AT THE BLACKBOARD I

The quadratic trinomial

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THE EXPRESSION

$$ax^2 + bx + c,$$

where a, b, and c are given numbers and $a \neq 0$, is called a quadratic trinomial in x. The values of x for which the quadratic trinomial vanishes (becomes zero) are called its *roots.*

Problems that require a knowledge of the properties of quadratic trinomials are often encountered in examinations. Many students can easily write various formulas, plot the function $y = ax^2 + bx + c$, and are familiar with its basic properties. This knowledge, however, is often superficial, and students don't know how to use it to solve problems.

In this article we show through examples the importance of combining algebraic and geometric reasoning to solve problems that involve quadratic trinomials.

1. *Find the maximum value of the quadratic trinomial* $y = -2x^2 + 4x - 5$.

A person familiar with the calculus can use the derivative to solve this problem. However, we can easily do without the calculus. Let us complete the square:

$$\begin{aligned} y &= -2x^2 + 4x - 5 \\ &= -2(x^2 - 2x + 1) + 2 - 5 \\ &= -2(x-1)^2 - 3. \end{aligned}$$

We see that the maximum value of this quadratic trinomial is –3, and it is attained for $x = 1$.

The method of completing the square is used to derive the formula for the roots of a quadratic equation. It can also be used to plot the graph of the general quadratic function $y = ax^2 + bx + c$. Indeed, if we complete the square, we have

$$y = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{2a}.$$

Thus, we see that the graph of the quadratic function is obtained by translating the parabola $y = ax^2$ by the vector

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{2a}\right).$$

2. *Figure 1 shows four parabolas. Each can be described by an equation of the form* $y = ax^2 + bx + c$. *In each case, determine the sign of the numbers* a, b, *and* c.

We consider case (i) in detail. The coefficient a is less than 0, since the branches of the parabola are directed downwards. The abscissa of parabola's vertex is $-b/2a$. Since this is negative, we know that $b < 0$. The ordinate of the point where the parabola intersects the y-axis is equal to the value of $f(x) = ax^2 + bx + c$ for $x = 0$. Therefore, $c = f(0)$ is positive. Thus, we have $a < 0$, $b < 0$, and $c > 0$.

The same result can be obtained by considering the sum and the product of the roots of the equation $ax^2 + bx + c = 0$. However, this

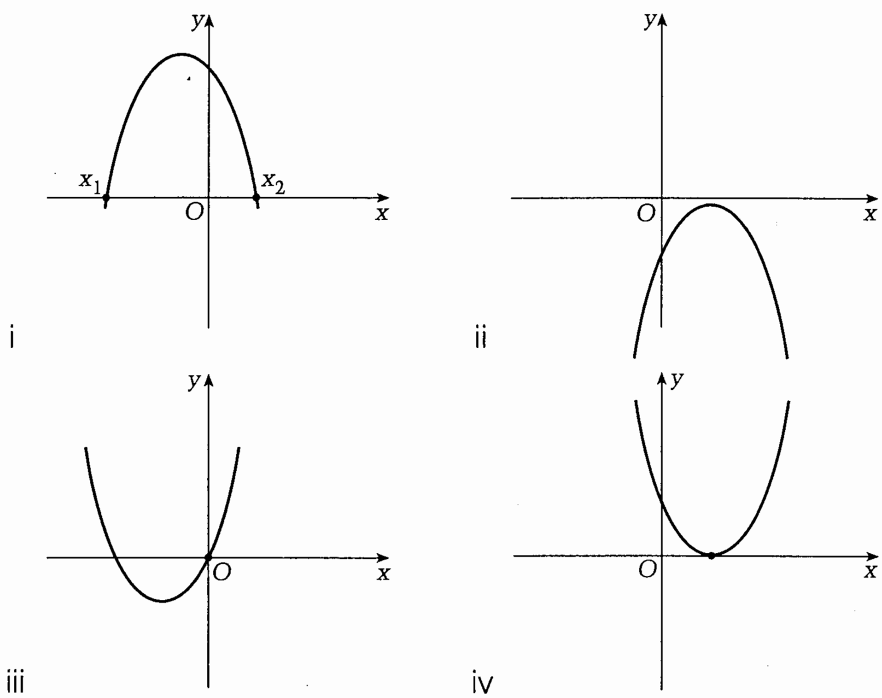

Figure 1

method cannot be used in case (ii), when the roots are complex.

We invite the reader to analyze cases (ii), (iii), and (iv).

3. *Suppose that the roots x_1 and x_2 of the quadratic equation $x^2 - 2rx - 7r^2 = 0$ satisfy the condition $x_1^2 + x_2^2 = 18$. Find the value of r.*

First, we represent $x_1^2 + x_2^2$ in terms of the sum and the product of the roots. We have

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2$$
$$= (2r)^2 - 2 \cdot (-7r^2) = 18r^2.$$

Thus, the problem reduces to solving the equation $r^2 = 1$, from which we find $r_1 = 1$ and $r_2 = -1$. We now need to check only that the roots exist for both values of r.

4. *Find necessary and sufficient conditions for the roots x_1 and x_2 of the equation $f(x) = x^2 + px + q = 0$ to have different signs and be greater than 1 in absolute value.*

A solution based on the discriminant and the quadratic formula is rather tedious. The problem, however, can be easily solved by using geometrical considerations.

First, we find a necessary condition. Let $x_1 < x_2$ (here the roots are different). We are given that $x_1 < -1$ and $x_2 > 1$; i.e., the interval $[-1, 1]$ belongs to the interval $[x_1, x_2]$ (see figure 2). This condition is equivalent to the following system of inequalities:

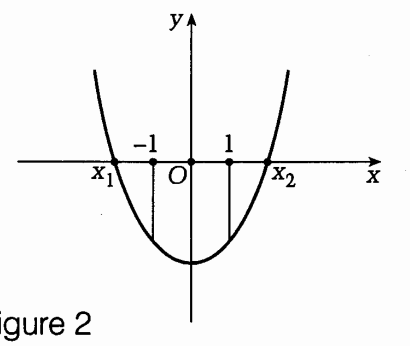

Figure 2

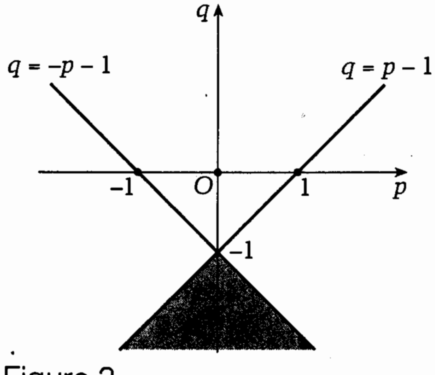

Figure 3

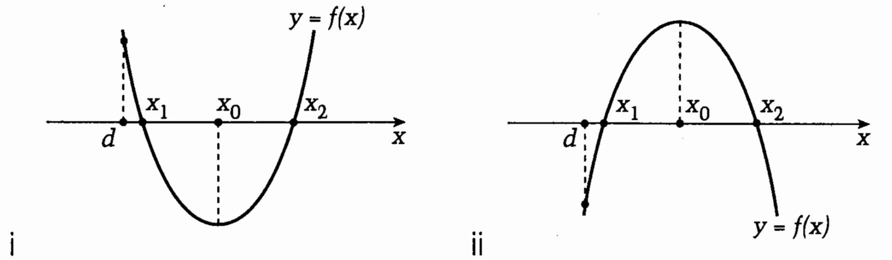

Figure 4

$$\begin{cases} f(-1) < 0, \\ f(1) < 0. \end{cases} \quad (1)$$

Substituting 1 and −1 into the expression for $f(x)$, we obtain the necessary conditions

$$\begin{cases} -p + q < -1, \\ p + q < -1. \end{cases} \quad (2)$$

The relationship between p and q can be illustrated graphically by a set of points (p, q), whose coordinates satisfy inequalities (2) (see figure 3).

Let us prove that the necessary conditions (2) are also sufficient. That is, if inequalities (2) are satisfied, the roots of the quadratic trinomial $f(x) = ax^2 + px + q$ satisfy the inequalities $x_1 < -1$ and $x_2 > 1$. Since conditions (2) are equivalent to conditions (1), the function $y = f(x)$ takes negative values at two different points ($x = 1$ and $x = -1$). Since the coefficient of x^2 is positive, the branches of the parabola $y = f(x)$ are directed upwards. Thus, the parabola intersects the x axis at two different points x_1 and x_2 ($x_1 < x_2$), and the points −1 and 1 belong to the interval $[x_1, x_2]$. Therefore, $x_1 < -1$ and $x_2 > 1$.

5. *Find all values of r for which the roots of the equation $(r - 4)x^2 - 2(r - 3)x + r = 0$ are greater than −1.*

Consider the case $r = 4$ separately. Then the equation becomes $-2x + 4 = 0$, so $x = 2$. Since $2 > -1$, the value $r = 4$ satisfies the condition of the problem. If $r \neq 4$, we have a quadratic equation.

We will solve a more general problem: *we will find necessary and sufficient conditions for the roots of the quadratic trinomial $f(x) = ax^2 + bx + c$ to be real and greater than a given real number d.*

Here is a geometric solution. The roots x_1 and x_2 must exist. Therefore,

$$D = b^2 - 4ac \geq 0. \quad (3)$$

Let us sketch the graph of $y = f(x)$. Figure 4 shows the two possible situations. Since both roots are greater than d, the abscissa of the parabola's vertex is greater than d. That is, $x_0 = (x_1 + x_2)/2 > d$. Using the formulas for the sum and product of the roots, we find:

$$-\frac{b}{2a} > d. \quad (4)$$

The point $x = d$ does not belong to the interval $[x_1, x_2]$. This means that parabola's branches are directed upwards in the case $a > 0$ and $f(d) > 0$ (figure 4i) or downwards in the case $a < 0$ and $f(d) < 0$ (figure 4ii). The numbers a and $f(d)$ are therefore identical in sign. That is,

$$a\,f(d) > 0. \quad (5)$$

We invite the reader to check that conditions (3)–(5) are not only necessary but also sufficient.

Here is an algebraic solution to the same problem. Two real numbers $x_1 - d$ and $x_2 - d$ are both positive if and only if their sum and product are both positive. Therefore, the condition given in the problem is equivalent to the following three conditions:

$$D = b^2 - 4ac > 0,$$
$$(x_1 - d) + (x_2 - d) > 0,$$
$$(x_1 - d)(x_2 - d) > 0.$$

Using the sum and product of the roots, we can rewrite the second condition as

$$x_1 + x_2 - 2d > 0,$$

$$\frac{x_1 + x_2}{2} > d,$$

$$-\frac{b}{2a} > d.$$

The third condition can be written as

$$x_1x_2 - (x_1 + x_2)d + d^2 > 0,$$

$$a(ad^2 + bd + c) > 0,$$

$$af(d) > 0.$$

Thus, we have again proved that the combination of conditions (1)–(3) is equivalent to the given conditions of the more general problem.

Returning to problem 5, we write conditions (1)–(3) in this case. We have the following system of inequalities:

$$\begin{cases} (r-3)^2 - r(r-4) = 9 - 2r \geq 0, \\ \dfrac{r-3}{r-4} > -1, \\ (r-4)(4r-10) > 0. \end{cases}$$

Solving these inequalities, we obtain $r < 5/2$, $4 < r \leq 9/2$. We also must add $r = 4$ to these solutions.

Answer:

$$\left]-\infty; \frac{5}{2}\right[\cup \left[4, \frac{9}{2}\right].$$

Problems.

6. Let x_1 and x_2 be the roots of the equation $x^2 + px + q = 0$. Find p and q if it is given that $x_1 + 1$ and $x_2 + 1$ are the roots of the equation $x^2 - p^2x + pq = 0$.

7. The graph of the quadratic function $y = ax^2 + bx + c$ cuts off segments AB and CD along two parallel lines. Prove that the line passing through the midpoints of these segments is parallel to the y-axis.

8. If the quadratic trinomial $f(x) = ax^2 + bx + c$ has no real roots, and if its coefficients satisfy the inequality $a - b + c < 0$, find the sign of c.

9. Let the roots x_1 and x_2 of the quadratic trinomial $ax^2 + bx + c$ be different. Prove that the number x_0 lies between x_1 and x_2 if and only if $a(ax_0^2 + bx_0 + c) < 0$.

10. Let the equation $ax^2 + bx + c = 0$ have no nonnegative roots, and suppose $a < 0$. Find the sign of c.

11. Let the coefficients of the equations $x^2 + p_1x + q_1 = 0$ and $x^2 + p_2x + q_2 = 0$ satisfy the equation $p_1p_2 = 2(q_1 + q_2)$. Prove that at least one of these equations has real roots.

12. Is it possible that the equation $x^2 + px + q = 0$, where p and q are rational numbers, has the following roots:

(a) $x_1 = \sqrt{3}$, $x_2 = \dfrac{1}{\sqrt{3}}$?

(b) $x_1 = \sqrt{3} + 2$, $x_2 = 2 - \sqrt{3}$?

13. Prove that any rational root of the equation $x^2 + px + q = 0$ with integer coefficients p and q is an integer.

14. Let the equations $x^2 + p_1x + q_1 = 0$ and $x^2 + p_2x + q_2 = 0$ with integer coefficients p_i and q_i $(i = 1, 2)$ have a common noninteger root. Prove that $p_1 = p_2$ and $q_1 = q_2$.

15. Let x_1 and x_2 be the roots of the quadratic equation $ax^2 + bx + c = 0$ and $S_m = x_1^m + x_2^m$ (where m is a positive integer). Prove the formula $aS_m + bS_{m-1} + cS_{m-2} = 0$.