Pythagoras). In the same way, the triangle $AB'C'$ contains 8^2 copies. Hence, the area of $AB'C'$ is $\frac{8^2}{5^2}$ times that of ABC.

We thus obtain the following result.[3]

Theorem 1.6 (Eucl. VI.19). *A similar triangle with q times longer sides has q^2 times larger area.*

1.6 A Remarkable Babylonian Document

Figure 1.13 displays a Babylonian tablet dating from 1900 B.C., hence much older than Nebuchadnezzar or Tutankhamun. This tablet shows a square with sides of length 30. On its diagonal the sexagesimal digits 1, 24 51 10 and 42, 25 35 are engraved (in Babylonian notation '𒁹' stands for 1, '𒌋' stands for 10).

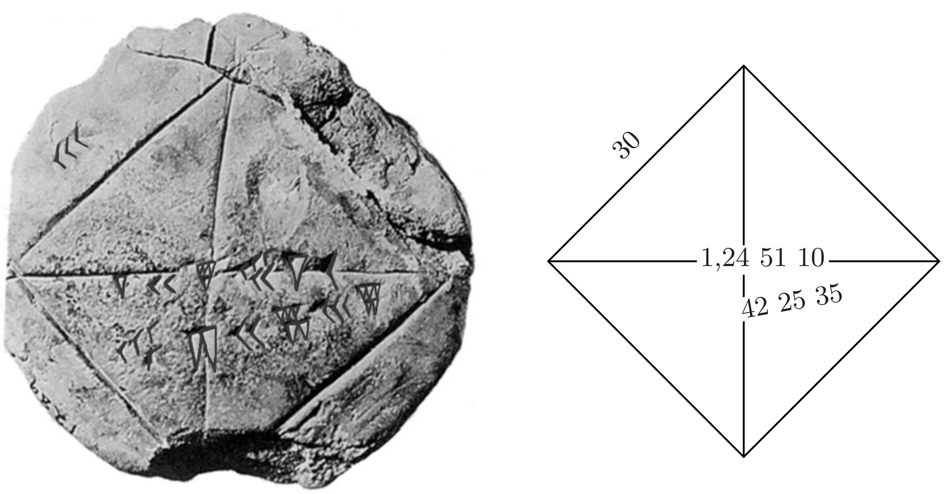


Fig. 1.13. Babylonian cuneiform tablet YBC7289 from 1900 B.C. (image enhanced by S. Cirilli)

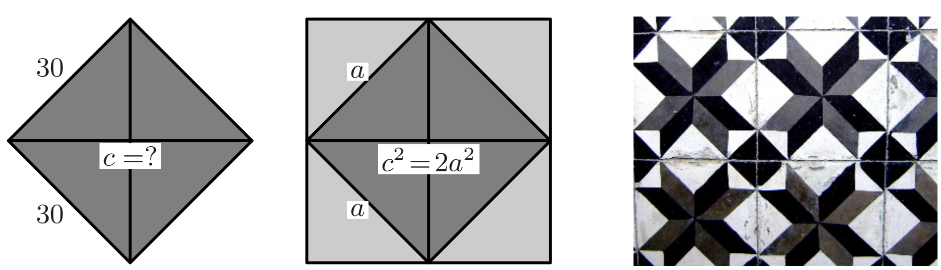


Fig. 1.14. Length of the diagonal of a square (left); ornamental tessellation seen in an old chapel in Crete (right)

[3]Another way of obtaining this result is based on (1.6).

Explanation. If we have a square of side length $a = 30$ and diagonal c (Fig. 1.14, left), the square on its diagonal is twice as large (composed of 8 triangles instead of 4; Fig. 1.14, middle). Thus, $c^2 = 2a^2$ and $c = a\sqrt{2}$. Another way of obtaining this result would be to meditate on one of the ornamental tessellations (Fig. 1.14, right) which were so frequent in antiquity. The above numbers, written in base 60, are

$$\sqrt{2} = 1, 24\ 51\ 10\ 7\ 46\ 6\ 4\ \ldots, \qquad 30 \cdot \sqrt{2} = 42, 25\ 35\ 3\ 53\ 3\ 2\ \ldots$$

and we see that the digits shown on the tablet are all correct (see Exercise 7 below for the computation).

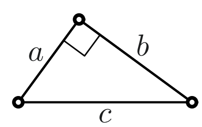
The tablet thus gives evidence that Pythagoras' theorem (for the case of an isosceles triangle) was already known to the Babylonians, as were the rules of proportions. This knowledge was combined with an admirable ability for calculation.

1.7 The Pythagorean Theorem

> "This great theorem is universally associated with the name of Pythagoras. Proclus says 'If we listen to those who wish to recount ancient history, we find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in celebration of his discovery.'" (T.L. Heath, *Euclid in Greek*, 1920, p. 219)

Millions of pupils around the world have had to learn the formula

$$a^2 + b^2 = c^2 \tag{1.8}$$



relating the three sides of a right-angled triangle; fewer by far know a proof, or even its precise meaning. This theorem, often considered the *first great theorem* of mankind, is attributed to Pythagoras (see the quotation), but it is not known how the original discovery was achieved.

Classical proofs. Figure 1.15 spans three civilisations: Chinese, Indian and Arabic. We start with the square of area c^2, slightly tilted as in Fig. 1.15 (a).

The Chinese proof. Adding four right-angled triangles with sides a and b, we arrive at Fig. 1.15 (b) and get the large square of area $(a+b)^2 = a^2+2ab+b^2$. Since the areas of the four triangles add up to $2ab$, the square of area c^2 also has area $a^2 + b^2$. This is the proof of Chou-pei Suan-ching (China, 250 B.C.; see van der Waerden, 1983, p. 27). In the pictures of Fig. 1.15 (right), this transformation is obtained by translating the three triangles 2, 3 and 4. The fact that the lower picture is precisely the picture of Eucl. II.4 on page 38 gives strong evidence that this was also Pythagoras' original proof.

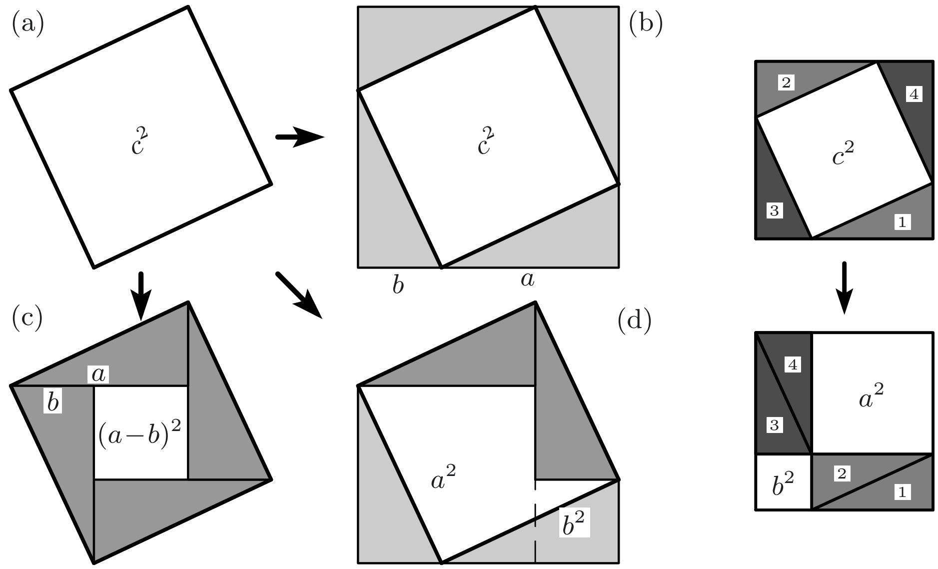


Fig. 1.15. Left: three classical proofs (Chou-pei Suan-ching (b), Bhāskara (c), Thābit ibn Qurra (d)); right: transforming Chou-pei's figure by translating triangles into Eucl. II.4

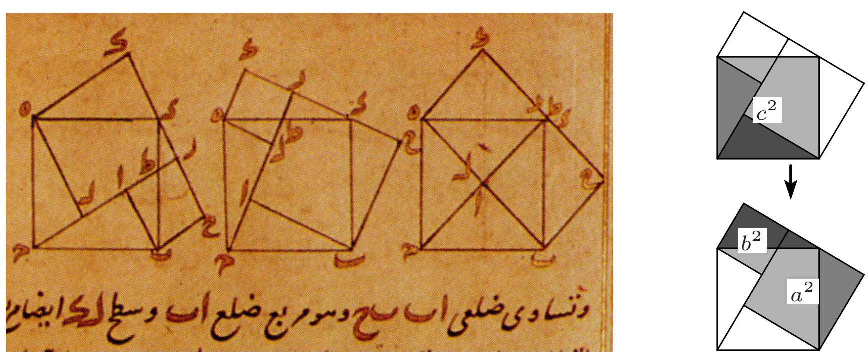


Fig. 1.16. Manuscript by Naṣīr al-Dīn al-Ṭūsī 1201–1274 with Thābit ibn Qurra's proof of Pythagoras' theorem (left); explanation (right)

The Indian proof. Bhāskara (born in 1114 A.D. in India) *removes* these four triangles to get $(a-b)^2$ and concludes the proof by saying simply "Look!", see Fig. 1.15 (c).

The Arabic proof. But why not remove *two* triangles and add them on the opposite sides, see Fig. 1.15 (d) and Fig. 1.16? By this construction, the square of area c^2 is transformed directly, without any additional triangle calculation, into two squares of total area $a^2 + b^2$. This elegant proof is attributed to Thābit ibn Qurra (826–901).

Proofs using tessellations. A legend says that Pythagoras discovered his theorem by observing a tiled floor in the palace of Polycrates, the tyrant of Samos. Since the legend does not describe the floor he considered, we have to rely on conjectures. Some possible patterns are displayed in Fig. 1.17. The

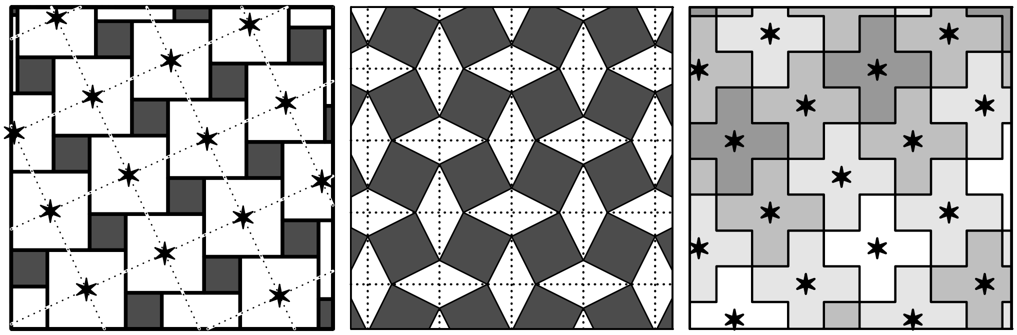


Fig. 1.17. Various patterns which could have tesselated Polycrates' palace

first picture shows a tessellation of a hypothetic hall in this palace by squares of two different areas, say a^2 and b^2, the two kinds in equal number. Looking at the dotted lines, we can also imagine this floor tiled by the same number of squares of area, say, c^2. It is thus intuitively clear that a^2+b^2 should be equal to c^2 (see also Penrose, 2005, pp. 26–27). In order to convert this intuition into a more convincing proof, we isolate *one* square of area c^2 and transform it by parallel translations of the quadrilaterals 2, 3 and 4 as in Fig. 1.18 into two squares of areas a^2 and b^2. The truth of Pythagoras' theorem is now immediately obvious. If we place the stars at one of the *vertices* of the squares c^2 (and not at their centres), we obtain in a similar way the Arabic proof (see also Exercise 11 below).

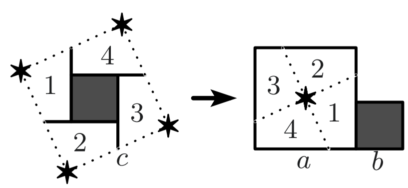


Fig. 1.18. Displacing tiles in Polycrates' tessellations for Pythagoras' theorem

The second pattern in Fig. 1.17 (proposed by Antje Kessler) might give the idea for the Chinese proof. Finally, the third pattern, with the Swiss crosses of area 5, indicates the truth of Pythagoras' theorem for a particular triangle, with sides 1, 2 and $\sqrt{5}$.

Euclid's proof. This brilliant proof was much admired by Proclus (see Heath, 1926, vol. I, p. 349). The idea is to attach the three squares of areas a^2, b^2 and c^2 to the right-angled triangle $AB\Gamma$ as in Fig. 1.19. The two grey triangles $BA\Delta$ and $BZ\Gamma$ are identical and just rotated by 90°. The triangle $BZ\Gamma$ has the same base and altitude as the square $BAHZ$; the triangle $BA\Delta$ has the same base and altitude as the rectangle $B\Delta\Lambda$. These two quadrilaterals thus have the same area. The same proof applies to the quadrilaterals on the right. The Pythagorean theorem now follows by adding the two results.

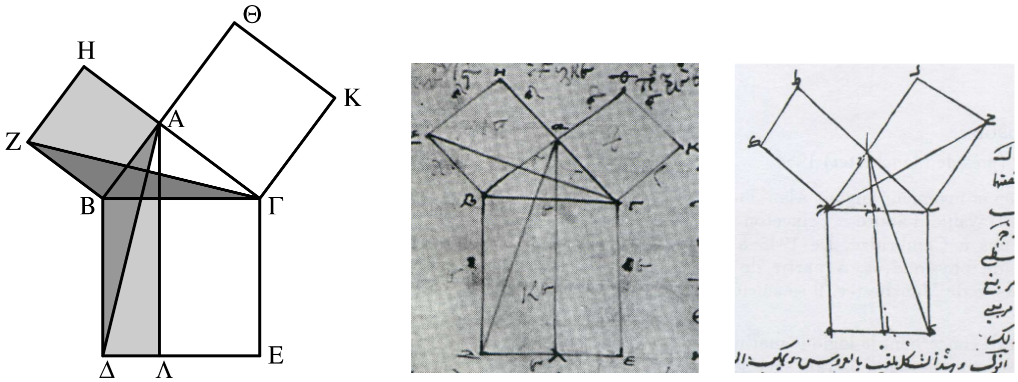


Fig. 1.19. Left: Euclid's proof; middle: Greek manuscript; right: Arabic manuscript (Thābit ibn Qurra, Baghdad 870)

Leonardo Pisano's proof. Leonardo Pisano (Fibonacci) proved Pythagoras' theorem in his *Practica Geometriae* (1220) by using Thales' theorem ("ut Euclides in sexto libro demonstravit") as follows (see Fig. 1.20).

Drawing the altitude through C gives two pairs of similar triangles: DBC, CBA and DAC, CAB; see Fig. 1.20. We thus get

$$\left.\begin{aligned} \frac{a}{p} = \frac{c}{a} \quad &\Longrightarrow \quad a^2 = pc \\ \frac{b}{q} = \frac{c}{b} \quad &\Longrightarrow \quad b^2 = qc \end{aligned}\right\} \quad \Longrightarrow \quad a^2 + b^2 = (p+q)c = c^2\,. \tag{1.9}$$

Note for later use that we also have

$$\frac{p}{h} = \frac{h}{q} \qquad \Longrightarrow \qquad h^2 = pq \qquad (\textit{the altitude theorem}). \tag{1.10}$$

Naber's proof. B.L. van der Waerden (1983, p. 30) attributes this proof to H.A. Naber (Haarlem 1908); Heath (1921, p. 148) presents it as one of the most probable original proofs of Pythagoras.

Without doubt, this proof is the most elegant of all. The four triangles in Fig. 1.21 are similar. If the area of the first, with hypotenuse 1, is denoted by k, the areas of the others are, by Theorem 1.6, equal to ka^2, kb^2, and kc^2,

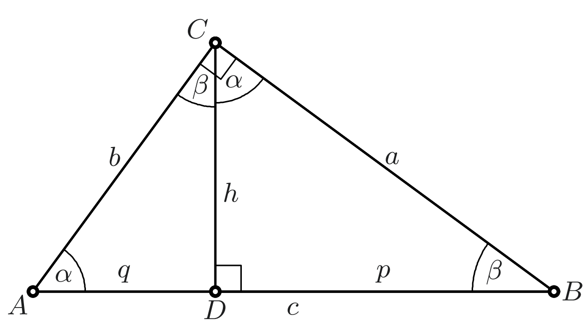


Fig. 1.20. A proof using Thales' theorem

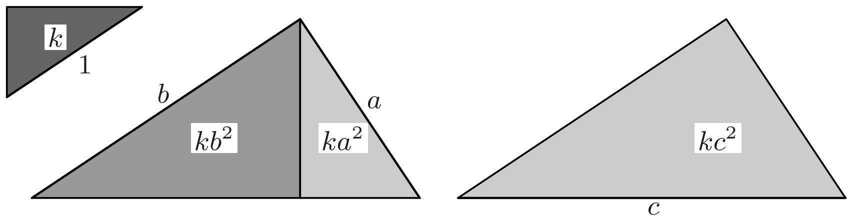


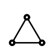
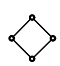
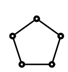
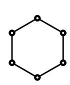
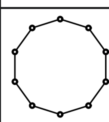
Fig. 1.21. Naber's proof

respectively. By comparing the figures, we see that, obviously, $ka^2+kb^2 = kc^2$, and it only remains to divide this formula by k.

For more proofs of the Pythagorean theorem, we recommend some exercises below and the book by Loomis (1940), which enumerates 370 proofs. One of these proofs is even due to a president of the United States (James Garfield), from those beautiful times when mathematics was more fascinating than oil.

Application to regular polygons. The above pre-Euclidean results allow us to demystify many regular polygons and to compute the radii ρ of their incircle and R of their circumcircle. The results are collected in Table 1.1.

Table 1.1. Radius of incircle (ρ) and radius of circumcircle (R) for regular polygons with side length 1

n		R	ρ
3		$R = \frac{\sqrt{3}}{3}$	$\rho = \frac{\sqrt{3}}{6}$
4		$R = \frac{\sqrt{2}}{2}$	$\rho = \frac{1}{2}$
5		$R = \frac{1}{\sqrt{3-\Phi}} = \frac{\sqrt{2+\Phi}}{\sqrt{5}}$	$\rho = \frac{\sqrt{3+4\Phi}}{2\sqrt{5}}$
6		$R = 1$	$\rho = \frac{\sqrt{3}}{2}$
10		$R = \Phi$	$\rho = \frac{\sqrt{3+4\Phi}}{2}$

Proofs. One always has $\rho = \sqrt{R^2 - \frac{1}{4}}$ by Pythagoras. For $n = 3$ and 5, the quantities h and ℓ (defined in Fig. 1.22) are calculated with Pythagoras; ℓ simplifies by using $\Phi^2 = \Phi + 1$. This gives

$$h = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}, \qquad \ell = \sqrt{1 - \frac{\Phi^2}{4}} = \frac{\sqrt{3-\Phi}}{2}.$$

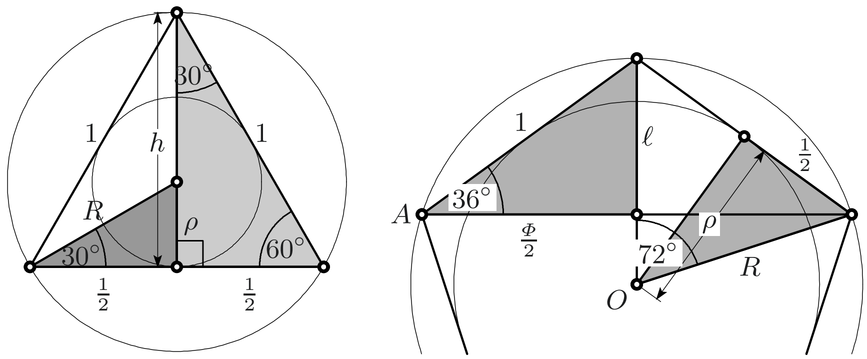


Fig. 1.22. Equilateral triangle and pentagon

The values for R are obtained by applying Thales' theorem to the grey triangles in Fig. 1.22. For $n = 10$, see Fig. 1.10 (c) ; since $\alpha = 36°$, ten of these triangles arranged as in a cake form a regular decagon. □

1.8 Three Famous Problems of Greek Geometry

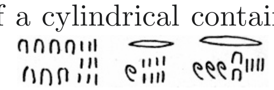
The following three problems appeared during the pre-Euclidean period and occupied the Greek geometers for at least three centuries. The new curves and algebraic tools which were needed to solve them contributed for another two millennia to the development of geometry, algebra and analysis.

Squaring the circle. Finding a square whose area is equal to that of a given rectangle was an easy exercise after the altitude theorem (1.10) was discovered. The next challenge was then to find areas of certain regions bounded by *curves.* In particular, the *squaring of a given circle* exercised great fascination throughout the centuries. The earliest known result is given in the examples No. 48 and 50 of the Rhind papyrus, see the pictures of Fig. 1.23 (left): a circle in a square of $9 \times 9 = 81$ units is squared by cutting off corners with two sides of length 3 units. This creates a surface of $81 - 18 = 63$ units. Since 63 is close to $64 = 8^2 = (9-1)^2$, we obtain the "Egyptian algorithm"

subtract one ninth of the diameter, then square .

This is demonstrated in No. 50, where (reproductions from Peet, 1923)

the area of a circle of diameter 9  is 64  .

In Rhind No. 42, while computing the volume of a cylindrical container, the area for diameter 10 is given as $79\frac{1}{108} + \frac{1}{324}$ or  , which is $79\frac{1}{81}$, the correct value. In modern notation these values correspond to the approximation $\pi \approx \frac{256}{81} = 3.1605$. Only during the Greek period were rigorous results obtained. Archimedes showed is his *celebrissimo* work (*Measurement of a circle*, Heath, 1897, p. 91), with virtuoso estimates from above and below that

$$3\,\frac{10}{71} < \pi < 3\,\frac{1}{7}\,. \tag{1.11}$$

The details are given in Exercise 22 on page 58. In Chap. 8 we will see why all the efforts of the Greek geometers to obtain an exact solution were doomed to failure.

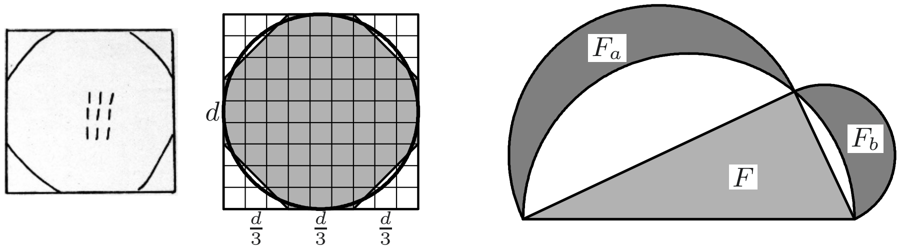


Fig. 1.23. Squaring the circle in Rhind papyrus (left pictures, reproduced from Rhind No. 48 in Peet, 1923); the quadrature of the lunes of Hippocrates (right)

The lunes of Hippocrates. However, *one* precise result in this direction was found during the Greek period, the squaring of the lunes by Hippocrates of Chios.[4] Let two lunes be cut out by three semicircles drawn on the sides of a right-angled triangle (see Fig. 1.23, right). Then their areas satisfy the relation

$$F_a + F_b = F \quad \text{(area of the triangle)}. \tag{1.12}$$

To see this, let F_a', F_b' and F_c' be the areas of the semicircles with diameters a, b and c. Then we see from the figure that $F_a' + F_b' + F = F_a + F_b + F_c'$. We have to know that Theorem 1.6, i.e. the fact that the areas of the semicircles are proportional to the squares of the diameters, remains valid here (this result will later be Eucl. XII.2). Then the terms $F_a' + F_b'$ and F_c' cancel by Pythagoras' theorem.

Doubling the cube. The problem is: *find a cube whose volume is twice that of a given cube* (see Fig. 1.24, left). Ancient sources give two different versions for the origin of the problem: according to one source, King Minos of Crete wanted Glaucus' tomb to be doubled (see Heath, 1921, p. 245); according to the other source, the *oracle of Delos* ordered the altar to be doubled in order to stop a plague epidemic. When the people went to Plato asking for help with the solution, he replied that the oracle did not mean that the actual doubling of the altar would heal the people, but that the advances in mathematics required for this construction would do so. For the geometers, who already knew how to *double a square* (see Section 1.6), this problem, which consists in

[4] who lived in the 5th century B.C., not to be mistaken with his contemporary Hippocrates of Kos, the famous physician.

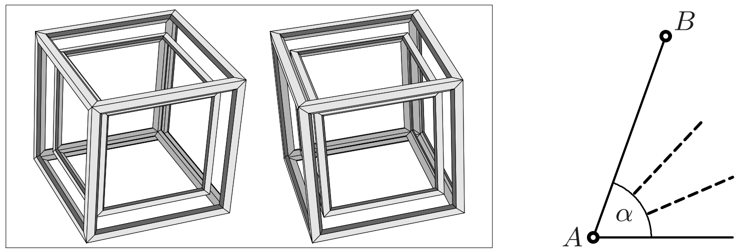


Fig. 1.24. Doubling the cube (left; the picture is a stereogram, if you stare at it by merging the two images with the left and right eye, you'll see it in 3d); trisecting an angle (right)

constructing $\sqrt[3]{2}$, was an interesting challenge. We will see how this problem led to the discovery of the first conic sections (Chap. 3) and many other new curves, one of which is the conchoid (see Chap. 4). Today's science would not be the same without the theory of conics (Chap. 5).

Trisecting an angle. The regular polygons with their divine beauty have fascinated geometers since time immemorial. The square and the equilateral triangle were known to the Babylonians, the regular pentagon was demystified by the Greeks (see above). Since it is easy to *bi*sect an angle (e.g. with Eucl. III.20), we have no difficulty in constructing a hexagon, octagon, decagon, dodecagon or any 2^k-gon. The next challenges are thus the regular heptagon (7-gon) and the regular enneagon (nonagon, 9-gon). This last problem would require one to *tri*sect the angle of 120°. From this question arose (probably) the challenge of trisecting *any* given angle (see Fig. 1.24, right). The solution of these problems contributed considerably to the development of algebra (see Chap. 6).

1.9 Exercises

1. Ptolemy gives the approximation

$$\sqrt{3} \approx 1, 43\ 55\ 23$$

in base 60 (see Heath, 1926, vol. II, p. 119). Check whether he is accurate.

2. Modify the proof of Theorem 1.4 for the case in which the points C and D are *not* on opposite sides of AB, see Fig. 1.25 (left). This time, α will be the *difference* of two angles β and γ.

3. Let ABC be a triangle inscribed in a circle, as in Fig. 1.25 (right). Show that the size of α is independent of the position of A on the circle (Eucl. III.21).

4. In order to approximate the golden ratio we consider the sequence of rational numbers given recursively by

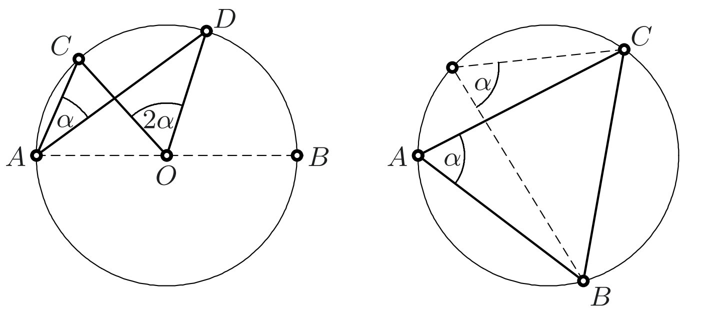


Fig. 1.25. Eucl. III.20 modified (left); Eucl. III.21 (right)

$$r_{k+1} = 1 + \frac{1}{r_k}, \qquad k \geq 0$$

with $r_0 = 1$. Find a relation to (1.3) and discover, by considering the denominators of the fractions r_k, an interesting sequence, the *Fibonacci numbers*.

5. Let a "golden" rectangle with sides Φ and 1 be given. Show that cutting off a square from this golden rectangle produces another golden rectangle with sides smaller by the factor $1/\Phi$ (see Fig. 1.26). The procedure can be repeated and produces an embedded sequence of golden rectangles. If we draw a quarter of a circle in each of these squares, we obtain a beautiful spiral which is said to possess great mystical power ...

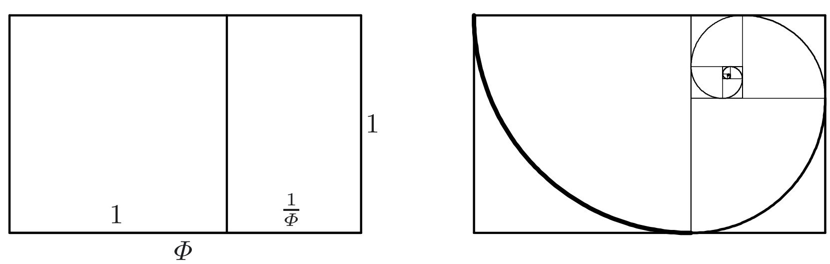


Fig. 1.26. A golden rectangle and its subdivisions

6. Find the error in the "proof" presented in Fig. 1.27, where different arrangements of identical pieces suggest that $273 = 272$.

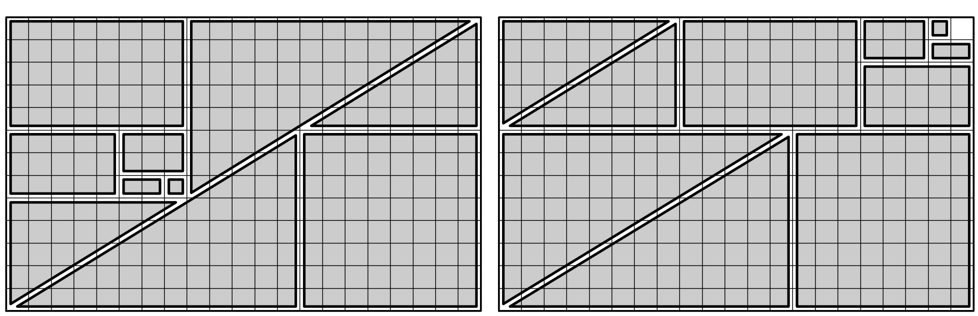
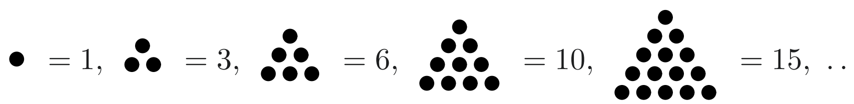


Fig. 1.27. A curious proof that $273 = 272$

7. The Ancients were skilled at extracting square roots as this was necessary for applying Pythagoras' theorem (see also Exercise 22 of Chap. 2). Rediscover the method which (probably) allowed the Babylonians, nearly 4000 years ago, to find the excellent value $1, 24\ 51\ 10$ in base 60 for $\sqrt{2}$.

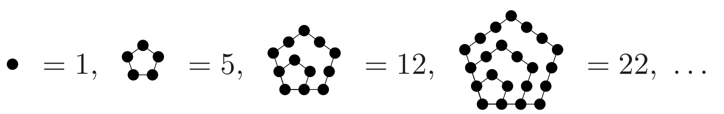
   Hint. On another Babylonian tablet, which lists squares of numbers, you will discover that a square of sides $1, 25$ has an area very close to 2, because $(1, 25)^2 = 2, 00\ 25$ in base 60. Cut two strips of width δ from this square in order to reduce the area to 2.

8. Triangular arrangements of dots of the form

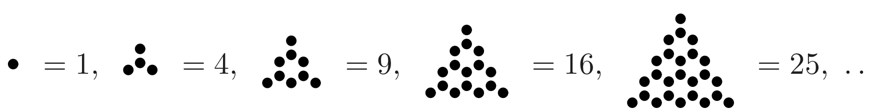


   were sacred figures for the Pythagoreans, especially the *holy tetractys* with 10 dots, by which the Pythagoreans used to swear. Find a general expression for t_n, the number of dots of the n-th figure.

9. Find a general formula for the *pentagonal numbers*



10. (Inspired by a picture of Eugen Jost, 2010.) Guess a formula for the number of dots forming an equilateral triangle on a hexagonal grid



    and explain the result.

11. Glue the drawings of Fig. 1.28 onto some cardboard (or make a Xerox copy if you want to preserve this beautiful book undamaged). Carefully cut out the pieces to obtain two jig-saw puzzles that allow one to *grasp* (literally) a 2500-year-old theorem. Which theorem is this?

12. Explain another version of Euclid's proof of Pythagoras' theorem (see Fig. 1.29, left and middle pictures): Produce ZH and $K\Theta$ to find a point Π such that Π, A, Λ are collinear and $\Pi A = B\Delta$ (why?). Move the area a^2 first upwards parallel to ZH and then downwards parallel to ΠA.

13. (A discovery of Heron.) Show that in Euclid's figure for the proof of the Pythagorean theorem the lines ΓZ, BK and $A\Lambda$ are concurrent, see Fig. 1.29 (right).

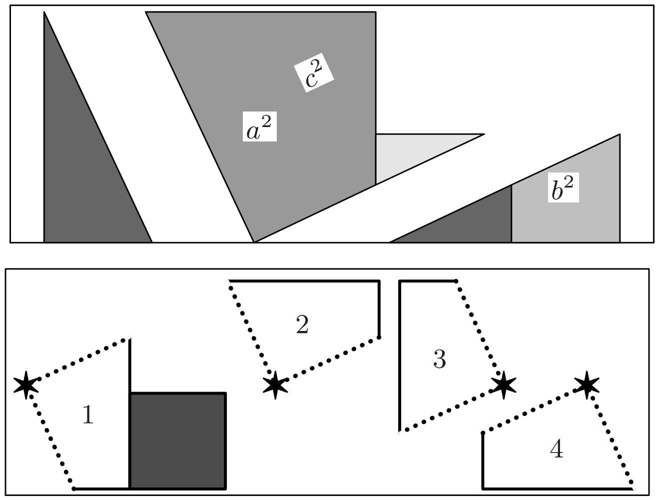


Fig. 1.28. Two jig-saw puzzles of high educational value

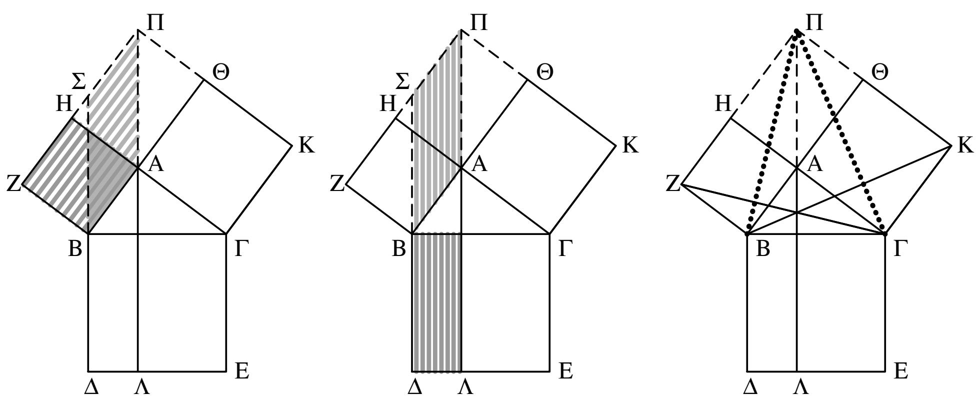


Fig. 1.29. Another proof of Pythagoras' theorem (left); Heron's discovery about Euclid's figure (right)

14. Given an angle AOB with vertex O and a point P inside the angle, construct perpendiculars PA, PB, and OC, PD, see Fig. 1.30 (left). Then show that $AC = BD$ (Hartshorne, 2000, p. 62).

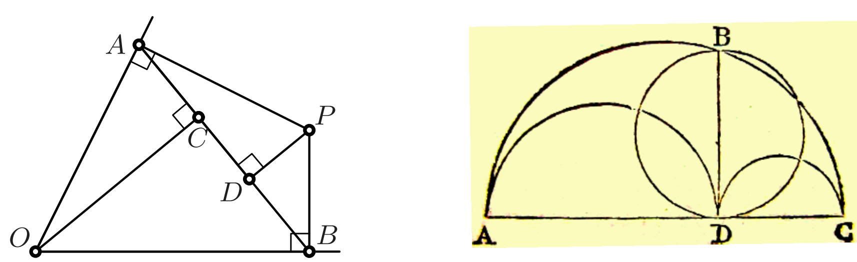


Fig. 1.30. Diagonal in a particular quadrilateral (left); Archimedes' Lemma (right; copied from Peyrard's edition of Archimedes' *Opera*, vol. 2, Paris 1808)

15. Prove one of Archimedes' Lemmata (see Fig. 1.30, right): *The area of the moon-like region bounded by the semicircles AC, CD and DA is equal to the area of the circle with diameter BD.*

16. A young couple, to celebrate their golden wedding, set up a tent whose base is a square of side length the golden ratio Φ (what else), held up by 5 tubes of length 1, see Fig. 1.31. Show that the polygons AEB, $BEFC$, CFD and $DFEA$ are parts of a regular pentagon. Further, show that the angles of the faces AEB and $BCEF$ with the base add up to a right angle ⊾. With these two results we at once understand the construction of the dodecahedron (Eucl. XIII.17) by attaching six of these tents to a *golden* cube, see Fig. 2.37 in Sect. 2.6.

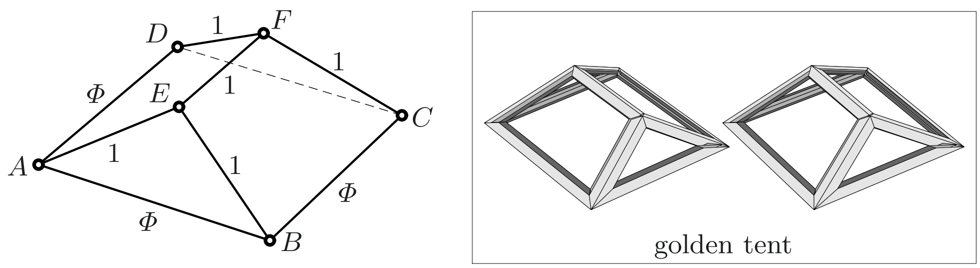
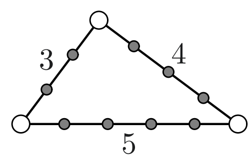


Fig. 1.31. The golden tent

17. (Pythagorean triples.) Find (all) right-angled triangles with all sides of integer length.



18. Show that

$$x = \frac{1-u^2}{1+u^2}, \quad y = \frac{2u}{1+u^2}, \quad u \in \mathbb{Q} \tag{1.13}$$

represent all points with rational coordinates on the unit circle, except $(-1, 0)$ (which corresponds to $u = \infty$).

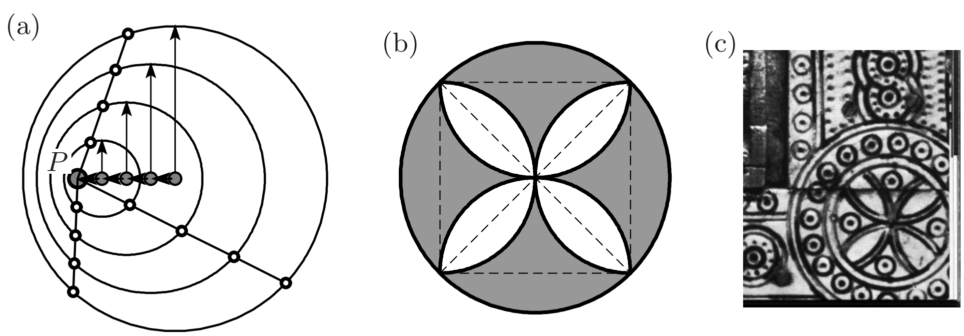


Fig. 1.32. Geneva duck theorem (a); ornamental figure (b); ornament from a reliquary casket, 8th century, Abbey church of Saint Ludger, Essen-Werden (c)

19. Prove the famous "Geneva duck theorem": A duck moves on the Lake of Geneva at constant speed towards a point P and creates circles at a constant rate (see Fig. 1.32 (a)). Prove (with Thales) that any half-line through P is cut by the circles into intervals of the same length (the situation is slightly more complicated if the movement is "supersonic").

20. An ancient ornamental figure (see Figs. 1.32 (b) and (c)) consists of a circle (which we take of radius 1) from which a cross is cut out. The cross is bordered by eight circular arcs which are either tangent to each other or cut orthogonally at the centre. Find the area of the part shaded in grey.

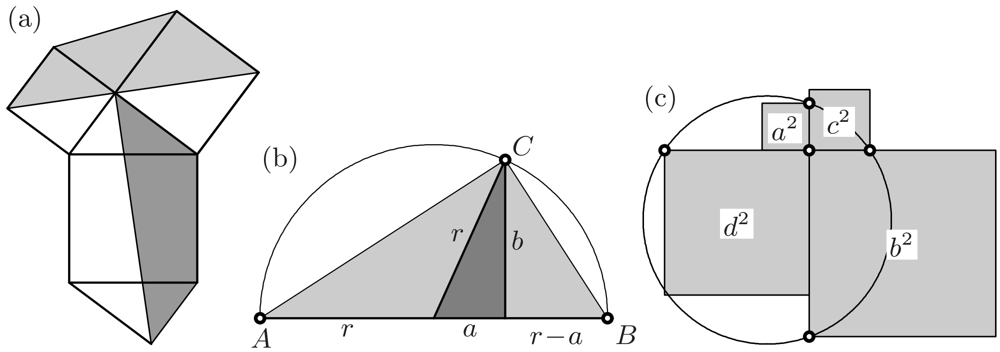


Fig. 1.33. Leonardo's proof (a); altitude theorem (b); four squares (c)

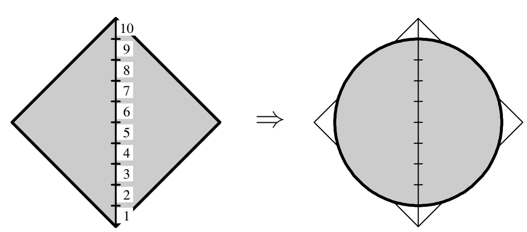
21. Explain Leonardo da Vinci's proof (see Fig. 1.33 (a)) of Pythagoras' theorem, which is striking by — its beauty!

22. Use Fig. 1.33 (b) to deduce the altitude theorem (1.10) for triangle ABC from Pythagoras' theorem for the small dark triangle, and conversely. (This will be Eucl. II.14 in the next chapter.)

23. Solve a "beau problème de géométrie", inspired by a serigraph of Max Bill (1908–1994) and communicated to the authors by P. Zabey, Geneva: Let $ABCD$ be a square whose side length is taken as 1. Let E be the midpoint of BC. Construct a square $EFGH$ such that D is the midpoint of FG. This creates six triangles whose angles and areas are requested.

24. The oldest theorems of humanity in this chapter provide nice discoveries even now in the 21st century. Prove the following result, due to Nelsen (2004): If two chords of a circle intersect at right angles forming four segments a, b, c, d, then $a^2+b^2+c^2+d^2 = D^2$, where D is the diameter of the circle (see Fig. 1.33 (c)).

25. Analyse Dürer's *circling of the square* (Underweysung, book 2) and its error for π.



2

The Elements of Euclid

"At age eleven, I began Euclid, with my brother as my tutor. This was one of the greatest events of my life, as dazzling as first love. I had not imagined that there was anything as delicious in the world." (B. Russell, quoted from K. Hoechsmann, *Editorial, π in the Sky*, Issue 9, Dec. 2005. A few paragraphs later K. H. added: An innocent look at a page of contemporary theorems is no doubt less likely to evoke feelings of "first love".)

"At the age of 16, Abel's genius suddenly became apparent. Mr. Holmboë, then professor in his school, gave him private lessons. Having quickly absorbed the *Elements*, he went through the *Introductio* and the *Institutiones calculi differentialis* and *integralis* of Euler. From here on, he progressed alone."
(Obituary for Abel by Crelle, *J. Reine Angew. Math.* 4 (1829) p. 402; transl. from the French)

"The year 1868 must be characterised as [Sophus Lie's] breakthrough year. ... as early as January, he borrowed [from the University Library] Euclid's major work, *The Elements* ..." (*The Mathematician Sophus Lie* by A. Stubhaug, Springer 2002, p. 102)

"There never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures ... from the plan laid down by Euclid."
(A. De Morgan 1848; copied from the *Preface* of Heath, 1926)

"Die Lehrart, die man schon in dem ältesten auf unsere Zeit gekommenen Lehrbuche der Mathematik (den Elementen des Euklides) antrifft, hat einen so hohen Grad der Vollkommenheit, dass sie von jeher ein Gegenstand der Bewunderung [war] ... [The style of teaching, which we already encounter in the oldest mathematical textbook that has survived (the *Elements* of Euclid), has such a high degree of perfection that it has always been the object of great admiration ...]" (B. Bolzano, *Grössenlehre*, p. 18r, 1848)

A. Ostermann and G. Wanner, *Geometry by Its History,*
Undergraduate Texts in Mathematics, DOI: 10.1007/978-3-642-29163-0_2,