Basic Calculus

From Archimedes to Newton to its Role in Science

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1

The Greeks Measure the Universe

This chapter is an exposition of fundamental matters that will be in use throughout this book. It introduces basic concepts of geometry and trigonometry and shows how the Greeks—who developed these disciplines into usable theories—put them to use.

1.1 The Pythagoreans Measure Length

The primary purpose of a number system is to count and measure things. This is where we will begin. The Pythagorean theorem[1] asserts that in a right triangle (see Figure 1.1), the equality $a^2 + b^2 = c^2$ relates the lengths of the sides. We are assuming that some unit of length (nowadays it would be a foot, yard, mile, inch, or meter, for example) is given. The Pythagorean theorem works "in reverse" also. Namely, if in some triangle the lengths of the sides satisfy the relationship $a^2 + b^2 = c^2$, then the triangle is a right triangle with c the length of the hypotenuse and a and b the lengths of the other two sides.

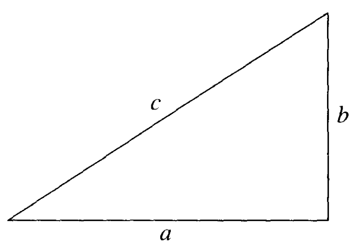

Figure 1.1

It appears that "rope stretchers" used this fact in early construction. Take, for example, a rope that has a length of 12 units (Figure 1.2): If the rope is stretched out horizontally on the ground in the triangular pattern indicated in Figure 1.3, then, since the lengths of the sides satisfy the relationship $4^2 + 3^2 = 16 + 9 = 25 = 5^2$, the Pythagorean theorem asserts that the angle at 4 is a right angle. The construction of two perpendicular walls can begin!

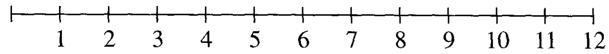

Figure 1.2

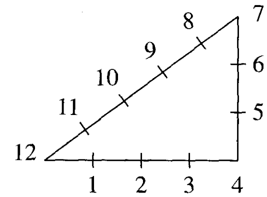

Figure 1.3

Assume that μ is some unit of length. We will say that a straight segment is *measurable* in μ if its length in units of μ is $\frac{m}{n}$, where m and n are both positive integers. So the segment is measurable if its length can be expressed precisely in whole or fractional units of μ. For instance, if the unit μ is the inch, then the segments of lengths $26\frac{4}{7} = \frac{186}{7}$ inches, as well as $53\frac{14}{29} = \frac{1551}{29}$ inches and $1726\frac{951}{3657} = \frac{6312933}{3657}$ inches are measurable.

The question presents itself as to whether all straight segments are measurable in the unit μ. Assume that the segment pictured directly below is measurable and that it has length $\frac{m}{n}$.

Take another segment of the same length, place them together at right angles, and form the right triangle pictured in Figure 1.4. Is the resulting hypotenuse h measurable? We will see (surprise?) that it is not.

Assume that it is. Then the length of h is some fraction $\frac{r}{s}$ of positive integers. By Pythagoras's theorem, $\left(\frac{r}{s}\right)^2 = \left(\frac{m}{n}\right)^2 + \left(\frac{m}{n}\right)^2 = 2\left(\frac{m}{n}\right)^2$. By clearing denominators, $r^2n^2 = 2m^2s^2$. Now set $x = rn$ and $y = ms$, and

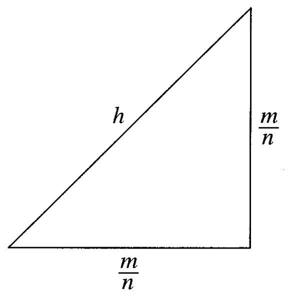

Figure 1.4

observe that $x^2 = 2y^2$. Factoring 2 out of x as many times as possible gives $x = 2^a x_1$ with x_1 odd. In the same way, $y = 2^b y_1$ with y_1 odd. (If, for instance, x were equal to 172, then $172 = 2\cdot 86 = 2\cdot 2\cdot 43 = 2^2\cdot 43$. So $a = 2$ and $x_1 = 43$.) By a substitution we get $(2^a x_1)^2 = 2(2^b y_1)^2$. So by the rules for exponentiating,

$$2^{2a}x_1^2 = 2^{2b+1}y_1^2.$$

Notice that $2a \neq 2b+1$, since $2a$ is even and $2b+1$ is odd. Therefore, either $2a > 2b+1$ or $2a < 2b+1$.

1. Assume that $2a > 2b+1$. So $2^{2a-(2b+1)}x_1^2 = y_1^2$. It follows that y_1^2 is even. Since y_1 is odd, it has the form $y_1 = 2k+1$ for some integer k. So $y_1^2 = 4k^2+4k+1 = 4(k^2+k)+1$, which is the sum of an even integer and 1. But this means that y_1^2 is odd. Thus y_1^2 is both even and odd. This is impossible.

2. Assume next that $2a < 2b+1$. This implies that $x_1^2 = 2^{(2b+1)-2a}y_1^2$. By the argument in Step 1, it is now x_1^2 that is both even and odd. This too is impossible.

Reflect over what has been done. The discussion started with the supposition that h is measurable in μ. Then the argument moved in a strictly logical way to impossible consequences. The inescapable conclusion is that h cannot be measurable in μ.

The preceding proof was not taken from an old Greek text. It is, however, from the point of view of the precision of the logic and the flow of its argument, very much in the spirit of Greek mathematics. The point is this: While the hypotenuse h is a perfectly valid geometric construction, it cannot be measured with the numbers of the Pythagoreans. In particular, their numerical considerations ran into limitations that geometrical ones did not. The suggestion presents itself that this was an important reason why Greek geometry and trigonometry flourished in a way that numerical analyses and algebra did not.

Pythagoras and his followers had founded a Greek colony in today's southern Italy in the 6th century B.C. They formed a cult based on the philosophical principle that mathematics is the underlying explanation of all things from the relationship between musical notes to the movements of the planets of the solar system. Indeed, they held that all reality finds its ultimate explanation in numbers and mathematics. As we have just seen, however, the numbers of the Pythagoreans were unable to come to grips with the very basic matter of measuring length. It is believed that when the Pythagoreans realized this, a crisis ensued within their ranks that contributed to their downfall.[2] In any case, it seems somewhat ironic that the Pythagorean theorem—an assertion about the lengths of certain segments—derives its name from a school (or person) that did not possess a number system with which length could always be measured.

Today we can put it this way: The Pythagorean number system did not have enough numbers. It consisted only of the numbers of the form $\frac{m}{n}$, i.e., the rational numbers, and it did not include other real numbers. In fact, the preceding demonstration of the nonmeasurability of h shows that $\sqrt{2}$ is a number that is not rational. (Take $m = n = 1$.) It is, in other words, irrational. On the other hand, we know that

$$\begin{aligned}\sqrt{2} &= 1.4142\ldots\\ &= 1 + 4\frac{1}{10} + 1\frac{1}{100} + 4\frac{1}{1000} + 2\frac{1}{10000} + \cdots,\end{aligned}$$

and hence that $\sqrt{2}$ can be constructed in terms of a decimal expansion. This infinite process gives rise to the number line. Fix a unit of length and take a straight line that runs infinitely in both directions.

Fix a point and label it 0. Mark off a point one unit to the right of 0 and label it 1. Continue in this way to get 2, 3, ... Do a similar thing on the left of 0, but use −1, −2, ... to label the points. Continue in this way with tenths of units, hundredths of units, and so on, to achieve the following relationship: Every point on the line corresponds to a real number, in other words, a number given by a decimal expansion, and every such number corresponds to a point on the line. See the illustration for $\sqrt{2} = 1.4142\ldots$ in Figure 1.5.

The Greeks shied away from infinite processes and did not hit upon the construction of the real numbers. (This did not occur until the 16th century!) This means that they did not have the coordinate—also called Cartesian—plane. Therefore, they could not graph algebraic equations and make use of the interplay between algebra and geometry that lies at the core of modern mathematical analysis. In this course we will not shy away from real numbers: we will make use of them throughout.

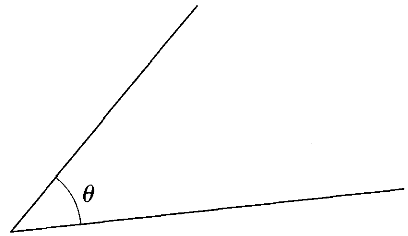

Figure 1.6

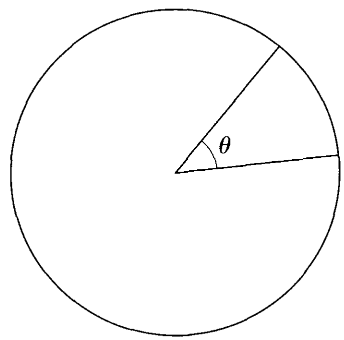

Figure 1.7

1.2 The Measure of Angles

We turn next to the study of angles. Take two line segments with a common endpoint; they form a wedge. The issue is to devise a numerical way of measuring the angle, i.e., the amount θ of the opening of the wedge. See Figure 1.6.

Place the wedge in a circle with the common point at the center. See Figure 1.7. One way of measuring θ is with degrees. This unit is designated by $^\circ$ and is a legacy of the base 60 number system of the Babylonians. Declare the entire circle to have 360 degrees and then apportion degrees proportionally. So a wedge consisting of half the circle will have 180°,

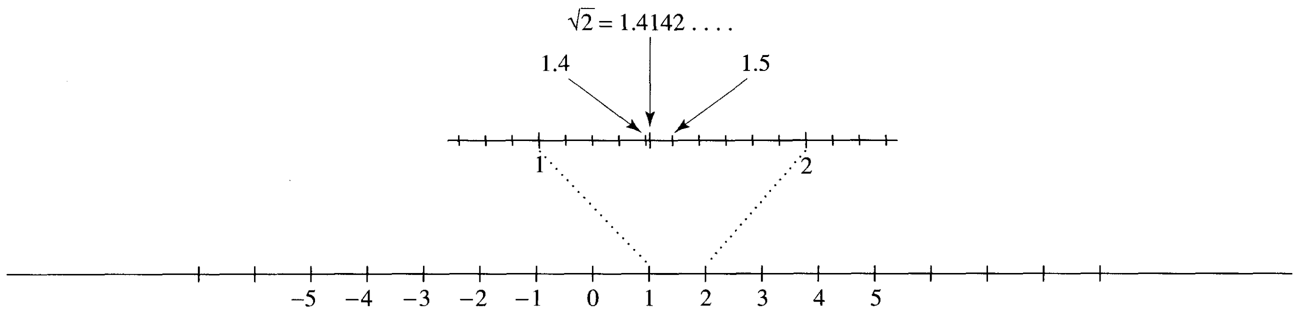

Figure 1.5

one consisting of a quarter circle will measure 90°, and so on. In this way, any angle can be assigned a certain number of degrees. It is known that the angles of any triangle add up to 180°.

A more useful numerical measure of an angle is the radian measure. It is based on the measurement of length. Let r be the radius of the circle in Figure 1.8 and let s be the length of the arc that is cut out of the circumference by the wedge. The radian measure of the angle θ is defined to be the ratio $\frac{s}{r}$. We will write $\theta = \frac{s}{r}$. An immediate question that arises is this: Does the radian measure of the angle depend on the size of the circle into which it is placed? If this measure is to be a meaningful concept, it should not.

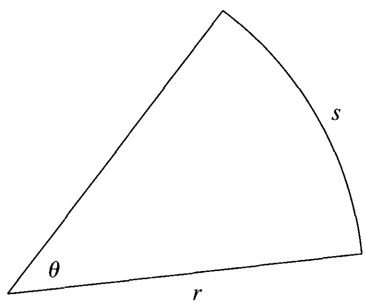

Figure 1.8

Place θ into another circle. Let R be its radius and S the length of the arc that θ cuts out. See Figure 1.9. Is $\frac{s}{r} = \frac{S}{R}$? This is the question that must be addressed. Consider what follows to be a "thought experiment," rather than something that is carried out in practice.

Let n be a positive integer (n can be, say, 5 or 500 or 40,000), and partition the wedge into n equal pieces. Connect the intersection points with straight line segments and let their respective lengths be d_n and D_n. (Note that these lengths depend on the n that you have picked.) The case $n = 4$ is shown in Figure 1.10. Observe that each of the smaller triangular wedges (those with sides of length r) is similar to each of the

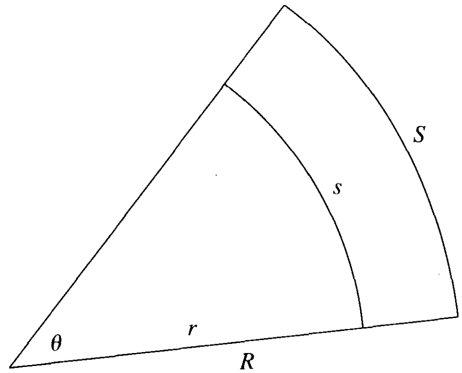

Figure 1.9

larger ones (with sides of length R). This is because the corresponding angles are equal. It follows that $\frac{d_4}{r} = \frac{D_4}{R}$. In the same way, $\frac{d_n}{r} = \frac{D_n}{R}$ for any n. Observe next that if n is taken to be large, then the lengths of the n segments each of length d_n add up to approximately s. So the number nd_n is nearly equal to s, and therefore $\frac{nd_n}{r}$ is nearly equal to $\frac{s}{r}$. If n is taken sequentially larger and larger, the numbers $\frac{nd_n}{r}$ close in on $\frac{s}{r}$. We abbreviate this by writing

$$\lim_{n\to\infty} \frac{nd_n}{r} = \frac{s}{r}.$$

The symbol lim is short for limit and refers to the closing-in process. The symbol ∞ represents "infinity"

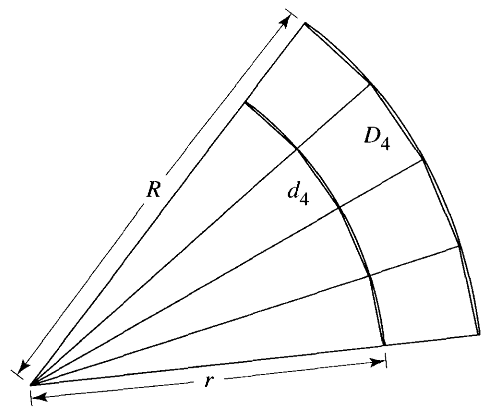

Figure 1.10

and $\lim_{n\to\infty}$ means that n, by being taken larger, is being "pushed to infinity." Proceeding in exactly the same way with the larger arc, we get that

$$\lim_{n\to\infty} \frac{nD_n}{R} = \frac{S}{R}.$$

Since $\frac{d_n}{r} = \frac{D_n}{R}$ for any n, we see that

$$\frac{s}{r} = \lim_{n\to\infty} \frac{nd_n}{r} = \lim_{n\to\infty} \frac{nD_n}{R} = \frac{S}{R}.$$

So our thought experiment has established the required equality $\frac{s}{r} = \frac{S}{R}$. Observe that the radian measure of an angle is a ratio of lengths. It is therefore a real number.

We will learn later that "limit" procedures such as the one we just considered are the cornerstone of calculus.

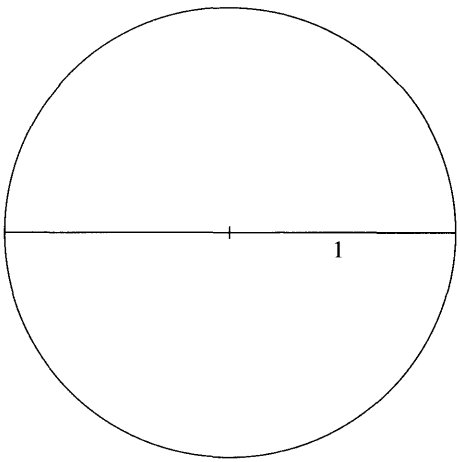

Figure 1.11

Now take a circle of radius 1 (Figure 1.11). The length of one-half of its circumference is a number that is denoted by π. Consider the angle 180° and observe that its radian measure is $\frac{\pi}{1} = \pi$. Now take a circle of radius r and let c be its circumference (Figure 1.12). Using this circle, we get that the radian measure of 180° is equal to $\dfrac{c/2}{r}$. Since the radian measure of an angle is the same regardless of which circle is used, it follows that $\dfrac{c/2}{r} = \pi$. This gives the well-known formula

$$c = 2\pi r$$

for the circumference of a circle of radius r.

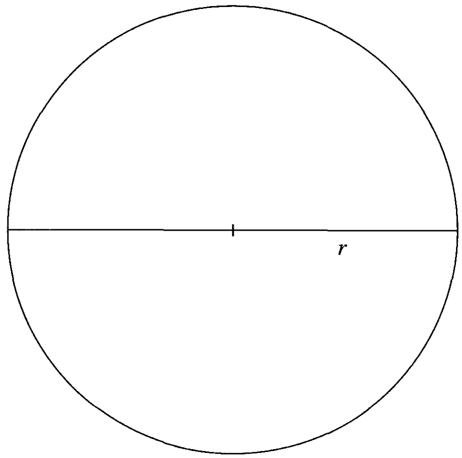

Figure 1.12

Consider the circle of Figure 1.11 and check that degrees and radians correspond as shown in Table 1.1.

Table 1.1

Degrees	Radians
360°	2π
180°	π
90°	$\frac{\pi}{2}$
60°	$\frac{\pi}{3}$
45°	$\frac{\pi}{4}$
30°	$\frac{\pi}{6}$
1°	$\frac{\pi}{180}$

Since π represents length, it is some real number. What is its decimal expansion?[3] Take a circle of radius 1 and center C and divide the upper semicircle into three equal parts, as shown in Figure 1.13. Each of the angles is equal to $\frac{\pi}{3}$. Concentrate on the wedge on the right. Let A and B be the indicated points of intersection, and observe that the isosceles triangle ΔCAB is equilateral. (Why?) Therefore, AB has length 1. Choose the point P on the circle such that the segment CP bisects the angle $\angle ACB$ and draw the tangent $A'B'$ to the circle at P. Since the bisector CP and the tangent $A'B'$ are perpendicular (by a basic property of the circle), the angles at A' and B' are both equal to $\frac{\pi}{3}$. It follows that the triangles $\Delta CA'P$ and $\Delta CB'P$ are equal. Put[4] $A'P = PB' = t$. Since the three angles of $\Delta CA'B'$ are equal, it follows that $\Delta CA'B'$ is equilateral. Therefore, CA' has length $2t$, and by the Pythagorean theorem applied to $\Delta CPA'$, it follows that $1 + t^2 = (2t)^2$. Thus $3t^2 = 1$, and hence $t = \frac{1}{\sqrt{3}}$. So $A'B'$ has length $2t = \frac{2}{\sqrt{3}}$. Since the radian measure of the angle $\frac{\pi}{3}$ is the length of the arc AB, it seems plausible (we will be content with plausibility in this discussion—see also Exercise 15) from Figure 1.13, that

$$1 = AB < \frac{\pi}{3} = \text{length arc } AB < A'B' = \frac{2}{\sqrt{3}}.$$

Now multiply through by 3 to get

$$3 < \pi < \frac{2 \cdot 3}{\sqrt{3}} = \frac{2\sqrt{3}\sqrt{3}}{\sqrt{3}} = 2\sqrt{3}.$$

This corresponds to $3 < \pi < 3.47$.

Archimedes (287–212 B.C.)—we will encounter him again soon—used an argument, similar in principle but much more involved (it uses many more triangles instead of our three), to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

This gives $3.1408 < \pi < 3.1429$. The correct expansion begins $\pi = 3.14159\ldots$. It turns out (and this is difficult to establish) that π is an irrational number.

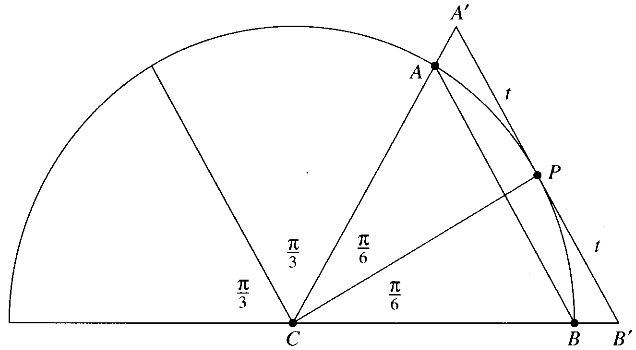

Figure 1.13

1.3 Eratosthenes Measures the Earth

Eratosthenes (276–194 B.C.) was director of the Museum of Alexandria, as Euclid had been before him. Under the sponsorship of Egypt's Ptolemaic rulers, this was an early version of a government funded research institute.

It was the commonly accepted view among Greek philosophers that the Earth is round. Eratosthenes was able to measure its size. He knew that at noon

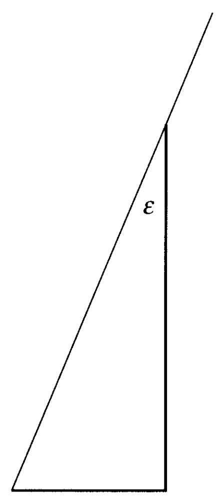

Figure 1.14

at the summer solstice of every year (this occurs on or about June 21 in the Northern Hemisphere) in the town of Syene (near today's Aswan in Egypt), the Sun shines into the very bottom of a deep well. This means that at precisely that time of the year the Sun is directly overhead at this location. In Alexandria, again precisely at noon at the summer solstice, Eratosthenes measured the length of the shadow of a gnomon[5] in vertical position (using a plumbline) and determined the angle ε in Figure 1.14 to be 7.5°. Eratosthenes knew also that Syene was (roughly) due south of Alexandria at a distance of about 500 miles.[6]

He next let r be the radius of the Earth and noticed that the basic situation is given by Figure 1.15 with $\varepsilon = 7.5°$ and $s = 500$. The rest was easy. On the one hand,

$$\varepsilon = \left(7\frac{1}{2}\right)\frac{\pi}{180} = \frac{15(3.14)}{360} = 0.13 \text{ radians.}$$

But on the other hand, $\varepsilon = \frac{s}{r} = \frac{500}{r}$. Therefore,

$$r = \frac{500}{\varepsilon} = \frac{500}{0.13} = 3850 \text{ miles.}$$

So Eratosthenes—quite literally with a stick, some observations, and geometry, cemented together by pure thought—had measured the size of the Earth! Of course, he had only an approximation. The correct value of the radius of the Earth is 3950 miles.

1.4 Right Triangles

Trigonometry is the study of the right triangle and the applications of this study. Indeed, the word trigonometry is the Greek rendition of "measuring the triangle." This section recalls some ba-

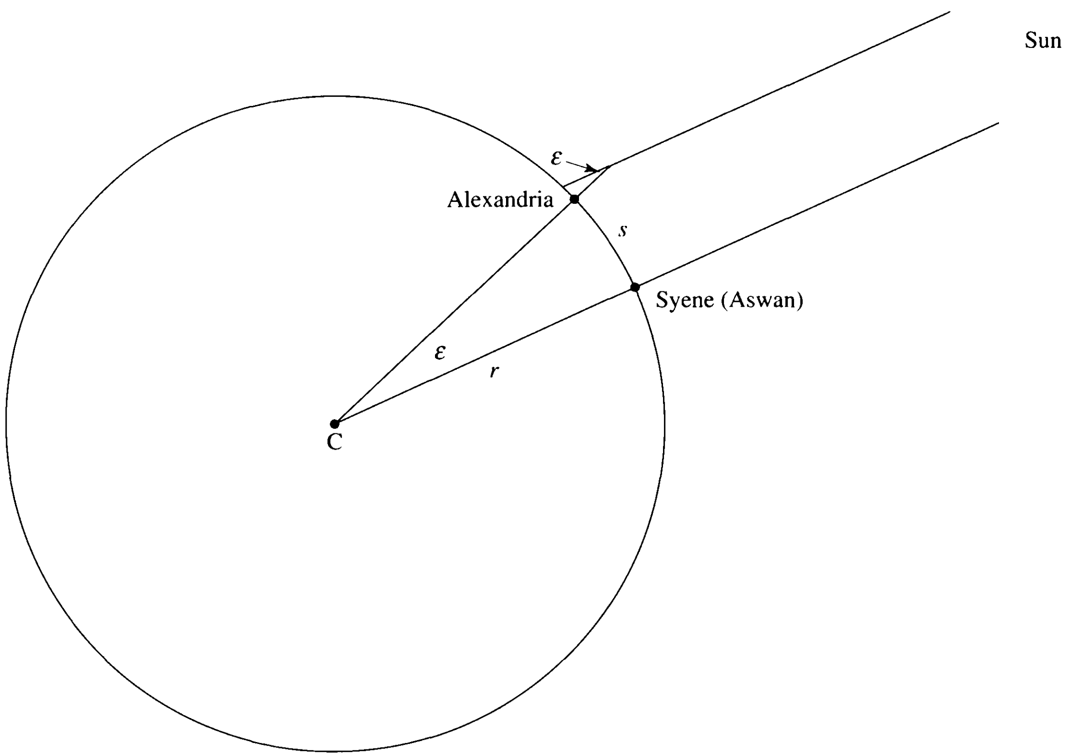

Figure 1.15

sic facts, all known—with different terminology and notation—to the Greeks.

Consider the right triangle in Figure 1.16. For the given angle θ, define the "sine," "cosine," and "tangent" to be the following ratios of lengths:

$$\sin\theta = \frac{a}{h}, \quad \cos\theta = \frac{b}{h},$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{h}}{\frac{b}{h}} = \frac{a}{b}.$$

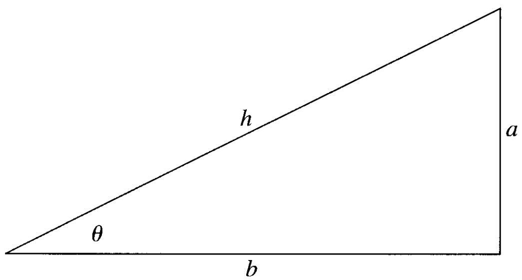

Figure 1.16

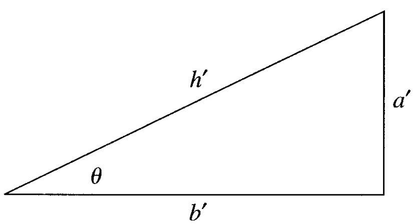

Figure 1.17

Only the size of the angle θ matters in these definitions and not the right triangle into which it is placed. This is easily seen as follows. Suppose another right triangle is used; see Figure 1.17. Superimposing the two triangles shows that they are similar. (See Figure 1.18.) Therefore, $\frac{a'}{h'} = \frac{a}{h}$, $\frac{b'}{h'} = \frac{b}{h}$, and $\frac{a'}{b'} = \frac{a}{b}$, so it doesn't matter which right triangle is used to compute $\sin\theta$, $\cos\theta$, and $\tan\theta$.

There are many identities that relate $\sin\theta$, $\cos\theta$, and $\tan\theta$. For example, by Pythagoras's theorem, $a^2 + b^2 = h^2$. So $\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1$, and therefore,

$$\sin^2\theta + \cos^2\theta = 1.$$

It is customary to write $\sin^2\theta$ instead of the less efficient $(\sin\theta)^2$.

We will now compute $\sin\theta$, $\cos\theta$, and $\tan\theta$ for some standard values of θ. Consider the right triangle in Figure 1.19. Since it is isosceles, the acute angles are each 45° or $\frac{\pi}{4}$ radians. Therefore,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \text{ and } \tan\frac{\pi}{4} = \frac{1}{1} = 1.$$

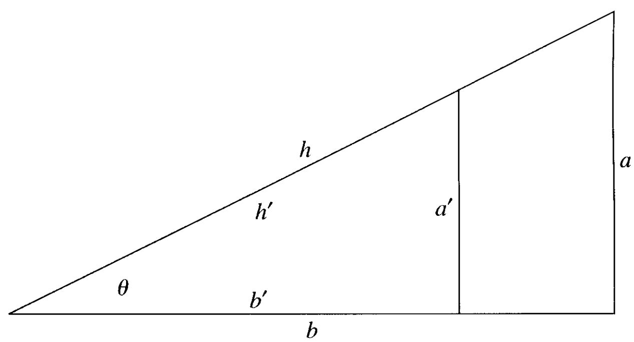

Figure 1.18

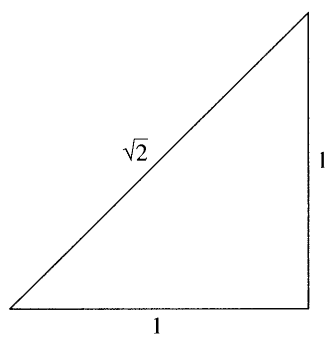

Figure 1.19

Next, take the equilateral triangle that has sides of length 2 and height h. By Pythagoras's theorem, $h^2 + 1^2 = 2^2$. So $h^2 = 3$ and $h = \sqrt{3}$. See Figure 1.20. Recalling that the angles of an equilateral triangle are each equal to 60° or $\frac{\pi}{3}$ radians, we see that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \ \cos\frac{\pi}{3} = \frac{1}{2}, \text{ and } \tan\frac{\pi}{3} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

Since each of the smaller angles at the top is $\frac{\pi}{6}$, it follows that

$$\sin\frac{\pi}{6} = \frac{1}{2}, \ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \text{ and } \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

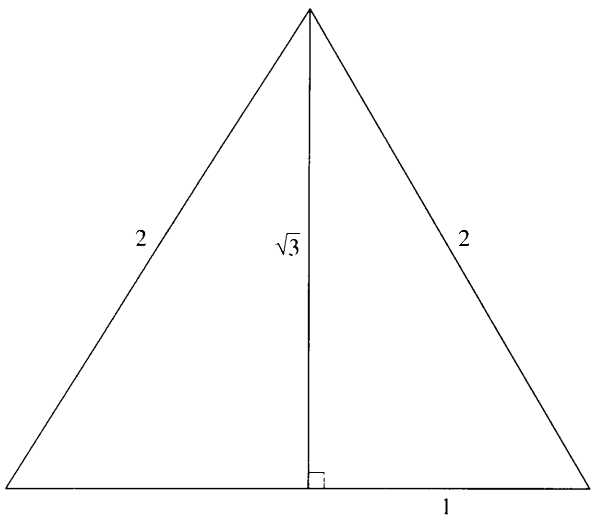

Figure 1.20

Observe that $\sin 0$, $\cos 0$, and $\tan 0$ do not make sense, for the simple reason that there is no right triangle with an angle of 0° or, equivalently, 0 radians. Suppose, however, that θ is very small. A look at the right triangle with hypotenuse 1 in Figure 1.21 shows that $\sin\theta = \frac{a}{1} = a$ is also very small. So if θ is close to 0, then $\sin\theta$ is also close to zero. Take θ sequentially smaller and smaller. This rotates the hypotenuse downwards and pushes a to zero. Therefore, $\sin\theta = a$ is pushed to zero. We summarize this by writing

$$\lim_{\theta\to 0} \sin\theta = 0.$$

It now makes sense to set $\sin 0 = 0$. For entirely similar reasons, $\sin\frac{\pi}{2}$ does not make sense either. This time focus on the angle ϕ in Figure 1.21. As the hypotenuse rotates, ϕ is pushed to $\frac{\pi}{2}$, and in the process, $\sin\phi = \frac{b}{1} = b$ is pushed to 1. In limit notation,

$$\lim_{\phi\to\frac{\pi}{2}} \sin\phi = 1.$$

So it makes sense to set $\sin\frac{\pi}{2} = 1$. A similar analysis for the cosine shows that $\lim_{\theta\to 0}\cos\theta = 1$ and $\lim_{\theta\to\frac{\pi}{2}}\cos\theta = 0$; so we set $\cos 0 = 1$ and $\cos\frac{\pi}{2} = 0$.

Table 1.2

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	?

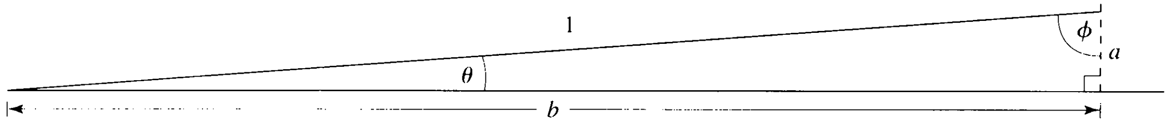

Figure 1.21

Table 1.2 summarizes this information. The reader is asked to fill in the question mark.

We conclude with some basic properties of the sine. Consider the circle of radius 1 and the right triangle shown in Figure 1.22. Observe that $\sin\theta = \frac{a}{1} = a$ and $\theta = \frac{s}{1} = s$ radians. Since $s > a$, it follows that

$$\theta > \sin\theta.$$

If θ is small, note that the lengths s and a will be close to each other. We symbolize this by writing $s \approx a$. Therefore, for small θ,

$$\sin\theta \approx \theta.$$

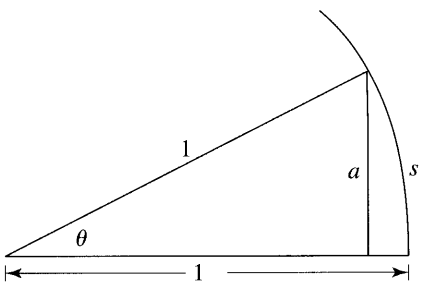

Figure 1.22

Table 1.3

θ **in degrees**	θ **in radians**	$\sin\theta$
30°	$\frac{\pi}{6} = 0.5237$	0.5000
15°	0.2618	0.2588
10°	0.1746	0.1737
3°	0.0524	0.0524
2°	0.0349	0.0349
1°	0.0175	0.0175

Take $\pi = 3.142$ and use your calculator to verify the data in Table 1.3. Observe that there is close agreement between $\sin\theta$ and θ, but only if θ is taken in radians. So when computing $\sin\theta$, make sure that your calculator is in "radian mode" and not "degree mode."

1.5 Aristarchus Sizes Up the Universe

Not much is known about Aristarchus of Samos (310–230 B.C.). He received his education directly or indirectly from Aristotle's Lyceum (the institute in Athens). The most important fact about him is this: He believed that the universe is Sun centered (or heliocentric), that the Sun is fixed, and that the Earth revolves around the Sun and rotates about its own axis in the process.

What will interest us is Aristarchus's use of "cosmic" trigonometry in his treatise *On the Magnitudes and Distances of the Sun and Moon*. His analysis rests on the following hypotheses and observations[7]:

A. The Moon receives its light from the Sun.

B. The Moon revolves in a circle about the Earth with the Earth at the center.

C. When an observer on Earth looks out at a precise half moon, the angle $\angle EMS$ in Figure 1.23 is 90°. At that moment the angle $\angle MES$ can be measured to equal 87°.

D. At the instant of a total eclipse of the Sun, the Moon and the Sun (as viewed from the Earth) subtend the same angle, and this angle can be measured to be 2°. Refer to Figure 1.24.

E. During a lunar eclipse, the shadow indicated in Figure 1.25 has width 4 times the radius r_M of the Moon. (This was based on how long the Moon was observed to be in the Earth's shadow.)

What did Aristarchus deduce from these observations? Let

r_E = the radius of the Earth

r_M = the radius of the Moon

r_S = the radius of the Sun

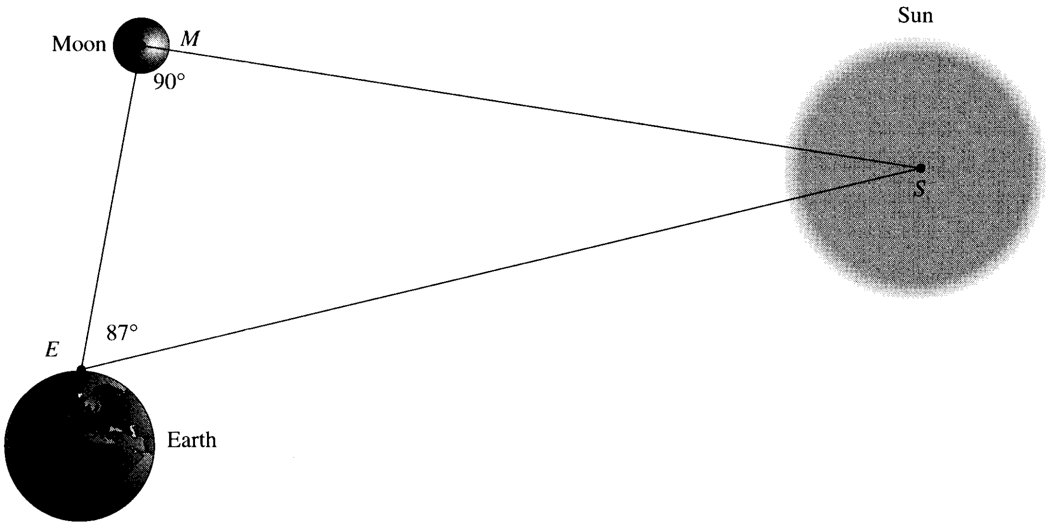

Figure 1.23

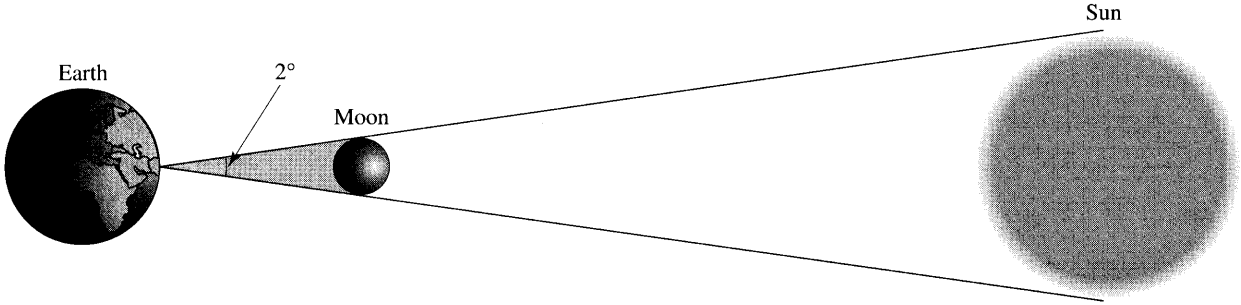

Figure 1.24

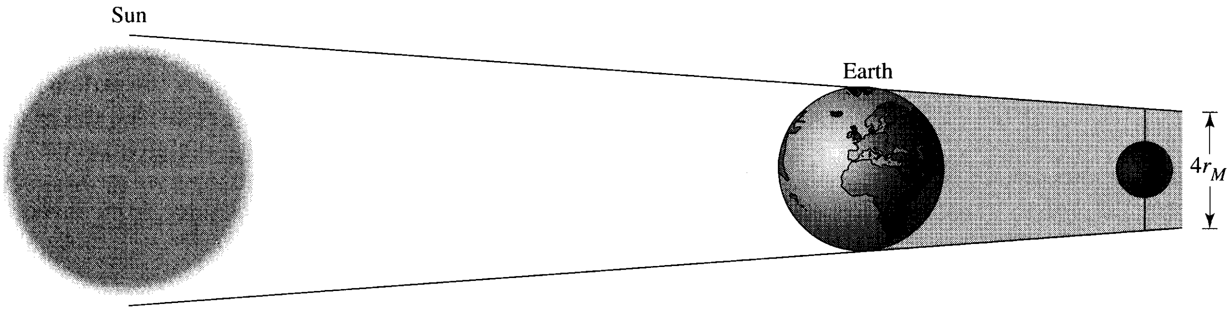

Figure 1.25

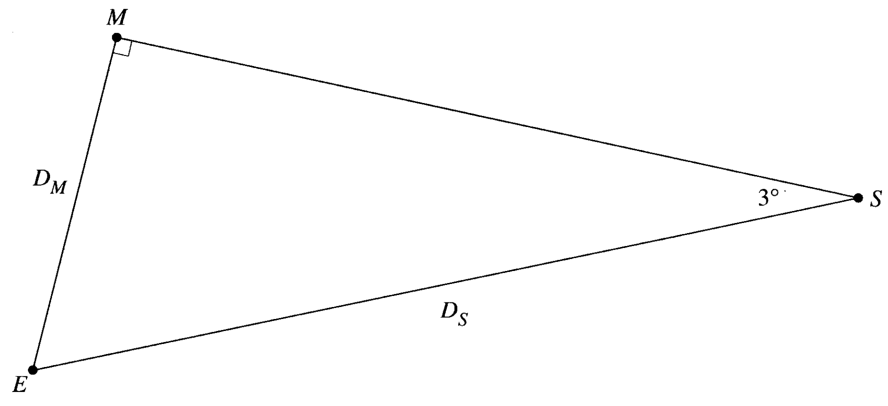

Figure 1.26

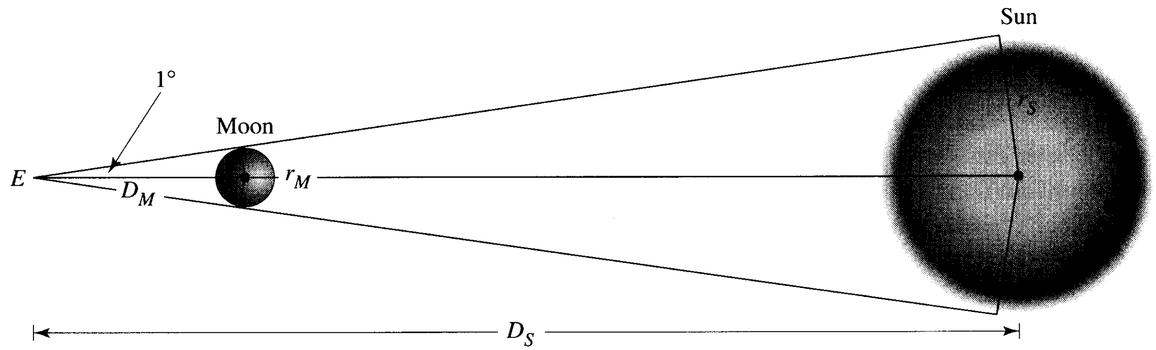

Figure 1.27

D_M = the distance from the Earth to the Moon

D_S = the distance from the Earth to the Sun

Figure 1.26 comes directly from observation **C**. Observe that $3^\circ = \frac{\pi}{60}$. Therefore, $\frac{D_M}{D_S} = \sin 3^\circ = \sin\frac{\pi}{60}$. Since $3^\circ = \frac{\pi}{60}$ radians is a small angle, $\sin\frac{\pi}{60}$ is approximately equal to $\frac{\pi}{60}$. (See Table 1.3.) In view of Archimedes's estimate, we take $\pi = 3.14$. (Aristarchus was a contemporary of Archimedes and would certainly have been aware of such estimates.) This gives $\frac{\pi}{60} = 0.052$. To make things simple, we will take $\sin\frac{\pi}{60} = \frac{\pi}{60} = 0.05 = \frac{1}{20}$. So $\frac{D_M}{D_S} = \sin\frac{\pi}{60} = \frac{1}{20}$. In this way Aristarchus arrived at the estimate

$$\frac{D_S}{D_M} = 20.$$

Refer next to observation **D** and Figure 1.24. This is the situation of the solar eclipse. Figure 1.27 is an elaboration of Figure 1.24. Radii of the Moon and the Sun have been inserted and some distances have been labeled. The angle at E indicated as being equal to 1° is obtained by bisecting the 2° angle of Figure 1.24. Notice that the Sun is larger than the Moon. By similar triangles, $\frac{r_S}{r_M} = \frac{D_S}{D_M}$, and therefore

$$\frac{r_S}{r_M} = 20.$$

Observe also that $\frac{r_M}{D_M} = \sin 1^\circ$. Since $1^\circ = \frac{\pi}{180}$ radians, $\frac{r_M}{D_M} = \sin\frac{\pi}{180} \approx \frac{\pi}{180}$. Taking $\pi = 3.14$, it follows that $\frac{\pi}{180} = 0.017$. Note that $\frac{1}{60}$ is very close to 0.017. So to make things simple, we will take $\frac{r_M}{D_M} = \sin\frac{\pi}{180} = \frac{1}{60}$.

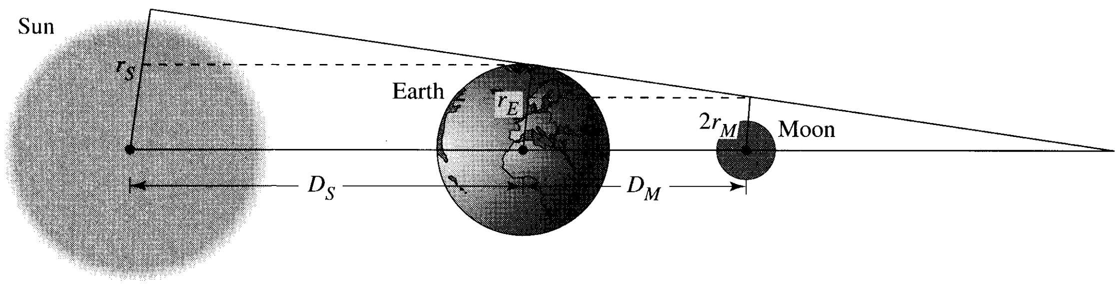

Figure 1.28

So Aristarchus obtained the approximation

$$\frac{D_M}{r_M} = 60.$$

From observation **E** and Figure 1.25, he obtained Figure 1.28. This figure shows a light ray that is tangent to both the Sun and the Earth. The radii of the Sun and the Earth drawn into the figure are both perpendicular to this light ray. The extension of the radius of the Moon indicated in the figure is perpendicular to this light ray as well. Because the two triangles with the "dotted" bases are similar, it follows that

$$\frac{r_E - 2r_M}{r_S - r_E} = \frac{D_M}{D_S}.$$

Recalling that

$$\frac{D_M}{D_S} = \frac{r_M}{r_S}$$

we get

$$\frac{r_E - 2r_M}{r_S - r_E} = \frac{r_M}{r_S}.$$

After cross-multiplying, $r_S r_E - 2r_S r_M = r_M r_S - r_M r_E$. So $r_S r_E + r_M r_E = 3r_S r_M$. Dividing this last equation through by $r_S r_M$ gives us $\frac{r_E}{r_M} + \frac{r_E}{r_S} = 3$. Since $\frac{r_S}{r_M} = 20$, we have $r_S = 20r_M$. By substitution, we get

$$3 = \frac{r_E}{r_M} + \frac{r_E}{20r_M} = \frac{20r_E + r_E}{20r_M} = \frac{21r_E}{20r_M}.$$

Therefore, $\frac{r_E}{r_M} = \frac{60}{21} = \frac{20}{7}$, and

$$r_M = \frac{7}{20} r_E.$$

Since $\frac{r_S}{r_E} = \frac{r_S}{r_M} \cdot \frac{r_M}{r_E} = 20 \cdot \frac{7}{20} = 7$, it follows that

$$r_S = 7r_E.$$

With Eratosthenes's value of $r_E = 3850$ miles, Aristarchus got the approximations

$$r_M = 1350 \text{ miles} \quad \text{and} \quad r_S = 27{,}000 \text{ miles}.$$

Since $\frac{D_M}{r_M} = 60$, he estimated that $D_M = 60r_M =$ 80,000 miles; and inserting this value into $\frac{D_S}{D_M} = 20$, he got

$$D_S = 1{,}600{,}000 \text{ miles}.$$

Aristarchus's argument was actually more elaborate and complicated than this. He used different approximations and got slightly different answers. For example, instead of $\frac{r_M}{r_S} = \frac{1}{20}$, he obtained $\frac{1}{20} < \frac{r_M}{r_S} < \frac{1}{18}$; and instead of $\frac{r_M}{D_M} = \frac{1}{60}$, he had $\frac{1}{60} < \frac{r_M}{D_M} < \frac{1}{45}$. However, the essence of his analysis has been retained.

Table 1.4 compares Aristarchus's estimates to modern values. Notice that Aristarchus's value for the radius of the Moon is rather accurate. However, his estimates for the distance to the Moon, the radius of the Sun, and its distance from the Earth are off by factors of 3, 16, and 50, respectively.

Table 1.4

	Aristarchus	Actual
r_E radius of Earth	3850 miles[a]	3950 miles
r_M radius of Moon	1350 miles	1080 miles
r_S radius of Sun	27,000 miles	432,000 miles[b]
D_M Earth to Moon	80,000 miles	238,868 miles[c]
D_S Earth to Sun	1,600,000 miles	93×10^6 miles[c]

[a] As already pointed out, this is taken from Eratosthenes. While Eratosthenes lived about 40 years after Aristarchus, similar estimates had been made earlier.

[b] The Sun consists of gas. The radius given is that of the photosphere, the illuminated part. The part from the center to 30% of its radius has in essence all the shining power and 60% of the mass. The part from the center to 60% of its radius has 95% of the mass.

[c] Radar measurements.

In any case, using only some basic observations and pure thought (i.e., mathematics), Aristarchus provided at least some idea of the magnitude of the distances in the solar system and began to unravel some of its mystery. In fact, Aristarchus's strategy is correct in principle. With more accurate measurements of the angles involved, he would have done much better; see Exercise 26. More serious is the problem of refraction. A ray of light bends as it moves from one medium to another of different density, and in particular, it bends when it moves through the atmosphere. The distance D_S was not computed with suitable accuracy until the 17th century, when the Italian astronomer Casini came within $7\frac{1}{2}\%$ of the correct value. More accurate values were not obtained until the latter part of the 18th. The calculations for these estimates used information about the observed path of Venus across the Sun.

1.6 The Sandreckoner

Archimedes—the most famous scientist of antiquity—was born around 285 B.C. in the Greek city state of Syracuse, a port on the Mediterranean island of Sicily. There is historical reference to the fact that he spent considerable time in Alexandria, and there seems little doubt that he studied with the successors of Euclid. After his studies, he returned to Syracuse and lived there in complete absorption with his mathematical investigations.

Late in the 3rd century B.C., Syracuse became embroiled in the struggle between Rome and Carthage for control of the western Mediterranean. In his *Parallel Lives*, the historian Plutarch (about 46–126 A.D.) recounts Archimedes's efforts in the defense of the city against the Romans:

> When, therefore, the Romans assaulted the walls in two places at once, fear and consternation stupefied the Syracusans, believing that nothing was able to resist that violence and those forces. But when Archimedes began to ply his engines, he at once shot against the land forces all sorts of missile weapons, and immense masses of stone that came down with incredible noise and violence, against which no man could stand; for they knocked down those upon whom they fell, in heaps, breaking all their ranks and files. In the meantime huge poles thrust out from the walls over the ships, sunk some by the great weights which they let down from on high upon them; others they lifted up into the air by an iron hand [and soon] such terror had seized upon the Romans, that, if they did but see a little rope or a piece of wood from the wall, instantly crying out, that there it was again, Archimedes was about to let fly some engine at them, they turned their backs and fled.

The Roman attack on Syracuse was repelled. A lengthy siege was later successful and Syracuse was conquered and destroyed in 212 B.C. Archimedes perished during the destruction. Plutarch relates several versions of his death. The one most widely cited finds Archimedes, oblivious to the city's capture, absorbed in the study of a particular diagram that he had sketched in the sand. When a Roman soldier confronted him, Archimedes requested time to complete his deliberations. The impatient soldier, however, ran him through with his sword.

The work of Archimedes is impressive—as we shall soon see. Plutarch speaks of Archimedes's

> purer speculations [and] studies, the superiority of which to all others is unquestioned, and in which the only doubt can be, whether the beauty and grandeur of the subjects examined, or the precision and cogency of the methods and means of proof, most deserve our admiration. It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanation. No amount of investigation of yours would succeed in attaining the proof, and yet once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required.

Archimedes was also the quintessential eccentric scientist. His deep absorption in thought

> made him forget his food and neglect his person, to that degree that when he was occasionally carried by absolute violence to bathe, or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil on his body.

A famous episode recounts how, after a particularly satisfying discovery, Archimedes ran through the streets of Syracuse in naked celebration shouting "Eureka, Eureka!" (Eureka, or *ευρεκα*, is Greek for "I have found it.")

It is, of course, difficult to separate fact from fiction and reality from legend in Plutarch's account of Archimedes's remarkable talents as inventor of machines of war. The ingenuity of Archimedes's speculations about geometry and physics, however, can be corroborated. Much of his work has survived in transmitted form.

We turn now to Archimedes's scheme for writing large numbers. You will encounter the basic Greek number system in the exercises. It allows for numbers as large as

$$99{,}999{,}999 = 9999(10{,}000) + 9999 = \,{}_{,}\delta\lambda\varsigma\delta M{}_{,}\overline{\delta\lambda\varsigma\delta}$$

but not larger. Archimedes enlarged the basic Greek number system into an incredible scheme. He introduced the number

$$MM = (10{,}000)(10{,}000) = 100{,}000{,}000 = 10^8.$$

and referred to numbers up to MM as numbers of the *first order*. He then built the following tower of "orders" and "periods" of numbers. In modern notation:

First order: The numbers N with $1 \le N < 10^8 = 10^{8\cdot\mathbf{1}}$

Second order: The numbers N with $10^{8\cdot 1} = 10^8 \le N < 10^{16} = 10^{8\cdot\mathbf{2}}$

Third order: The numbers N with $10^{8\cdot 2} = 10^{16} \le N < 10^{24} = 10^{8\cdot\mathbf{3}}$

Fourth order: The numbers N with $10^{8\cdot 3} = 10^{24} \le N < 10^{32} = 10^{8\cdot\mathbf{4}}$

⋮

$\mathbf{10^8}$th order: The numbers N with $10^{8(10^8-1)} \le N < 10^{8\cdot\mathbf{10^8}}$.

These are the numbers of the **first period**. This is only the beginning. The numbers of the **second period** are also partitioned into orders:

First order: The numbers N with $10^{8\cdot 10^8} \le N < 10^{8(10^8+\mathbf{1})}$

Second order: The numbers N with $10^{8(10^8+1)} \le N < 10^{8(10^8+\mathbf{2})}$

Third order: The numbers N with $10^{8(10^8+2)} \le N < 10^{8(10^8+\mathbf{3})}$

Fourth order: The numbers N with
$$10^{8(10^8+3)} \leq N < 10^{8(10^8+\mathbf{4})}$$

$\vdots$

$\mathbf{10^8}$th order: The numbers N with
$$10^{8(10^8+10^8-1)\cdot 8} \leq N < 10^{8(10^8+\mathbf{10^8})}.$$

The **second period** ends with (one less than) the number $10^{8(10^8+10^8)} = (10^{8\cdot 10^8})^{\mathbf{2}}$. The **third**, **fourth**, **fifth**, etc.... **periods** follow. Finally, with the $\mathbf{10^8}$**th period** and the number $(10^{8(10^8)})^{\mathbf{10^8}} = 10^{8\cdot 10^{16}}$, the array stops.

To put all of this into perspective, note that the visible universe is thought to have on the order of 10^{80} atoms (about 99.9% of them hydrogen and helium). This is Eddington's number, after the British astrophysicist Arthur Stanley Eddington. Since $10^{80} = 10^{8\cdot 10}$, this is the first number of the 11th order of the first period. The point is not so much the usefulness of Archimedes's scheme, but rather the grandiose nature of his speculations. Archimedes, in other words, thought big!

Archimedes looked for a context in which to illustrate the utility of his cosmic array of numbers. Having found it, he was evidently very pleased to address his manuscript *The Sandreckoner* to his benefactor the king of Syracuse. He began by giving the king an astronomy lesson:

> Aristarchus brought out a book consisting of some of the hypotheses, in which the premises lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves about the Sun in the circumference of a circle, the Sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same center as the Sun, is so great ...

Then he stated his purpose:

> I say then that, even if a sphere were made up of sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that, of the numbers named by me, some exceed in multitude the number of grains of sand in a mass which is equal in magnitude to the sphere referred to, provided that the following assumptions are made ...

In other words, Archimedes imagined the entire cosmos to be packed with sand and proposed to count the number of grains of sand in question!

Our description of Archimedes's discussion will use earlier notation: r_E for the radius of the Earth, r_M for the radius of the Moon, r_S for the radius of the Sun, and D_S for the distance between the Sun and the Earth.

What assumptions did Archimedes make? The first concerned r_E:

$$r_E \leq 47{,}500 \text{ miles.}$$

We have seen that Eratosthenes's rather accurate estimate was 3850 miles. So here Archimedes thought too big. Remember, however, that it was his purpose to display the vastness of his number scheme. Following Aristarchus, Archimedes supposed that

$$r_M < r_E.$$

Recalling that Aristarchus had shown that r_S is about 20 times greater than r_M, he next assumed that

$$r_S \leq 30 r_M.$$

This is in fact too small. A look at Table 1.4 shows, after a quick calculation, that $r_S = 400 r_M$.

Because $r_S \leq 30 r_M < 30 r_E \leq (30)(47{,}500)$, Archimedes got

$$r_S \leq 1{,}425{,}000 \text{ miles.}$$

While these inequalities are based on a combination of earlier estimates and pure speculation, Archimedes next turned to careful observation and delicate geometrical arguments. He used a long rod

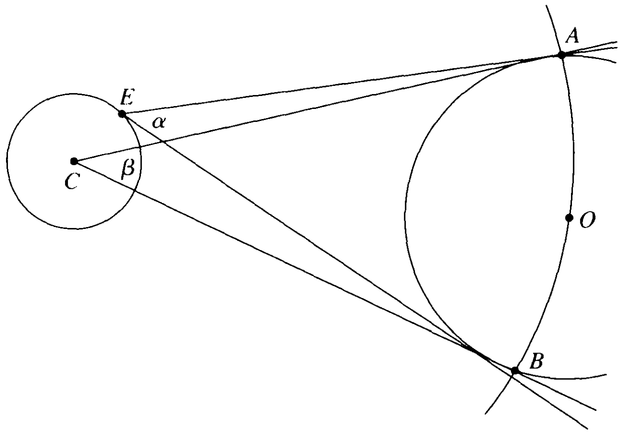

Figure 1.29

with a small disc fastened at its end. He pointed it in the direction of the Sun just after sunrise and carefully measured the angle α that the Sun subtends in the sky. He determined that $(\frac{90}{200})^\circ < \alpha < (\frac{90}{164})^\circ$ or, in radian measure,

$$\frac{1}{200}\frac{\pi}{2} < \alpha < \frac{1}{164}\frac{\pi}{2}.$$

The implied estimate $\alpha \approx \frac{1}{2}^\circ$ for the so-called angular diameter of the Sun is accurate. (Aristarchus took it to be 2°. Refer to Figure 1.24.)

Archimedes imagined an observer on the Earth looking out at the Sun just after sunrise. He constructed the diagram of Figure 1.29 and positioned the observer at the point E. Two tangent lines are drawn from E to the Sun. Note that α is the angle that these tangent lines determine at E. From C, the center of the Earth, he placed two more tangent lines to the Sun and let β be the angle that they determine. Taking the center of the Sun to be O, he obtained a circular arc by rotating CO, and he let A and B be the two points of intersection of this arc with the tangents from C. Since $\beta = \frac{\text{arc}\, AB}{D_S}$,

$$D_S = \frac{\text{arc}\, AB}{\beta} \approx \frac{2AO}{\beta} = \frac{2r_S}{\beta}.$$

Since the Sun is far away and E and C relatively close, observe that $\alpha \approx \beta$. (Archimedes showed much more. By a very careful argument that used delicate geometry and trigonometry, for example, a formula equivalent to $\frac{\sin\varepsilon}{\sin\gamma} < \frac{\varepsilon}{\gamma} < \frac{\tan\varepsilon}{\tan\gamma}$ for $0 < \gamma < \varepsilon < \frac{\pi}{2}$, he verified that $\beta < \alpha < \frac{100}{99}\beta$.) So Archimedes obtained the approximations

$$D_S \approx \frac{\text{arc}\, AB}{\alpha} \approx \frac{2r_S}{\alpha}.$$

Inserting the inequalities $r_S \leq (30)(47{,}500)$ miles and $\frac{1}{200}\frac{\pi}{2} < \alpha$, he found that $D_S < \frac{(60)(47{,}500)(400)}{\pi} < 160 \times 10^6$ miles. He therefore obtained

$$D_S < 160 \times 10^6 \text{ miles}.$$

Let D_* be the distance to the stars. Speculating about another of Aristarchus's assertions, Archimedes took

$$D_* < 1.6 \times 10^{12} \text{ miles}.$$

It is a strange curiosity that these last two inequalities correspond in a sense to correct values (Table 1.5).

Archimedes could now turn to the computation of the number of grains of sand needed to fill the sphere of the universe, i.e., the sphere of radius D_*. See Figure 1.30. He began with a poppy seed and assumed that a sphere the size of a poppy seed could hold 10,000 grains of sand. (The sand that Archimedes has in mind is evidently very finely grained.) He next estimated that the diameters of 40 poppy seeds added to one "fingerbreadth." Now, the volume of a sphere of radius r is

Table 1.5

	Archimedes[a]	**Actual**
D_S Earth to Sun	160×10^6 miles	93×10^6 miles
D_* Earth to Stars	1.6×10^{12} miles	24×10^{12} miles[b]

[a]The numbers in the column are to be understood to be bigger than the distances to which they refer. For example, D_S is less than the indicated number. Note also that these numbers were based mostly on speculation. These distances were not determined with any finality until more than 2000 years later.

[b]This is the approximate distance to the nearest stars; see the discussion in Section 1.7 (Postscript).

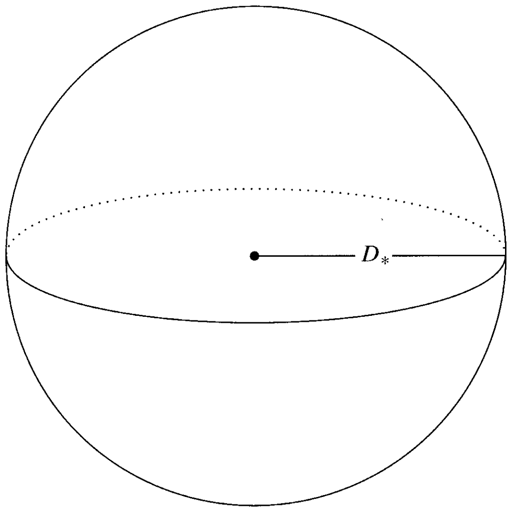

Figure 1.30

$\frac{4}{3}\pi r^3$, or $\frac{4}{3}\pi(\frac{d}{2})^3 = \frac{\pi}{6}d^3$ in terms of its diameter d. It follows that if the diameter of a sphere is increased by a factor of $2, 3, 4, \ldots, k, \ldots$, then its volume is increased by the factor 2^3, 3^3, 4^3, $\ldots$, $k^3, \ldots$. So a sphere of diameter one finger-breadth has a volume of 40^3 times greater than that of a poppy seed. Therefore, it can be filled with $10{,}000 \times 40^3 = 4^3 \times 10^3 \times 10^4 \approx 10^9$ grains of sand. Continuing in this way, Archimedes concluded that 10^{63} grains of sand will fill a sphere of radius 1.6×10^{12} miles. Since $D_* < 1.6 \times 10^{12}$ miles, this is more than enough to fill the sphere of the universe. Because Archimedes's number system handles 10^{63} with ease, his goal was accomplished!

By a remarkable coincidence, the 10^{63} grains of sand correspond to about 10^{80} atoms. So Archimedes arrived at a total amount of cosmic matter that isn't far from Eddington's 20th century estimate.

1.7 Postscript

Trigonometry—on a bigger scale yet—is used (*is* used today!) to determine the distances to the near stars. The principle is simple. Imagine yourself in a moving car looking out at the scenery. As the car moves, your perspective is constantly changing. The objects that are near will zoom past, those that are farther will move past more slowly, and the distant ones, say, the Moon, will hardly move at all. The Earth, too, is moving. As it does, it affords different perspectives on the heavens. Plotting the positions of stars carefully and regularly month after month reveals that some stars change their position in a detectable and measurable way and that others remain fixed. The greater the change in a star's position, the closer it is. The smaller the change in position the farther away it is.

To measure the distance to a near star A, proceed as follows. Consider the Earth in the two opposed positions E and E' in its orbit about the Sun. (See Figure 1.31.) When the Earth is at E, make note of the position C_1 of A against the fixed pattern of distant stars in a constellation with a telescope. Since A is near, its position in the constellation is observed to have shifted to, say, C_2 when the Earth is at E'. When the Earth is back at E, measure the angle $\theta = \angle C_1EC_2$

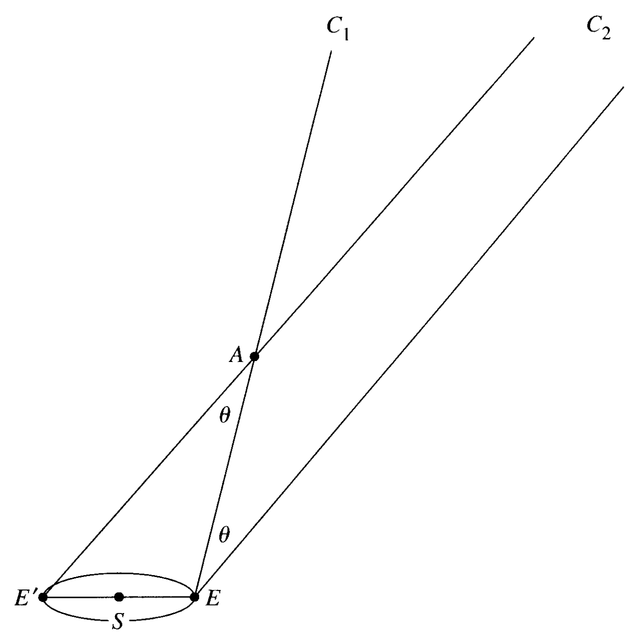

Figure 1.31

between the positions C_1 and C_2. Since the stars in the constellation are very far, the lines $E'C_2$ and EC_2 are parallel. So $\angle EAE' = \theta$.

The angle $p = \frac{1}{2}\theta$ is called the *stellar parallax* of the star A. Let D_A be the distance from the Earth to the star A. This distance is in essence the same as that from the Sun S to A. The distance between the Earth and the Sun is some 93 million miles. See Table 1.5. Astronomers refer to this distance as the *astronomical unit*. So 1 AU = 93×10^6 miles. So the triangle ΔASE can be approximated by the circular sector shown in Figure 1.32. Letting p_{rad} be the stellar parallax of A in radian measure, it follows that

$$p_{\text{rad}} = \frac{1}{D_A},$$

where D_A is given in AUs.

Since stellar parallax is always extremely small, it is customary to measure it in seconds, where 1 second is $\frac{1}{60}$ of 1 minute, which is $\frac{1}{60}$ of 1 degree. So $1^\circ = 3600$ seconds. Therefore,

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$$

$$= \frac{3600 \cdot 180}{\pi} \text{ seconds} = 2 \times 10^5 \text{ seconds}.$$

So the parallax p is converted from radians to seconds by $p_{\text{sec}} = 2 \times 10^5 p_{\text{rad}}$.

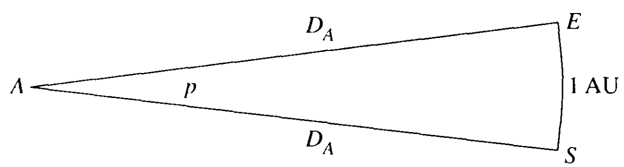

Figure 1.32

The distances to the stars are so vast that they are not measured in AUs but in *light years*. One light year, 1 LY for short, is the distance that light travels in one year. Since light travels 186,000 miles in one second, it travels approximately 6×10^{12} miles in one year. So we take 1 LY = 6×10^{12} miles. Since 1 AU = 93×10^6 miles, it follows that 1 LY = $\frac{6\times10^{12}}{93\times10^6}$ AU. Therefore, 1 AU = $\frac{93\times10^6}{6\times10^{12}}$ LY = 1.5×10^{-5} LY. The equality $p_{\text{rad}} = \frac{1}{D_A}$ can now be converted as follows:

$$p_{\text{sec}} = 2 \times 10^5 p_{\text{rad}} = 2 \times 10^5 \frac{1}{D_A} \text{ AU}$$

$$= (2 \times 10^5)(1.5 \times 10^{-5})\frac{1}{D_A} \text{LY} = \frac{3}{D_A} \text{LY}.$$

Therefore, the distance D_A from the Earth to the star A in light years is

$$D_A = \frac{3}{p_{\text{sec}}},$$

where p_{sec} is the parallax of A measured in seconds.

The parallax of the faint star Proxima Centauri is about 0.76 seconds, or $\frac{1}{5000}^\circ$. Since this is the nearest star, its parallax is the largest. Inserting $p_{\text{sec}} = 0.76$ into the preceding equation shows that the distance to Proxima Centauri is approximately $\frac{3}{0.76} = 4$ LY. Since 1 LY = 6×10^{12} miles, Proxima Centauri is about 24×10^{12} miles distant (see Table 1.5). Stellar parallaxes were measured with good accuracy in the 19th century. For example, Wilhelm Bessel, the German mathematician and astronomer, measured the stellar parallax of 61 Cygni in 1838 and came within 10% of the modern value of 0.27 seconds. Popping $p_{\text{sec}} = 0.27$ into the equation just derived, we find that 61 Cygni is 10.9 LY away. Sirius, the brightest star, has a parallax of about 0.38 seconds. This yields a distance of about 8 LY.

Modern methods for measuring distances to the more distant stars and galaxies include analyzing the convergence or divergence of the motion of individual stars in a cluster to determine the distance of the cluster; and analyzing the light of a particular star to determine its intrinsic luminosity, which, in combination with its brightness as measured from Earth, gives an indication of its distance. The size of the universe is, of course, considerably bigger than even Archimedes speculated. Some galaxies are millions of LYs distant and some have a size of hundreds of thousands of LYs.

Exercises

1A. The Greek Number System

One version (there are variations) of the traditional number system of the Greeks is the following:

Units	Tens	Hundreds
$\alpha = 1$ (alpha)	$\iota = 10$ (iota)	$\rho = 100$ (rho)
$\beta = 2$ (beta)	$\kappa = 20$ (kappa)	$\sigma = 200$ (sigma)
$\gamma = 3$ (gamma)	$\lambda = 30$ (lambda)	$\tau = 300$ (tau)
$\delta = 4$ (delta)	$\mu = 40$ (mu)	$\upsilon = 400$ (upsilon)
$\varepsilon = 5$ (epsilon)	$\nu = 50$ (nu)	$\phi = 500$ (phi)
ϛ $= 6$ (digamma)	$\xi = 60$ (xi)	$\chi = 600$ (chi)
$\zeta = 7$ (zeta)	$o = 70$ (omicron)	$\psi = 700$ (psi)
$\eta = 8$ (eta)	$\pi = 80$ (pi)	$\omega = 800$ (omega)
$\theta = 9$ (theta)	ϙ $= 90$ (koppa)	ϡ $= 900$ (sampi)

(The digamma, koppa, and sampi were taken from an older alphabet of the Phoenicians.)

Other numbers are formed by juxtaposition, using the rule that larger numbers go on the left and smaller ones on the right. For example: $\kappa\varepsilon = 25$, $\lambda\zeta = 37$, $\upsilon\pi\eta = 488$. To designate thousands, the units symbols were used with a stroke before the letter to avoid confusion. For example: $,\gamma = 3000$ and $,\beta\tau\pi\delta = 2384$. (To distinguish between numerals and letters, the Greeks sometimes put a bar over the numerals: $\overline{,\varepsilon\chi o} = 5670$.) For 10,000 an M (for $\mu\upsilon\rho\iota\alpha\sigma$ = myriad) was used. This was combined with other symbols as follows:

$$\beta\text{M} = 20{,}000,$$

$$\iota\delta\text{M},\eta\phi\xi\zeta = 14(10{,}000) + 8567 = 148{,}567$$

$$\upsilon\text{M} = (400)(10{,}000) = 4{,}000{,}000$$

$$\omega\mu\varepsilon\text{M},\beta\tau\pi\delta = 845(10{,}000) + 2384 = 8{,}452{,}384$$

Addition and multiplication were cumbersome; this is probably one reason why Greek algebra lagged behind Greek geometry.

1. Write the numbers 85; 842; 34,547; 2,875,739 using the Greek system.

1B. Greek Algebra

Problems (2)-(5) are taken from *The Greek Anthology*.[8]

2. The Muses stole and divided among themselves, in different proportions, the apples I was bringing from Helicon. Clio got the fifth part, and Euterpe the twelfth, but divine Thalia the eighth. Melpomene carried off the twentieth part, and Terpsichore the fourth, and Erato the seventh; Polyhymnia robbed me of thirty apples, and Urania of a hundred and twenty, and Calliope went off with a load of three hundred apples. So I come to thee with lighter hands, bringing these fifty apples that the goddesses left me. How many apples did I bring from Helicon?

3. Make me a crown weighing sixty minae, mixing gold and brass, and with them tin and much wrought iron. Let the gold and brass together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, how much brass, how much tin, and how much iron, so as to make the whole crown weigh sixty minae. [A number of references describe the mina as a unit of weight roughly equal to one pound. So it seems that this crown was intended for no ordinary mortal.]

4. Throw me in, silversmith, besides the bowl itself, the third of its weight, and the fourth, and the twelfth; and casting them into the furnace stir them, and mixing them all up take out, please, the mass, and let it weigh one mina. [The first thing is to decide what the question is.]

5. Brick-maker, I am in a great hurry to erect this house. Today is cloudless, and I do not require many more bricks, but I have all I want but three hundred. Thou alone in one day couldst make as many, but thy son left off working when he had finished two hundred, and thy son-in-law when he had made two hundred and fifty. Working all together, in how many hours can you make these? [Hint: If there is not enough information, supply it.]

1C. The Quadratic Formula

6. Consider the quadratic polynomial $x^2 - 5x + 4$. Take the x coefficient -5; divide it by 2 to get $-\frac{5}{2}$; squaring gives $\left(-\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$. Now rewrite $x^2 - 5x + 4$ as follows:

$$x^2 - 5x + 4 = x^2 - 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 4.$$

Check that $x^2 - 5x + \left(\frac{5}{2}\right)^2 = \left(x - \frac{5}{2}\right)^2$. Therefore

$$\begin{aligned} x^2 - 5x + 4 &= \left(x - \frac{5}{2}\right)^2 + 4 - \left(\frac{5}{2}\right)^2 \\ &= \left(x - \frac{5}{2}\right)^2 + \frac{16}{4} - \frac{25}{4} \end{aligned}$$

$$= \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}.$$

We have done what is called "completing the square" for the polynomial $x^2 - 5x + 4$. Now answer the following:

i. For which values of x is $x^2 - 5x + 4 = 0$?
ii. What is the least value that $x^2 - 5x + 4$ can have?

7. Solve the equation $3x^2 + 21x + 12 = 0$ by completing the square for the polynomial $x^2 + 7x + 4$.

8. Let a, b, and c be constants and consider the equation $ax^2 + bx + c = 0$. What is x equal to if $a = 0$? If $a \neq 0$, use the strategy of Problems 6 and 7 to show that the solutions are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1D. Rational and Real Numbers

9. The matter of measurability is perhaps most concretely illustrated by the consideration of the numbers that underlie our monetary system. All dollar amounts are expressed in the form: $\$x\frac{yz}{100}$. So only rational numbers, indeed only certain rational numbers, are allowed. In a supermarket one will occasionally find, say, 3 items for a dollar. A single item is not measurable within the system. Why not?

10. Consider the numbers $1.333333\ldots$, $2.676767\ldots$, and $4.728728728\ldots$. Show that they are rational numbers. [Hint: Let r be the number. In the first case, consider $r - 10r$.]

11. What are the decimal expansions of the rational numbers $\frac{5}{4}$ and $\frac{468}{198}$? [Hint: Carry out the divisions.]

Note: It turns out that a real number r is rational precisely if its decimal expansion has a repetitive pattern after some point. For example,

$$234.5\underline{9}999999999\ldots = 234.6 = 234\frac{6}{10}$$

and

$$52.\underline{36}36\ldots = 52\frac{468}{198}$$

are rational numbers. So is

$$35.34672\underline{638}638638\ldots.$$

12. Recall that a prime number is a positive integer $p > 1$ that has no divisors except 1 and p. Euclid pursued the study of prime numbers in Book 9 of the *Elements*. The following fact about prime numbers is known as the Fundamental Theorem of Arithmetic: Every positive integer n is a product $n = p_1^{k_1} \cdots p_i^{k_i}$ of powers of distinct prime numbers $p_1, \ldots, p_i$; and there is only one way of doing this (aside from rearranging the order of the factors). For example, $54 = 2{\cdot}3{\cdot}3{\cdot}3 = 2^1{\cdot}3^3 = 2{\cdot}3^3$ is the unique factorization of 54 into prime powers.

i. Determine the factorizations of 28, 192, and 143 into powers of distinct primes.
ii. Use the Fundamental Theorem of Arithmetic to show that $\sqrt{3}$ is irrational.
iii. Show that a positive integer n is a square precisely when all the exponents of its factorization into primes are even integers.
iv. Let n be a positive integer. Use the Fundamental Theorem of Arithmetic to show that $\sqrt{n}$ is rational only if n is square.

1E. Angles and Circular Arcs

13. Fill in the following blanks.

i. 1 radian = ________ degrees
ii. 1 degree = ________ radians
iii. 78.5° = ________ radians
iv. 1.238 radians = ________ degrees

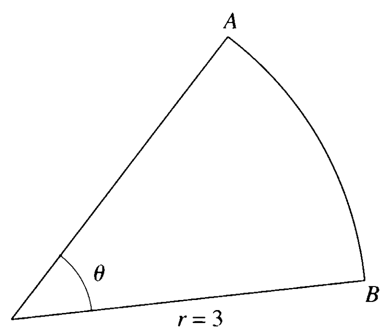

Figure 1.33

14. In the circular sector of Figure 1.33, $\theta = 57.3^\circ$. What is the length of the arc AB?

15. Start with Figure 1.13 of the text. Take the tangent to the circle at A and let P' be its point of intersection with $A'P$. By Figure 1.34, it is plausible that arc $AP \leq AP' + P'P$. Show that $AP' < A'P'$. Conclude that the

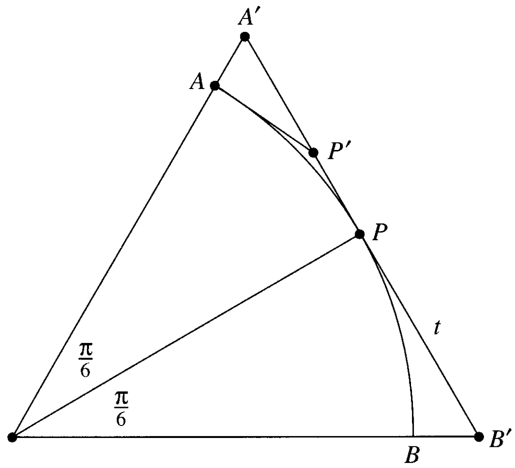

Figure 1.34

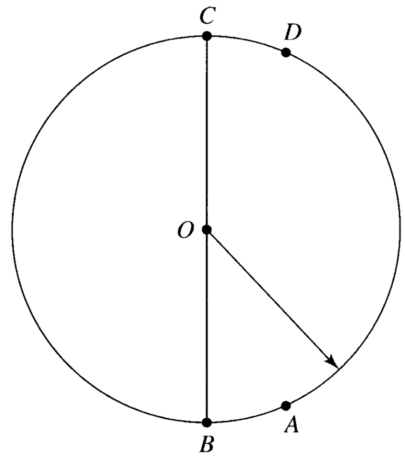

Figure 1.35

inequalities arc $AP < A'P = t$ and arc $AB < A'B' = 2t$ are plausible.

16. A rock is hurled with a sling. Just before it flies off, it is twirled in a circular arc of radius 3 feet at 4 revolutions per second. With what speed (in feet per second) does the rock fly off?

17. The circle in Figure 1.35 has center O and a radius of 2 feet. The arrow is rotating clockwise at a rate of one revolution in twelve hours (so think of it as the hour hand of a clock). The points A and D are positioned in such a way that the segment connecting them is parallel to the diameter BC of the circle. Determine the length of the arc CD, given that the arrow requires 7.5 hours to rotate from A to D.

1F. Basic Trigonometry

18. Use the appropriate triangles to fill in the values:

i. $\cos\frac{\pi}{6} =$ ________	**ii.** $\cos\frac{\pi}{4} =$ ________
iii. $\cos\frac{\pi}{3} =$ ________	**iv.** $\tan\frac{\pi}{6} =$ ________
v. $\tan\frac{\pi}{4} =$ ________	**vi.** $\tan\frac{\pi}{3} =$ ________

19. Use Figure 1.21 of the text to determine the following limits:

i. $\lim\limits_{\theta\to 0} \cos\theta =$ ________
ii. $\lim\limits_{\phi\to \frac{\pi}{2}} \cos\theta =$ ________
iii. $\lim\limits_{\theta\to 0} \tan\theta =$ ________
iv. $\lim\limits_{\phi\to \frac{\pi}{2}} \tan\theta =$ ________

20. Illustrate with a diagram that if $\theta' > \theta > 0$, then $\cos\theta' < \cos\theta$ and $\tan\theta < \tan\theta'$.

21. The secant of θ is defined by $\sec\theta = \frac{1}{\cos\theta}$. Verify the identity $\sec^2\theta = \tan^2\theta + 1$.

22. Compare (use a calculator) the values of α, $\sin\alpha$, $\tan\alpha$ for

i. $\alpha = 0.1$ radians:
$\sin\alpha =$ ________, $\tan\alpha =$ ________.
ii. $\alpha = 0.01$ radians:
$\sin\alpha =$ ________, $\tan\alpha =$ ________.
iii. $\alpha = 0.001$ radians:
$\sin\alpha =$ ________, $\tan\alpha =$ ________.

1G. Distances and Sizes in the Universe

23. Compute r_M, r_S, D_M, and D_S using Aristarchus's argument and

a. Keep $r_E = 3850$ miles.
b. In hypothesis **C**, take $89°50'$ instead of $87°$. (The angle measure ′ is called minute; $1'$ is equal to $\frac{1}{60}^\circ$.)
c. In hypothesis **D**, take $\frac{1}{2}^\circ$ instead of $2°$. (So the angle in Figure 1.27 is $\frac{1}{4}^\circ$ instead of $1°$.)
d. In hypothesis **E**, take $5r_M$ instead of $4r_M$.

Round off to get an accuracy up to 4 decimal places. Compare your conclusions with the modern values from Table 1.4.

24. Both Aristarchus and Archimedes assumed that the Earth is a sphere. Is this reasonable in view of the mountain ranges on its surface? You are given that the radius r_E of the Earth is 3950 miles, that the height of Mount Everest is 29,028 feet, that 1 mile = 5280 feet,[9] and that the radius of a basketball is 4.7 inches. If the Earth were shrunk to the size of a basketball, how high would Mount Everest be? Is this higher than one of the little mounds—officially called a pebble—on a basketball? These have a height of about 0.02 inches.[10]

25. Since 3950 miles $= 20,856,000$ feet and 4.7 inches $=$ 0.39 feet, the shrinking factor in Problem 24 is $\frac{0.39}{20,856,000}$, or about $\frac{1}{50,000,000}$. Show that if the radius r_M of the Moon, the distance D_M to the Moon, the radius r_S of the Sun, the distance D_S to the Sun, and the distance D_* to the nearest star were shrunk by this factor, we would have (approximately):

$r_E = 4.7$ inches (the radius of a basketball, as we have already seen)

$r_M = 1.37$ inches (about the radius of a baseball which is 1.43 inches)

$D_M = 25$ feet

$r_S = 45$ feet

$D_S = 1.86$ miles

$D_* = \frac{1}{2}$ million miles

26. Shrink the Sun to the size of a basketball. What is the shrinking factor? Shrink the rest of the solar system by this factor and compute r_E, r_M, D_M, r_S, D_S, and D_* in this case.

27. Reconstruct Archimedes's argument that 10^{63} grains of sand more than fill the sphere of the universe. Assume that 1 finger breadth is $\frac{2}{3}$ of an inch.

28. The near stars Barnard, G51-15, and Ross 780 have stellar parallaxes of 0.55, 0.27, and 0.21 seconds, respectively. Determine their distances in light years.

Notes

[1]This theorem was probably already known to the Babylonians in 1700 B.C., more than 1000 years before the time of Pythagoras (about 570 to 500 B.C.).

[2]The Pythagorean philosophy was not without later influence, however. The book by G.L. Hershey, *Pythagorean Palaces* (Cornell University Press, Ithaca N.Y. and London, 1976), was written to establish the fact that in "the Italian Rennaissance domestic architecture was largely ruled by Pythagorean principles."

[3]This is a question that occupied the mathematicians of antiquity and is still relevant today. Our discussion will give only a very elementary perspective.

[4]When a segment, such as $A'P$ or PB', appears in a mathematical expression, the reference here and elsewhere in this text will always be to its length.

[5]A gnomon is simply a straight stick or rod. The word comes from gno, "to know," in ancient Greek. Our words prognosis and physiognomy are derived from it.

[6]The Greeks used stadia, not miles. Ten stadia are the equivalent of about one mile.

[7]The detail on the Earth in the figures that follow should not mislead the reader into thinking that the Greeks of the third century B.C. understood the scope and shape of Africa, Northern Europe, and the Atlantic Ocean. The Greeks were very familiar with the territory near the Mediterranean Sea and the armies of Alexander the Great advanced all the way to today's India and Afghanistan. However, a true concept of the extent of the continental land masses and oceans began to develop only with the voyages of discovery in the 15th century.

[8]*The Greek Anthology* is a collection of Greek poems, songs, and riddles. Some of these were compiled as early as the 7th century B.C. and others as late as the 10th century A.D. Harvard University Press produced a new edition of *The Greek Anthology* in 1993.

[9]So Mount Everest is about 5.5 miles high. As an aside, note the Earth is actually not quite a sphere. The Earth's rotation has caused an "equatorial bulge," so that the Earth's diameter through the equator is about 26 miles more than that through the poles.

[10]According to Rawlings Sporting Goods, the official circumference of a basketball is from 29.5 to 30 inches. This translates to a radius r of 4.695 to 4.775 inches. A pebble has an official height from 0.013 to 0.025 inches.