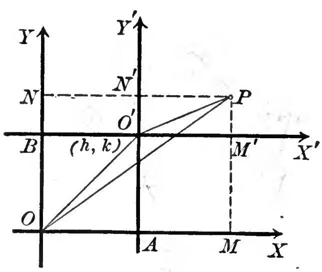
CHAPTER VII

TRANSFORMATION OF COÖRDINATES

60. When we are at liberty to choose the axes as we please we generally choose them so that our results shall have the simplest possible form. When the axes are given it is important that we be able to find the equation of a given curve referred to some other axes. The operation of changing from one pair of axes to a second pair is known as a **transformation of coördinates.** We regard the axes as moved from their given position to a new position and we seek formulas which express the old coördinates in terms of the new coördinates.

61. Translation of the axes. If the axes be moved from a first position OX and OY to a second position $O'X'$ and $O'Y'$ such that $O'X'$ and $O'Y'$ are respectively parallel to OX and OY, then the axes are said to be **translated** from the first to the second position.

Let the new origin be $O'(h, k)$ and let the coördinates of any point P before and after the translation be respectively (x, y) and (x', y'). Projecting OP and $OO'P$ on OX, we obtain (Theorem XI, p. 41)



$$x = x' + h.$$

Similarly, $y = y' + k$.

Hence,

Theorem I. *If the axes be translated to a new origin (h, k), and if (x, y) and (x', y') are respectively the coördinates of any point P before and after the translation, then*

$$\text{(I)} \qquad \begin{cases} x = x' + h, \\ y = y' + k. \end{cases}$$

Equations (I) are called the **equations for translating the axes.** To find the equation of a curve referred to the new axes when its equation referred to the old axes is given, we substitute the values of x and y given by (I) in the given equation. For the given equation expresses the fact that $P(x, y)$ lies on the given curve, and since equations (I) are true for *all* values of (x, y), the new equation gives a relation between x' and y' which expresses that $P(x', y')$ lies on the curve and is therefore (p. 46) the equation of the curve in the new coördinates.

Ex. 1. Transform the equation

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

when the axes are translated to the new origin $(3, -2)$.

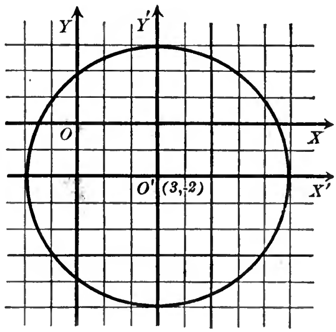
Solution. Here $h = 3$ and $k = -2$, so equations (I) become

$$x = x' + 3,\ y = y' - 2.$$

Substituting in the given equation, we obtain

$$(x' + 3)^2 + (y' - 2)^2 - 6(x' + 3) + 4(y' - 2) - 12 = 0,$$

or, reducing, $x'^2 + y'^2 = 25.$



This result could easily be foreseen. For the locus of the given equation is (Theorem I, p. 116) a circle whose center is $(3, -2)$ and whose radius is 5. When the origin is translated to the center the equation of the circle must necessarily have the form obtained (Corollary, p. 51).

PROBLEMS

1. Find the new coördinates of the points $(3, -5)$ and $(-4, 2)$ when the axes are translated to the new origin $(3, 6)$.

2. Transform the following equations when the axes are translated to the new origin indicated and plot both pairs of axes and the curve.

(a) $3x - 4y = 6$, $(2, 0)$. *Ans.* $3x' - 4y' = 0.$

(b) $x^2 + y^2 - 4x - 2y = 0$, $(2, 1)$. *Ans.* $x'^2 + y'^2 = 5.$

(c) $y^2 - 6x + 9 = 0$, $(\frac{3}{2}, 0)$. *Ans.* $y'^2 = 6x'.$

(d) $x^2 + y^2 - 1 = 0$, $(-3, -2)$. *Ans.* $x'^2 + y'^2 - 6x' - 4y' + 12 = 0.$

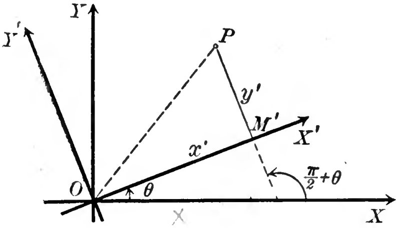
(e) $y^2 - 2kx + k^2 = 0$, $\left(\frac{k}{2}, 0\right)$. *Ans.* $y'^2 = 2kx'.$

(f) $x^2 - 4y^2 + 8x + 24y - 20 = 0$, $(-4, 3)$. *Ans.* $x'^2 - 4y'^2 = 0.$

3. Derive equations (I) if O' is in (a) the second quadrant; (b) the third quadrant; (c) the fourth quadrant.

62. Rotation of the axes. Let the axes OX and OY be rotated about O through an angle θ to the positions OX' and OY'. The equations giving the coördinates of any point referred to OX and OY in terms of its coördinates referred to OX' and OY' are called the **equations for rotating the axes.**

Theorem II. *The equations for rotating the axes through an angle θ are*



$$\text{(II)} \quad \begin{cases} x = x'\cos\theta - y'\sin\theta, \\ y = x'\sin\theta + y'\cos\theta. \end{cases}$$

Proof. Let P be any point whose old and new coördinates are respectively (x, y) and (x', y'). Draw OP and draw PM' perpendicular to OX'. Project OP and $OM'P$ on OX.

The proj. of OP on $OX = x$. (Theorem III, p. 24)

The proj. of OM' on $OX = x'\cos\theta$. (Theorem II, p. 23)

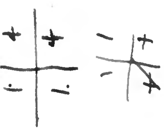
The proj. of $M'P$ on $OX = y'\cos\left(\frac{\pi}{2}+\theta\right)$. (Theorem II, p. 23)

$= - y'\sin\theta$. (by 6, p. 13)

Hence (Theorem XI, p. 41)

$$x = x'\cos\theta - y'\sin\theta.$$

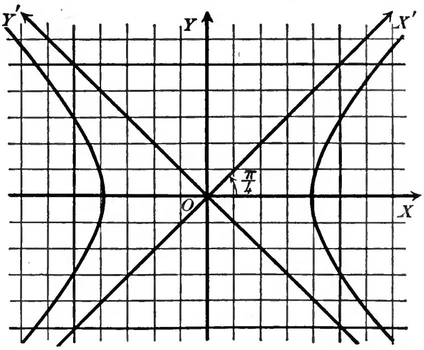
In like manner, projecting OP and $OM'P$ on OY, we obtain



$$y = x'\cos\left(\frac{\pi}{2}-\theta\right) + y'\cos\theta$$
$$= x'\sin\theta + y'\cos\theta. \qquad \text{Q.E.D.}$$

If the equation of a curve in x and y is given, we substitute from (II) in order to find the equation of the same curve referred to OX' and OY'.

Ex. 1. Transform the equation $x^2 - y^2 = 16$ when the axes are rotated through $\frac{\pi}{4}$.



Solution. Since

$$\sin\frac{\pi}{4} = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$$

and $\quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$

equations (II) become

$$x = \frac{x' - y'}{\sqrt{2}},\quad y = \frac{x' + y'}{\sqrt{2}}.$$

Substituting in the given equation, we obtain

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 16,$$

or, simplifying, $\qquad x'y' + 8 = 0.$

PROBLEMS

1. Find the coördinates of the points (3, 1), (– 2, 6), and (4, – 1) when the axes are rotated through $\frac{\pi}{2}$.

2. Transform the following equations when the axes are rotated through the indicated angle. Plot both pairs of axes and the curve.

(a) $x - y = 0,\ \frac{\pi}{4}$. *Ans.* $y' = 0$.

(b) $x^2 + 2xy + y^2 = 8,\ \frac{\pi}{4}$. *Ans.* $x'^2 = 4$.

(c) $y^2 = 4x,\ -\frac{\pi}{2}$. *Ans.* $x'^2 = 4y'$.

(d) $x^2 + 4xy + y^2 = 16,\ \frac{\pi}{4}$. *Ans.* $3x'^2 - y'^2 = 16$.

(e) $x^2 + y^2 = r^2,\ \theta$. *Ans.* $x'^2 + y'^2 = r^2$.

(f) $x^2 + 2xy + y^2 + 4x - 4y = 0,\ -\frac{\pi}{4}$. *Ans.* $\sqrt{2}y'^2 + 4x' = 0$.

3. Derive equations (II) if θ is obtuse.

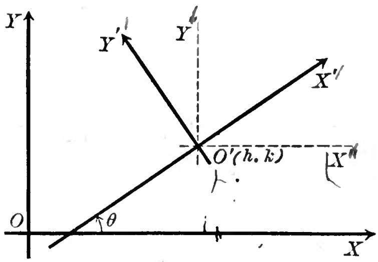
63. General transformation of coördinates. If the axes are moved in any manner, they may be brought from the old position to the new position by translating them to the new origin and then rotating them through the proper angle.

Theorem III. *If the axes be translated to a new origin (h, k) and then rotated through an angle θ, the equations of the transformation of coördinates are*

$$\text{(III)} \qquad \begin{cases} x = x' \cos\theta - y' \sin\theta + h, \\ y = x' \sin\theta + y' \cos\theta + k. \end{cases}$$

Proof. To translate the axes to $O'X''$ and $O'Y''$ we have, by (I),

$$x = x'' + h,$$
$$y = y'' + k,$$



where (x'', y'') are the coördinates of any point P referred to $O'X''$ and $O'Y''$.

To rotate the axes we set, by (II),

$$x'' = x'\cos\theta - y'\sin\theta,$$
$$y'' = x'\sin\theta + y'\cos\theta.$$

Substituting these values of x'' and y'', we obtain (III). Q.E.D.

64. Classification of loci. The loci of algebraic equations (p. 10) are classified according to the *degree* of the equations. This classification is justified by the following theorem, which shows that the degree of the equation of a locus is the same no matter how the axes are chosen.

Theorem IV. *The degree of the equation of a locus is unchanged by a transformation of coördinates.*

Proof. Since equations (III) are of the first degree in x' and y', the degree of an equation cannot be *raised* when the values of x and y given by (III) are substituted. Neither can the degree be *lowered;* for then the degree must be raised if we transform back to the old axes, and we have seen that it cannot be raised by changing the axes.*

As the degree can neither be raised nor lowered by a transformation of coördinates, it must remain unchanged. Q.E.D.

* This also follows from the fact that when equations (III) are solved for x' and y' the results are of the first degree in x and y.

65. Simplification of equations by transformation of coördinates. The principal use made of transformation of coördinates is to discuss the various forms in which the equation of a curve may be put. In particular, they enable us to deduce *simple* forms to which an equation may be reduced.

Rule *to simplify the form of an equation.*

First step. Substitute the values of x and y given by (I) [*or* (II)] *and collect like powers of x' and y'.*

Second step. Set equal to zero the coefficients of two terms obtained in the first step which contain h and k (or one coefficient containing θ).

Third step. Solve the equations obtained in the second step for h and k (or θ).*

Fourth step. Substitute these values for h and k (or θ) in the result of the first step. The result will be the required equation.

In many examples it is necessary to apply the rule twice in order to rotate the axes, and then translate them, or *vice versa.* It is usually simpler to do this than to employ equations (III) in the Rule and do both together. Just what coefficients are set equal to zero in the second step will depend on the object in view.

It is often convenient to drop the primes in the new equation and remember that the equation is referred to the new axes.

Ex. 1. Simplify the equation $y^2 - 8x + 6y + 17 = 0$ by translating the axes.

Solution. First step. Set $x = x' + h$ and $y = y' + k$.

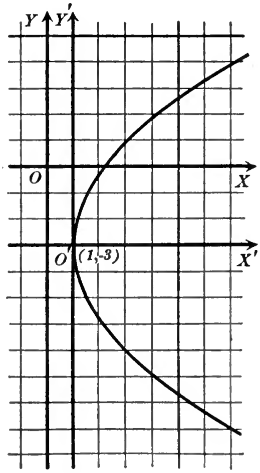
This gives $(y' + k)^2 - 8(x' + h) + 6(y' + k) + 17 = 0$, or

(1)
$$y'^2 - 8x' + \left.\begin{array}{r}2k\\+6\end{array}\right| y' + \left.\begin{array}{r}k^2\\-8h\\+6k\\+17\end{array}\right|^{\dagger} = 0.$$

* It may not be possible to solve these equations (Theorem IV, p. 81).

† These vertical bars play the part of parentheses. Thus $2k + 6$ is the coefficient of y' and $k^2 - 8h + 6k + 17$ is the constant term. Their use enables us to collect like powers of x' and y' *at the same time* that we remove the parentheses in the preceding equation.

Second step. Setting the coefficient of y' and the constant term, the only coefficients containing h and k, equal to zero, we obtain



(2) $$2k + 6 = 0,$$

(3) $$k^2 - 8h + 6k + 17 = 0.$$

Third step. Solving (2) and (3) for h and k, we find

$$k = -3, \ h = 1.$$

Fourth step. Substituting in (1), remembering that h and k satisfy (2) and (3), we have

$$y'^2 - 8x' = 0.$$

The locus is the parabola plotted in the figure which shows the new and old axes.

Ex. 2. Simplify $x^2 + 4y^2 - 2x - 16y + 1 = 0$ by translating the axes.

Solution. First step. Set $x = x' + h$, $y = y' + k$. This gives

(4) $$x'^2 + 4y'^2 + \left.\begin{array}{r} 2h \\ -2 \end{array}\right| x' + \left.\begin{array}{r} 8k \\ -16 \end{array}\right| y' + \left.\begin{array}{r} h^2 \\ +4k^2 \\ -2h \\ -16k \\ +1 \end{array}\right| = 0.$$

Second step. Set the coefficients of x' and y' equal to zero. This gives

$$2h - 2 = 0, \ 8k - 16 = 0.$$

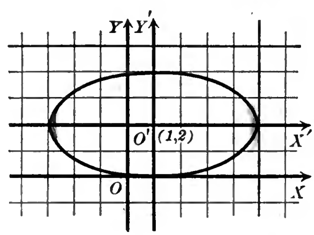
Third step. Solving, we obtain

$$h = 1, \ k = 2.$$

Fourth step. Substituting in (4), we obtain

$$x'^2 + 4y'^2 = 16.$$

Plotting on the new axes, we obtain the figure.



Ex. 3. Remove the xy-term from $x^2 + 4xy + y^2 = 4$ by rotating the axes.

Solution. First step. Set $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, whence

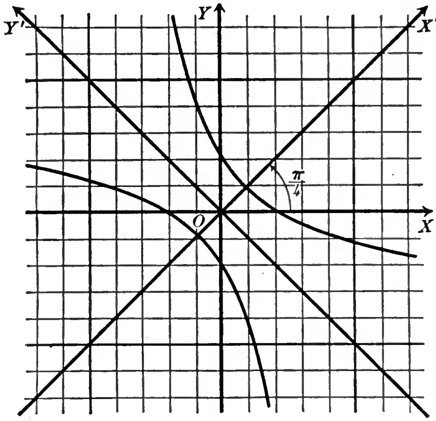
$$\left.\begin{array}{r} \cos^2\theta \\ +4\sin\theta\cos\theta \\ +\sin^2\theta \end{array}\right| x'^2 \left.\begin{array}{r} -2\sin\theta\cos\theta \\ +4(\cos^2\theta - \sin^2\theta) \\ +2\sin\theta\cos\theta \end{array}\right| x'y' \left.\begin{array}{r} +\sin^2\theta \\ -4\sin\theta\cos\theta \\ +\cos^2\theta \end{array}\right| y'^2 = 4.$$

or, by 3, p. 12, and 14, p. 13,

(5) $$(1 + 2\sin 2\theta)x'^2 + 4\cos 2\theta \cdot x'y' + (1 - 2\sin 2\theta)y'^2 = 4.$$

Second step. Setting the coefficient of $x'y'$ equal to zero, we have

$$\cos 2\theta = 0.$$



Third step. Hence

$$2\theta = \frac{\pi}{2}. \quad \therefore \theta = \frac{\pi}{4}.$$

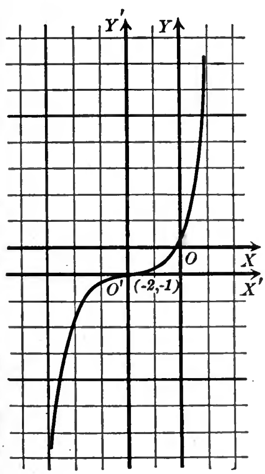
Fourth step. Substituting in (5), we obtain, since $\sin\frac{\pi}{2} = 1$ (p. 14),

$$3x'^2 - y'^2 = 4.$$

The locus of this equation is the hyperbola plotted on the new axes in the figure.

From $\cos 2\theta = 0$ we get, in general, $2\theta = \frac{\pi}{2} + n\pi$, where n is any positive or negative integer, or zero, and hence $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$. Then the xy-term may be removed by giving θ any one of these values. For most purposes we choose the smallest positive value of θ as in this example.

Ex. 4. Simplify $x^3 + 6x^2 + 12x - 4y + 4 = 0$ by translating the axes.



Solution. First step. Set

$$x = x' + h, \quad y = y' + k.$$

We obtain

$$(6) \quad \begin{array}{l|l|l|l} x'^3 + 3h & x'^2 + \ 3h^2 & x' - 4y' + & h^3 \\ \quad + 6 & \quad + 12h & & + 6h^2 \\ & \quad + 12 & & + 12h \\ & & & - 4k \\ & & & + 4 \end{array} = 0.$$

Second step. Set equal to zero the coefficient of x'^2 and the constant term. This gives

$$3h + 6 = 0,$$
$$h^3 + 6h^2 + 12h - 4k + 4 = 0.$$

Third step. Solving,

$$h = -2, \quad k = -1.$$

Fourth step. Substituting in (6), we obtain

$$x'^3 - 4y' = 0,$$

whose locus is the cubical parabola in the figure.

PROBLEMS

1. Simplify the following equations by translating the axes. Plot both pairs of axes and the curve.

(a) $x^2 + 6x + 8 = 0$. *Ans.* $x'^2 = 1$.
(b) $x^2 - 4y + 8 = 0$. *Ans.* $x'^2 = 4y'$.
(c) $x^2 + y^2 + 4x - 6y - 3 = 0$. *Ans.* $x'^2 + y'^2 = 16$.
(d) $y^2 - 6x - 10y + 19 = 0$. *Ans.* $y'^2 = 6x'$.
(e) $x^2 - y^2 + 8x - 14y - 33 = 0$. *Ans.* $x'^2 - y'^2 = 0$.
(f) $x^2 + 4y^2 - 16x + 24y + 84 = 0$. *Ans.* $x'^2 + 4y'^2 = 16$.
(g) $y^3 + 8x - 40 = 0$. *Ans.* $8x' + y'^3 = 0$.
(h) $x^3 - y^2 + 14y - 49 = 0$. *Ans.* $y'^2 = x'^3$.
(i) $4x^2 - 4xy + y^2 - 40x + 20y + 99 = 0$. *Ans.* $(2x' - y')^2 - 1 = 0$.

2. Remove the xy-term from the following equations by rotating the axes. Plot both pairs of axes and the curve.

(a) $x^2 - 2xy + y^2 = 12$. *Ans.* $y'^2 = 6$.
(b) $x^2 - 2xy + y^2 + 8x + 8y = 0$. *Ans.* $\sqrt{2}y'^2 + 8x' = 0$.
(c) $xy = 18$. *Ans.* $x'^2 - y'^2 = 36$.
(d) $25x^2 + 14xy + 25y^2 = 288$. *Ans.* $16x'^2 + 9y'^2 = 144$.
(e) $3x^2 - 10xy + 3y^2 = 0$. *Ans.* $x'^2 - 4y'^2 = 0$.
(f) $6x^2 + 20\sqrt{3}xy + 26y^2 = 324$. *Ans.* $9x'^2 - y'^2 = 81$.

66. Application to equations of the first and second degrees. In this section we shall apply the Rule of the preceding section to the proof of some general theorems.

Theorem V. *By moving the axes the general equation of the first degree,*

$$Ax + By + C = 0,$$

may be transformed into $x' = 0$.

Proof. Apply the Rule on p. 141, using equations (III).

Set
$$x = x'\cos\theta - y'\sin\theta + h,$$
$$y = x'\sin\theta + y'\cos\theta + k.$$

This gives

$$(1)\qquad \begin{array}{r|r|r|l} A\cos\theta & x' - A\sin\theta & y' + Ah & = 0. \\ +B\sin\theta & +B\cos\theta & +Bk & \\ & & +C & \end{array}$$

Setting the coefficient of y' and the constant term equal to zero gives

(2) $$-A\sin\theta + B\cos\theta = 0,$$

(3) $$Ah + Bk + C = 0.$$

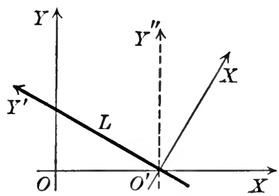
From (2), $$\tan\theta = \frac{B}{A}, \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{B}{A}\right).$$

From (3) we can determine many pairs of values of h and k. One pair is

$$h = -\frac{C}{A}, \quad k = 0.$$

Substituting in (1) the last two terms drop out, and dividing by the coefficient of x' we have left $x' = 0$. Q.E.D.

We have moved the origin to a point (h, k) on the given line L, since (3) is the condition that (h, k) lies on the line, and then rotated the axes until the new axis of y coincides with L. The particular point chosen for (h, k) was the point O' where L cuts the X-axis.



This theorem is evident geometrically. For $x' = 0$ is the equation of the new Y-axis, and evidently any line may be chosen as the Y-axis. But the theorem may be used to prove that the locus of every equation of the first degree is a straight line, if we prove it as above, for it is evident that the locus of $x' = 0$ is a straight line.

Theorem VI. *The term in xy may always be removed from an equation of the second degree,*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

by rotating the axes through an angle θ such that

(VI) $$\tan 2\theta = \frac{B}{A - C}.$$

Proof. Set $$x = x' \cos\theta - y' \sin\theta$$
and $$y = x' \sin\theta + y' \cos\theta.$$

This gives

$$(4)\qquad \begin{array}{r|l|l|l}
A\cos^2\theta & x'^2 - 2A\sin\theta\cos\theta & x'y' + A\sin^2\theta & y'^2 \\
+B\sin\theta\cos\theta & \quad +B(\cos^2\theta - \sin^2\theta) & \quad -B\sin\theta\cos\theta & \\
+C\sin^2\theta & \quad +2C\sin\theta\cos\theta & \quad +C\cos^2\theta &
\end{array}$$

$$\begin{array}{r|l|l}
+D\cos\theta & x' - D\sin\theta & y' + F = 0. \\
+E\sin\theta & \quad +E\cos\theta &
\end{array}$$

Setting the coefficient of $x'y'$ equal to zero, we have

$$(C - A)\,2\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = 0,$$

or (14, p. 13), $$(C - A)\sin 2\theta + B\cos 2\theta = 0.$$

Hence $$\tan 2\theta = \frac{B}{A - C}.$$

If θ satisfies this relation, on substituting in (4) we obtain an equation without the term in xy. Q.E.D.

Corollary. *In transforming an equation of the second degree by rotating the axes the constant term is unchanged unless the new equation is multiplied or divided by some constant.*

For the constant term in (4) is the same as that of the given equation.

Theorem VII. *The terms of the first degree may be removed from an equation of the second degree,*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

by translating the axes, provided that the discriminant of the terms of the second degree, $\Delta = B^2 - 4AC$, is not zero.

Proof. Set $$x = x' + h,\quad y = y' + k.$$

This gives

$$(5)\qquad \begin{array}{r|l|l|l}
Ax'^2 + Bx'y' + Cy'^2 + 2Ah & x' + Bh & y' + Ah^2 & = 0. \\
+Bk & \quad +2Ck & \quad +Bhk & \\
+D & \quad +E & \quad +Ck^2 & \\
 & & \quad +Dh & \\
 & & \quad +Ek & \\
 & & \quad +F &
\end{array}$$

Setting equal to zero the coefficients of x' and y', we obtain

$$2Ah + Bk + D = 0, \quad (6)$$

$$Bh + 2Ck + E = 0. \quad (7)$$

These equations can be solved for h and k unless (Theorem IV, p. 81)

$$\frac{2A}{B} = \frac{B}{2C},$$

or

$$B^2 - 4AC = 0.$$

If the values obtained be substituted in (5), the resulting equation will not contain the terms of the first degree. Q.E.D.

Corollary I. *If an equation of the second degree be transformed by translating the axes, the coefficients of the terms of the second degree are unchanged unless the new equation be multiplied or divided by some constant.*

For these coefficients in (5) are the same as in the given equation.

Corollary II. *When Δ is not zero the locus of an equation of the second degree has a center of symmetry.*

For if the terms of the first degree be removed the locus will be symmetrical with respect to the new origin (Theorem V, p. 66).

If $\Delta = B^2 - 4AC = 0$, equations (6) and (7) may still be solved for h and k if (Theorem IV, p. 81) $\frac{2A}{B} = \frac{B}{2C} = \frac{D}{E}$, when the new origin (h, k) may be any point on the line $2Ax + By + D = 0$. In this case every point on that line will be a center of symmetry.

For example, consider $x^2 + 4xy + 4y^2 + 4x + 8y + 3 = 0$. For this equation equations (6) and (7) become

$$2h + 4k + 4 = 0,$$

$$4h + 8k + 8 = 0.$$

In these equations the coefficients are all proportional and there is an infinite number of solutions. One solution is $h = -2$, $k = 0$. For these values the given equation reduces to

$$x^2 + 4xy + 4y^2 - 1 = 0,$$

or

$$(x + 2y + 1)(x + 2y - 1) = 0.$$

The locus consists of two parallel lines and evidently is symmetrical with respect to any point on the line midway between those lines.

MISCELLANEOUS PROBLEMS

1. Simplify and plot.

(a) $y^2 - 5y + 6 = 0.$

(b) $x^2 + 2xy + y^2 - 6x - 6y + 5 = 0.$

(c) $y^2 + 6x - 10y + 2 = 0.$

(d) $x^2 + 4y^2 - 8x - 16y = 0.$

(e) $x^2 + 4xy + y^2 = 8.$

(f) $x^2 - 9y^2 - 2x - 36y + 4 = 0.$

(g) $25y^2 - 16x^2 + 50y - 119 = 0.$

(h) $x^2 + 2xy + y^2 - 8x = 0.$

2. Find the point to which the origin must be moved to remove the terms of the first degree from an equation of the second degree (Theorem VII).

3. To what point (h, k) must we translate the axes to transform

$(1 - e^2)x^2 + y^2 - 2px + p^2 = 0$ into $(1 - e^2)x^2 + y^2 - 2e^2px - e^2p^2 = 0$?

4. Simplify the second equation in problem 3.

5. Derive from a figure the equations for rotating the axes through $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$, and verify by substitution in (II), p. 138.

6. Prove that every equation of the first degree may be transformed into $y' = 0$ by moving the axes. In how many ways is this possible?

7. The equation for rotating the polar axis through an angle ϕ is $\theta = \theta' + \phi$.

8. The equations of transformation from rectangular to polar coördinates, when the pole is the point (h, k) and the polar axis makes an angle of ϕ with the X-axis, are

$$x = h + \rho \cos(\theta + \phi),$$
$$y = k + \rho \sin(\theta + \phi).$$

9. The equations of transformation from rectangular coördinates to oblique coördinates are

$$x = x' + y' \cos \omega,$$
$$y = y' \sin \omega,$$

if the X-axes coincide and the angle between OX' and OY' is ω.

10. The equations of transformation from one set of oblique axes to any other set with the same origin are

$$x = x' \frac{\sin(\omega - \phi)}{\sin \omega} + y' \frac{\sin(\omega - \psi)}{\sin \omega},$$
$$y = x' \frac{\sin \phi}{\sin \omega} + y' \frac{\sin \psi}{\sin \omega},$$

where ω is the angle between OX and OY, ϕ is the angle from OX to OX', and ψ is the angle from OX to OY'.