Chapter 7

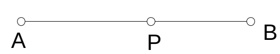
The Menelaus and Ceva Theorems

7.1

7.1.1 Sign convention

Let A and B be two distinct points. A point P on the line AB is said to divide the segment AB in the ratio $AP : PB$, positive if P is between A and B, and negative if P is outside the segment AB.

-1 < AP/PB < 0.



AP/PB > 0.

AP/PB < -1.

7.1.2 Harmonic conjugates

Two points P and Q on a line AB are said to divide the segment AB *harmonically* if they divide the segment in the same ratio, one externally and the other internally:

$$\frac{AP}{PB} = -\frac{AQ}{QB}.$$

We shall also say that P and Q are *harmonic conjugates* with respect to the segment AB.

7.1.3

Let P and Q be harmonic conjugates with respect to AB. If $AB = d$, $AP = p$, and $AQ = q$, then d is the harmonic mean of p and q, namely,

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{d}.$$

Proof. This follows from

$$\frac{p}{d-p} = -\frac{q}{d-q}.$$

7.1.4

We shall use the abbreviation $(A, B; P, Q)$ to stand for the statement P, Q divide the segment AB harmonically.

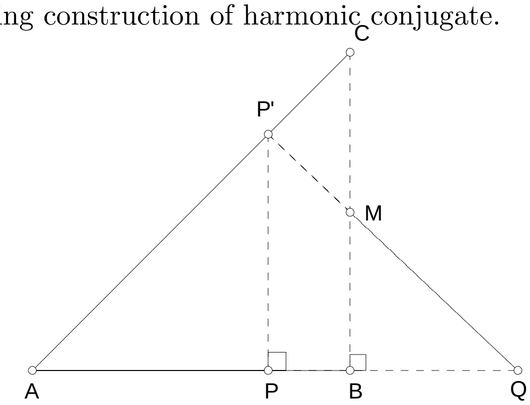
Proposition

If $(A, B; P, Q)$, then $(A, B; Q, P)$, $(B, A; P, Q)$, and $(P, Q; A, B)$.

Therefore, we can speak of two collinear (undirected) segments dividing each other harmonically.

Exercise

1. Justify the following construction of harmonic conjugate.



Given AB, construct a right triangle ABC with a right angle at B and $BC = AB$. Let M be the midpoint of BC.

For every point P (except the midpoint of AB), let P' be the point on AC such that $PP' \perp AB$.

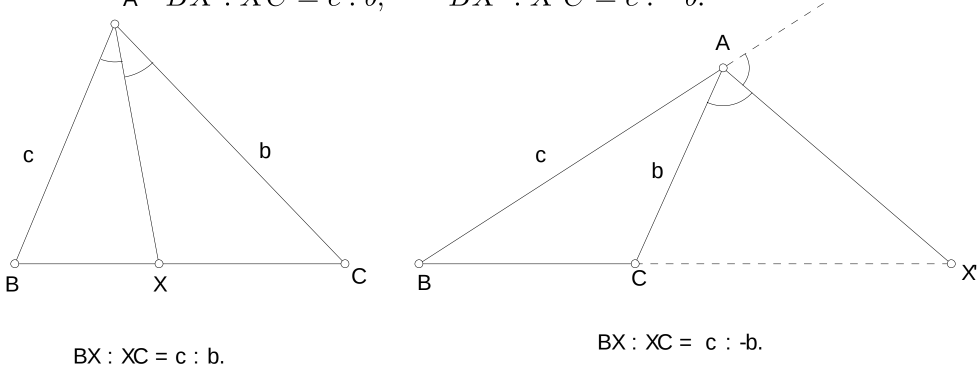
The intersection Q of the lines $P'M$ and AB is the harmonic conjugate of P with respect to AB.

7.2 Apollonius Circle

7.2.1 Angle bisector Theorem

If the internal (repsectively external) bisector of angle BAC intersect the line BC at X (respectively X'), then

$$BX : XC = c : b; \qquad BX' : X'C = c : -b.$$



7.2.2 Example

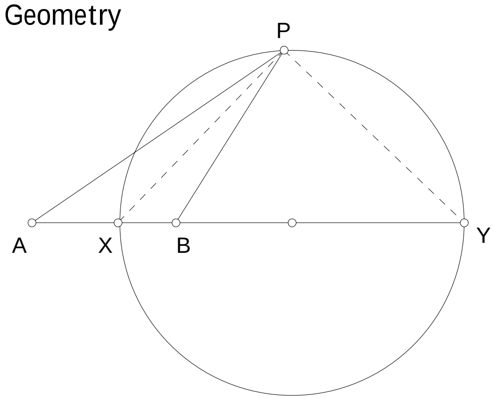
The points X and X' are harmonic conjugates with respect to BC, since

$$BX : XC = c : b, \quad \text{and} \quad BX' : X'C = c : -b.$$

7.2.3

A and B are two fixed points. For a given positive number $k \neq 1$, [1] the locus of points P satisfying $AP : PB = k : 1$ is the circle with diameter XY, where X and Y are points on the line AB such that $AX : XB = k : 1$ and $AY : YB = k : -1$.

[1] If $k - 1$, the locus is clearly the perpendicular bisector of the segment AB.



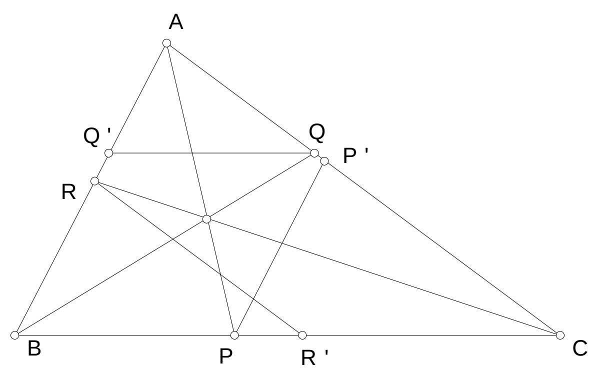
Proof. Since $k \neq 1$, points X and Y can be found on the line AB satisfying the above conditions.

Consider a point P not on the line AB with $AP : PB = k : 1$. Note that PX and PY are respectively the internal and external bisectors of angle APB. This means that angle XPY is a right angle.

Exercise

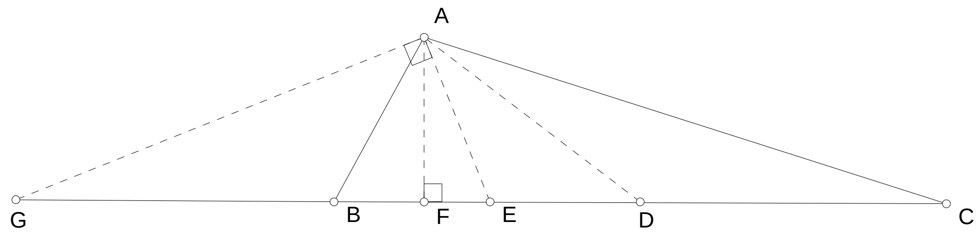
1. The bisectors of the angles intersect the sides BC, CA, AB respectively at P, Q, and R. P', Q', and R' on the sides CA, AB, and BC respectivley such that $PP'//BC$, $QQ'//CA$, and $RR'//AB$. Show that

$$\frac{1}{PP'} + \frac{1}{QQ'} + \frac{1}{RR'} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$



2. Suppose ABC is a triangle with $AB \neq AC$, and let D, E, F, G be points on the line BC defined as follows: D is the midpoint of BC, AE is the bisector of $\angle BAC$, F is the foot of the perpeandicular from

A to BC, and AG is perpendicular to AE (i.e. AG bisects one of the exterior angles at A). Prove that $AB \cdot AC = DF \cdot EG$.



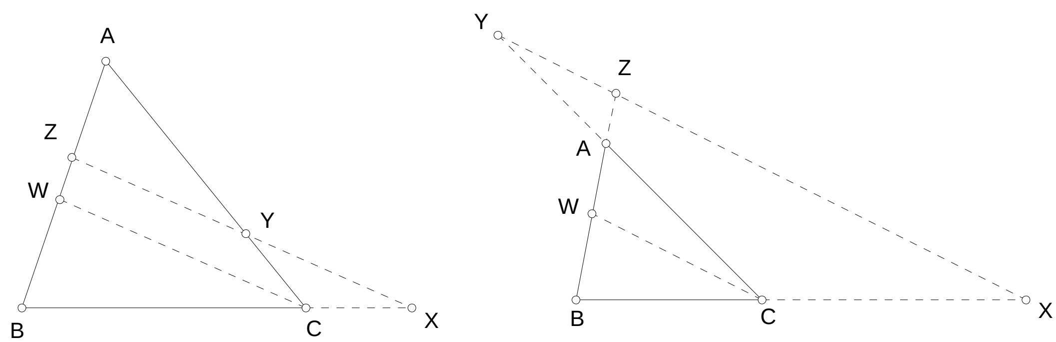
3. If $AB = d$, and $k \neq 1$, the radius of the Apollonius circle is $\frac{k}{k^2-1}d$.

4. Given two disjoint circles (A) and (B), find the locus of the point P such that the angle between the pair of tangents from P to (A) and that between the pair of tangents from P to (B) are equal. [2]

7.3 The Menelaus Theorem

Let X, Y, Z be points on the lines BC, CA, AB respectively. The points X, Y, Z are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$



[2]Let a and b be the radii of the circles. Suppose each of these angles is 2θ. Then $\frac{a}{AP} = \sin\theta = \frac{b}{BP}$, and $AP : BP = a : b$. From this, it is clear that the locus of P is the circle with the segment joining the centers of similitude of (A) and (B) as diameter.

Proof. ($\Longrightarrow$) Let W be the point on AB such that $CW//XY$. Then,

$$\frac{BX}{XC} = \frac{BZ}{ZW}, \quad \text{and} \quad \frac{CY}{YA} = \frac{WZ}{ZA}.$$

It follows that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{BZ}{ZW} \cdot \frac{WZ}{ZA} \cdot \frac{AZ}{ZB} = \frac{BZ}{ZB} \cdot \frac{WZ}{ZW} \cdot \frac{AZ}{ZA} = (-1)(-1)(-1) = -1.$$

($\Longleftarrow$) Suppose the line joining X and Z intersects AC at Y'. From above,

$$\frac{BX}{XC} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} = -1 = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}.$$

It follows that

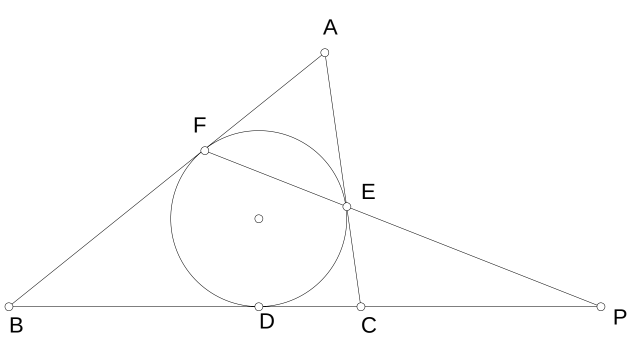
$$\frac{CY'}{Y'A} = \frac{CY}{YA}.$$

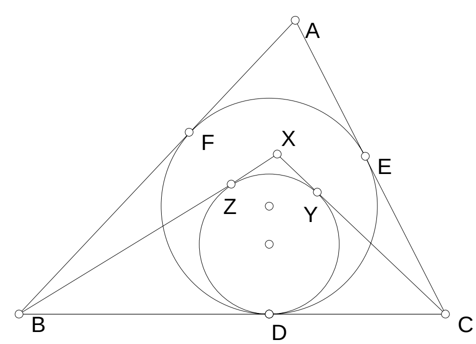
The points Y' and Y divide the segment CA in the same ratio. These must be the same point, and X, Y, Z are collinear.

Exercise

1. M is a point on the median AD of $\triangle ABC$ such that $AM : MD = p : q$. The line CM intersects the side AB at N. Find the ratio $AN : NB$. [3]

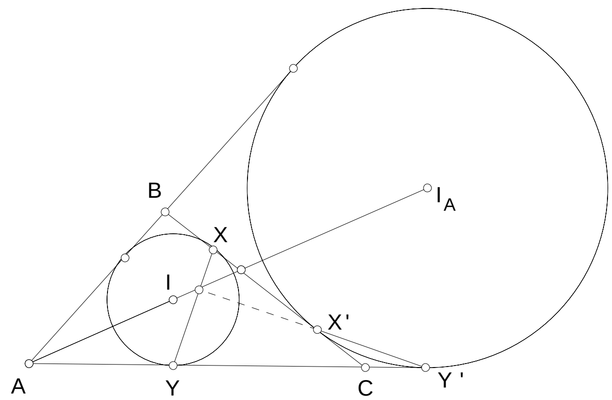
2. The incircle of $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. Suppose $AB \neq AC$. The line joining E and F meets BC at P. Show that P and D divide BC harmonically.





[3]Answer: $AN : NB = p : 2q$.

3. The incircle of $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D also, and touches CX and XB at Y and Z respectively. Show that E, F, Z, Y are concyclic. [4]



4. Given a triangle ABC, let the incircle and the ex-circle on BC touch the side BC at X and X' respectively, and the line AC at Y and Y' respectively. Then the lines XY and $X'Y'$ intersect on the bisector of angle A, at the projection of B on this bisector.

7.4 The Ceva Theorem

Let X, Y, Z be points on the lines BC, CA, AB respectively. The lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$

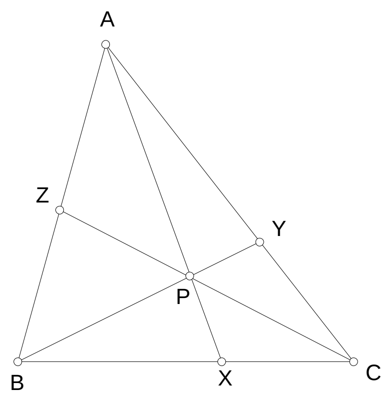
Proof. ($\Longrightarrow$) Suppose the lines AX, BY, CZ intersect at a point P. Consider the line BPY cutting the sides of $\triangle CAX$. By Menelaus' theorem,

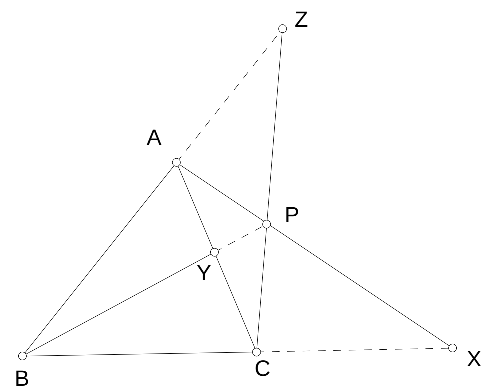
$$\frac{CY}{YA} \cdot \frac{AP}{PX} \cdot \frac{XB}{BC} = -1, \quad \text{or} \quad \frac{CY}{YA} \cdot \frac{PA}{XP} \cdot \frac{BX}{BC} = +1.$$

[4]IMO 1996.

Also, consider the line CPZ cutting the sides of $\triangle ABX$. By Menelaus' theorem again,

$$\frac{AZ}{ZB}\cdot\frac{BC}{CX}\cdot\frac{XP}{PA}=-1,\quad\text{or}\quad\frac{AZ}{ZB}\cdot\frac{BC}{XC}\cdot\frac{XP}{PA}=+1.$$





Multiplying the two equations together, we have

$$\frac{CY}{YA}\cdot\frac{AZ}{ZB}\cdot\frac{BX}{XC}=+1.$$

($\Longleftarrow$) Exercise.

7.5 Examples

7.5.1 The centroid

If D, E, F are the midpoints of the sides BC, CA, AB of $\triangle ABC$, then clearly

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1.$$

The medians AD, BE, CF are therefore concurrent (at the *centroid* G of the triangle).

Consider the line BGE intersecting the sides of $\triangle ADC$. By the Menelau theorem,

$$-1=\frac{AG}{GD}\cdot\frac{DB}{BC}\cdot\frac{CE}{EA}=\frac{AG}{GD}\cdot\frac{-1}{2}\cdot\frac{1}{1}.$$

It follows that $AG : GD = 2 : 1$. *The centroid of a triangle divides each median in the ratio 2:1.*

7.5.2 The incenter

Let X, Y, Z be points on BC, CA, AB such that

$$\begin{array}{ccc} AX & & \angle BAC \\ BY & \text{bisects} & \angle CBA, \\ CZ & & \angle ACB \end{array}$$

then

$$\frac{AZ}{ZB} = \frac{b}{a}, \quad \frac{BX}{XC} = \frac{c}{b}, \quad \frac{CY}{YA} = \frac{a}{c}.$$

It follows that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = +1,$$

and AX, BY, CZ are concurrent, at the *incenter* I of the triangle.

Exercise

1. Use the Ceva theorem to justify the existence of the excenters of a triangle.

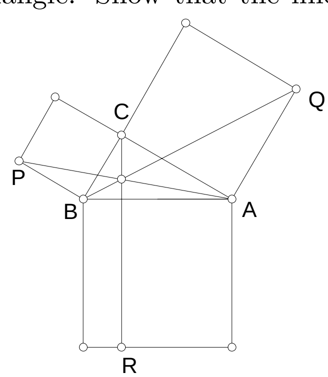
2. Let AX, BY, CZ be cevians of $\triangle ABC$ intersecting at a point P.

   (i) Show that if AX bisects angle A and $BX \cdot CY = XC \cdot BZ$, then $\triangle ABC$ is isosceles.

   (ii) Show if if AX, BY, CZ are bisectors and $BP \cdot ZP = BZ \cdot AP$, then $\triangle ABC$ is a right triangle.

3. Suppose three cevians, each through a vertex of a triangle, trisect each other. Show that these are the medians of the triangle.

4. ABC is a right triangle. Show that the lines AP, BQ, and CR are concurrent.



5. [5] If three equal cevians divide the sides of a triangle in the same ratio and the same sense, the triangle must be equilateral.

6. Suppose the bisector of angle A, the median on the side b, and the altitude on the side c are concurrent. Show that [6]

$$\cos\alpha = \frac{c}{b+c}.$$

7. Given triangle ABC, construct points A', B', C' such that ABC', BCA' and CAB' are isosceles triangles satisfying

$$\angle BCA' = \angle CBA' = \alpha, \quad \angle CAB' = \angle ACB' = \beta, \quad \angle ABC' = \angle BAC' = \gamma.$$

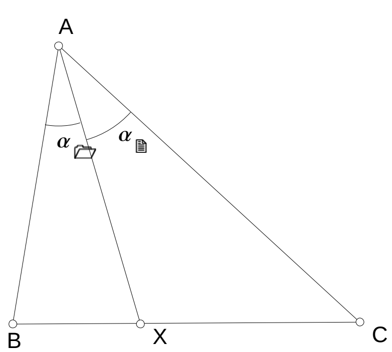
Show that AA', BB', and CC' are concurrent. [7]

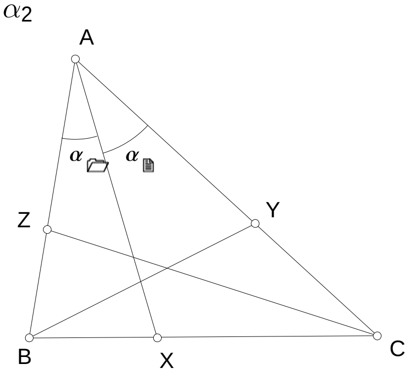
7.6 Trigonmetric version of the Ceva Theorem

7.6.1

Let X be a point on the side BC of triangle ABC such that the directed angles $\angle BAX = \alpha_1$ and $\angle XAC = \alpha_2$. Then

$$\frac{BX}{XC} = \frac{c}{b} \cdot \frac{\sin\alpha_1}{\sin\alpha_2}.$$





Proof. By the sine formula,

$$\frac{BX}{XC} = \frac{BX/AX}{XC/AX} = \frac{\sin\alpha_1/\sin\beta}{\sin\alpha_2/\sin\gamma} = \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin\alpha_1}{\sin\alpha_2} = \frac{c}{b} \cdot \frac{\sin\alpha_1}{\sin\alpha_2}.$$

[5]Klamkin
[6]AMME 263; CMJ 455.
[7]$A'B'C'$ is the tangential triangle of ABC.

7.6.2

Let X, Y, Z be points on the lines BC, CA, AB respectively. The lines AX, BY, CZ are concurrent if and only if

$$\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}=+1.$$

Proof. Analogous to

$$\frac{BX}{XC}=\frac{c}{b}\cdot\frac{\sin\alpha_1}{\sin\alpha_2}$$

are

$$\frac{CY}{YA}=\frac{a}{c}\cdot\frac{\sin\beta_1}{\sin\beta_2},\qquad \frac{AZ}{ZB}=\frac{b}{a}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}.$$

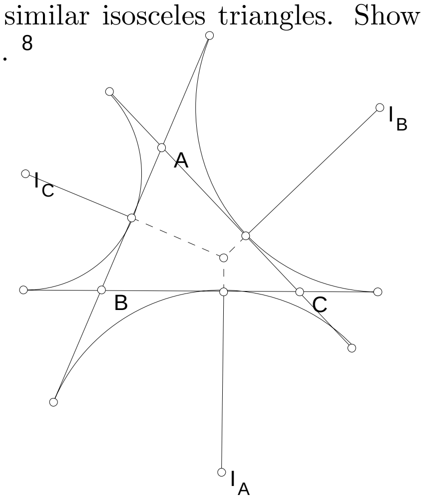
Multiplying the three equations together,

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YB}=\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}.$$

Exercise

1. Show that the three altitudes of a triangle are concurrent (at the *orthocenter* H of the triangle).

2. Let A', B', C' be points outside $\triangle ABC$ such that $A'BC$, $B'CA$ and $C'AB$ are similar isosceles triangles. Show that AA', BB', CC' are concurrent. [8]

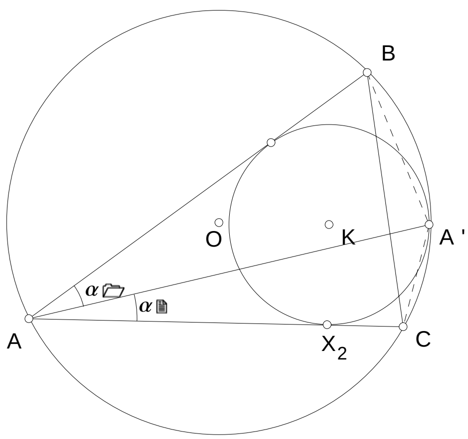


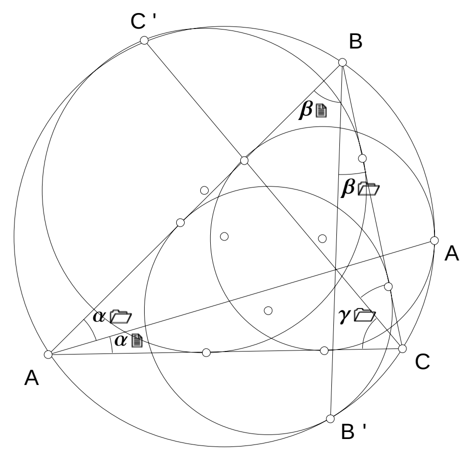
[8] Solution. Let X be the intersection of AA' and BC. Then $\frac{BX}{XC}=\frac{\sin(\beta+\omega)}{\sin(\gamma+\omega)}\cdot\frac{\sin\gamma}{\sin\beta}$.

3. Show that the perpendiculars from I_A to BC, from I_B to CA, and from I_C to AB are concurrent. [9]

7.7 Mixtilinear incircles

Suppose the mixtilinear incircles in angles A, B, C of triangle ABC touch the circumcircle respectively at the points A', B', C'. The segments AA', BB', and CC' are concurrent.





Proof. We examine how the mixtilinear incircle divides the minor arc BC of the circumcircle. Let A' be the point of contact. Denote $\alpha_1 := \angle A'AB$ and $\alpha_2 := \angle A'AC$. Note that the circumcenter O, and the points K, A' are collinear. In triangle KOC, we have

$$OK = R - \rho_1, \qquad OC = R, \qquad \angle KOC = 2\alpha_2,$$

where R is the circumradius of triangle ABC. Note that $CX_2 = \frac{b(s-c)}{s}$, and $KC^2 = \rho_1^2 + CX_2^2$. Applying the cosine formula to triangle KOC, we have

$$2R(R - \rho_1)\cos 2\alpha_2 = (R - \rho_1)^2 + R^2 - \rho_1^2 - \left(\frac{b(s-c)}{s}\right)^2.$$

Since $\cos 2\alpha_2 = 1 - 2\sin^2\alpha_2$, we obtain, after rearrangement of the terms,

$$\sin\alpha_2 = \frac{b(s-c)}{s} \cdot \frac{1}{\sqrt{2R(R-\rho_1)}}.$$

[9] Consider these as cevians of triangle $I_A I_B I_C$.

Similarly, we obtain

$$\sin \alpha_1 = \frac{c(s-b)}{s} \cdot \frac{1}{\sqrt{2R(R-\rho)}}.$$

It follows that

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{c(s-b)}{b(s-c)}.$$

If we denote by B' and C' the points of contact of the circumcircle with the mixtilinear incircles in angles B and C respectively, each of these divides the respective minor arcs into the ratios

$$\frac{\sin \beta_1}{\sin \beta_2} = \frac{a(s-c)}{c(s-a)}, \qquad \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{b(s-a)}{a(s-b)}.$$

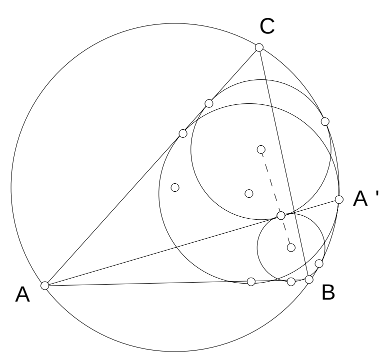
From these,

$$\frac{\sin \alpha_1}{\sin \alpha_2} \cdot \frac{\sin \beta_1}{\sin \beta_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{a(s-c)}{c(s-a)} \cdot \frac{b(s-a)}{a(s-b)} \cdot \frac{c(s-b)}{b(s-c)} = +1.$$

By the Ceva theorem, the segments AA', BB' and CC' are concurrent.

Exercise

1. The mixtilinear incircle in angle A of triangle ABC touches its circumcircle at A'. Show that AA' is a common tangent of the mixtilinear incircles of angle A in triangle $AA'B$ and of angle A in triangle $AA'C$. [10]



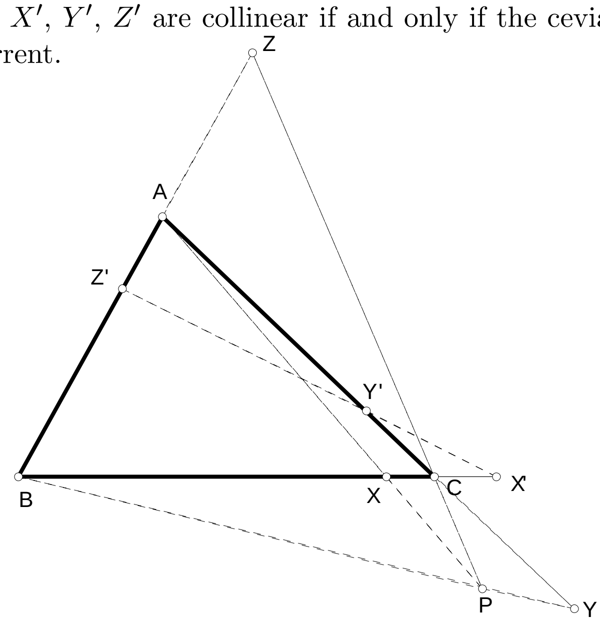
[10] Problem proposal to Crux Mathematicorum.

7.8 Duality

Given a triangle ABC, let

$$\begin{array}{lll} X, & X' & & BC \\ Y, & Y' & \text{be harmonic conjugates with respect to the side} & CA. \\ Z, & Z' & & AB \end{array}$$

The points X', Y', Z' are collinear if and only if the cevians AX, BY, CZ are concurrent.



Proof. By assumption,

$$\frac{BX'}{X'C} = -\frac{BX}{XC}, \quad \frac{CY'}{Y'A} = -\frac{CY}{YA}, \quad \frac{AZ'}{Z'B} = -\frac{AZ}{ZB}.$$

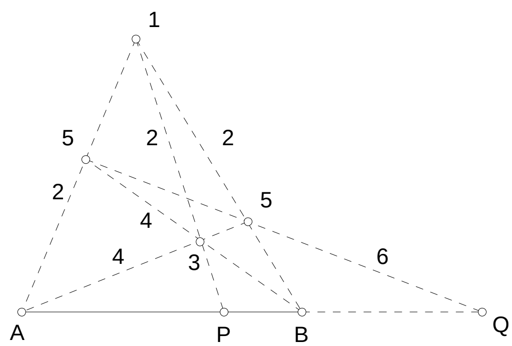
It follows that

$$\frac{BX'}{X'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B} = -1 \quad \text{if and only if} \quad \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$

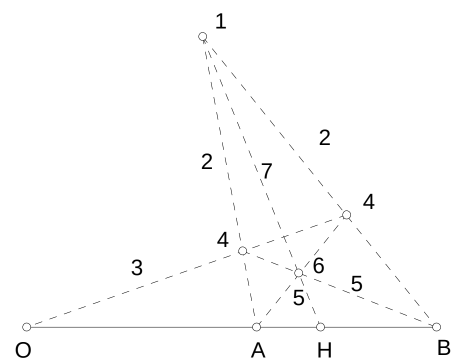
The result now follows from the Menelaus and Ceva theorems.

7.8.1 Ruler construction of harmonic conjugate

Given two points A and B, the harmonic conjugate of a point P can be constructed as follows. Choose a point C outside the line AB. Draw the lines CA, CB, and CP. Through P draw a line intersecting CA at Y and CB at X. Let Z be the intersection of the lines AX and BY. Finally, let Q be the intersection of CZ with AB. Q is the harmonic conjugate of P with respect to A and B.



Harmonic conjugate



harmonic mean

7.8.2 Harmonic mean

Let O, A, B be three collinear points such that $OA = a$ and $OB = b$. If H is the point on the same ray OA such that $h = OH$ is the harmonic mean of a and b, then $(O, H; A, B)$. Since this also means that $(A, B; O, H)$, the point H is the harmonic conjugate of O with respect to the segment AB.

7.9 Triangles in perspective

7.9.1 Desargues Theorem

Given two triangles ABC and $A'B'C'$, the lines AA', BB', CC' are concurrent if and only if the intersections of the pairs of lines $\begin{array}{l} AB, A'B' \\ BC, B'C' \\ CA, C'A' \end{array}$ are collinear.

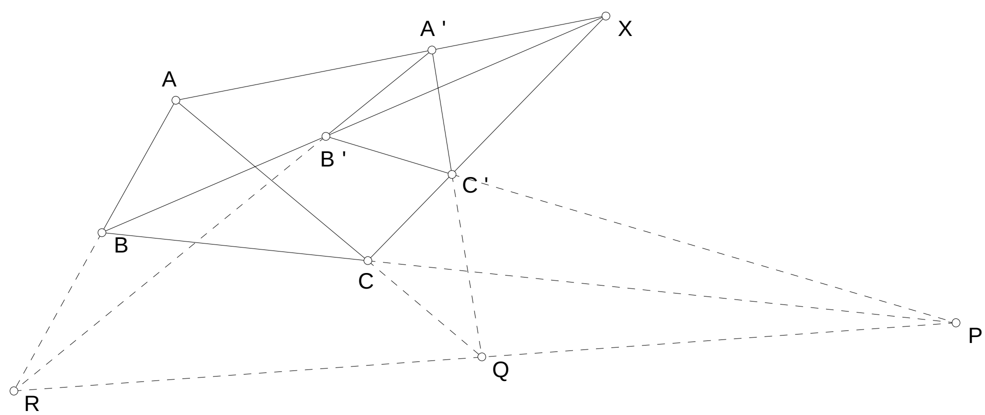
Proof. Suppose AA', BB', CC' intersect at a point X. Applying Menelaus' theorem to the triangle $\begin{array}{l} XAB \\ XBC \\ XCA \end{array}$ and transversal $\begin{array}{l} A'B'R \\ B'C'P \\ C'A'Q \end{array}$, we have

$$\frac{XA'}{A'A}\cdot\frac{AR}{RB}\cdot\frac{BB'}{B'X}=-1,\quad \frac{XB'}{B'B}\cdot\frac{BP}{PC}\cdot\frac{CC'}{C'X}=-1,\quad \frac{XC'}{C'C}\cdot\frac{CQ}{QA}\cdot\frac{AA'}{A'X}=-1.$$

Multiplying these three equation together, we obtain

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1.$$

By Menelaus' theorem again, the points P, Q, R are concurrent.

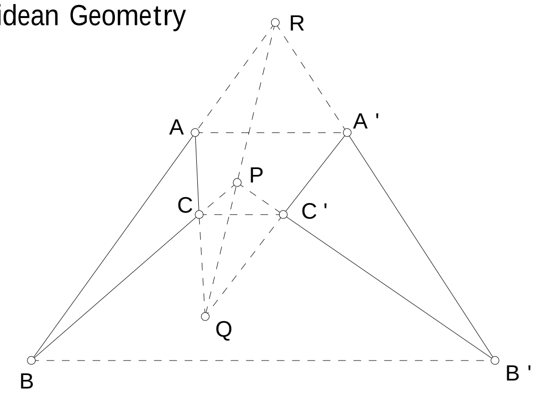


7.9.2

Two triangles satisfying the conditions of the preceding theorem are said to be *perspective*. X is the center of perspectivity, and the line PQR the axis of perspectivity.

7.9.3

Given two triangles ABC and $A'B'C'$, if the lines AA', BB', CC' are parallel, then the intersections of the pairs of lines $\begin{array}{ll} AB & A'B' \\ BC & B'C' \\ CA & C'A' \end{array}$ are collinear.



Proof.

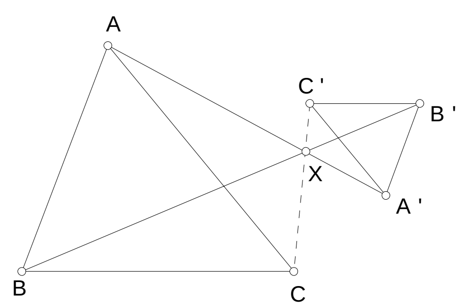
$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = \left(-\frac{BB'}{CC'}\right)\left(-\frac{CC'}{AA'}\right)\left(-\frac{AA'}{BB'}\right) = -1.$$

7.9.4

If the correpsonding sides of two triangles are pairwise parallel, then the lines joining the corresponding vertices are concurrent.
Proof. Let X be the intersection of BB' and CC'. Then

$$\frac{CX}{XC'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

The intersection of AA' and CC' therefore coincides with X.



7.9.5

Two triangles whose sides are parallel in pairs are said to be homothetic. The intersection of the lines joining the corresponding vertices is the homothetic center. Distances of corresponding points to the homothetic center are in the same ratio as the lengths of corresponding sides of the triangles.