CHAPTER II

GRAPHICAL REPRESENTATION OF EQUATIONS

9. Variables and constants.

I. DEFINITION. *A* ***variable*** *is a number whose value may change, arbitrarily, or in accordance with some law.*

The speed of a train changes as it gathers headway on leaving a station, hence the number which expresses the speed is a variable. A stone thrown into the air changes its distance from the ground from moment to moment, hence the number which expresses this distance is a variable.

II. DEFINITION. *A number which does not change in value in the course of any discussion is called a* ***constant.***

In analytic geometry the equations employed contain variables and constants. The constants are either definite numbers, as 3, $\frac{5}{8}$, π, log 2, $\sqrt{2}$, etc., or they may be represented by letters which stand for quantities whose values are assumed to be known in the problem under discussion.

10. Graphical representation of equations. Examples.

1. Let $x + y = 2$ be an equation in two variables x and y. The equation states in algebraic language that two variable numbers, x and y, are so related to each other that their sum is always 2. Evidently any value may be assigned to either x or y, and then the value of the other is determined. Thus if $x = 3$, $y = -1$; if $x = 1$, $y = 1$; if $x = -4$, $y = 6$; if $x = \frac{5}{2}$, $y = -\frac{1}{2}$; etc. If this be done systematically and the results arranged in a table we have, taking at first only integral values:

$x =$	-4	-3	-2	-1	0	1	2	3	4	etc.
$y =$	6	5	4	3	2	1	0	-1	-2	

Draw a pair of coordinate axes and locate the points which have these corresponding pairs of values of x and y as abscissas and ordinates respectively, thus obtaining the points P_1, P_2, $\cdots$, P_9, Fig. 14 (a). Through the points thus obtained draw a line, as in Fig. 14 (b). This is found to be a straight line. As

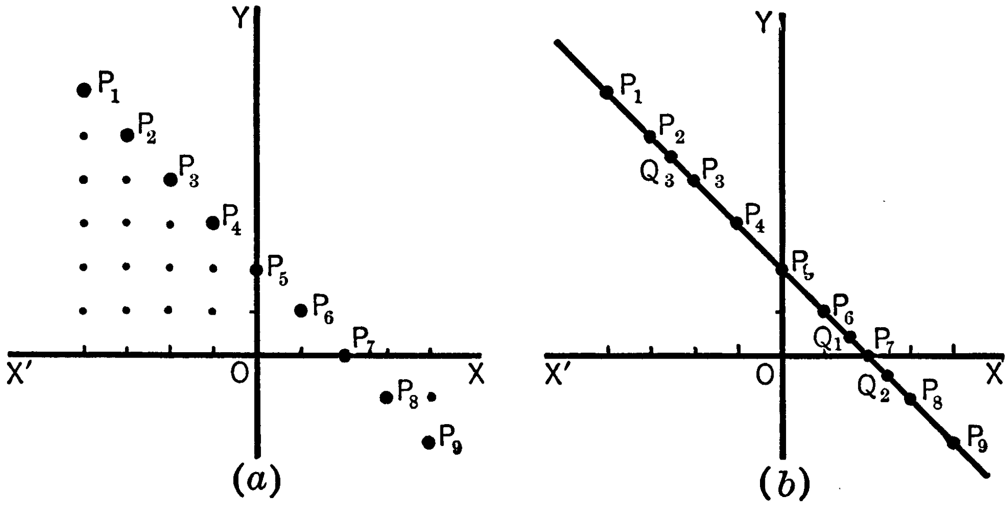

FIG. 14

many additional pairs of values of x and y as we please can be found from the equation, and the corresponding points located. Thus $x = \frac{3}{2}$, $y = \frac{1}{2}$ satisfy the equation and give the point Q_1; $x = 1 + \sqrt{2}$, $y = 1 - \sqrt{2}$ satisfy the equation and give the point Q_2; $x = -\sqrt{7}$, $y = 2 + \sqrt{7}$ satisfy the equation and give the point Q_3.

If it were possible to construct all of the infinitely many points whose coordinates satisfy the equation, we should find that they all lie on the line which has been drawn. This straight line is therefore a graphical representation (called briefly the **graph**) of the equation $x + y = 2$.

2. Construct the graph of $x^2 - 6x - 4y - 5 = 0$.

First solve the equation for y, thus

$$y = \tfrac{1}{4}(x^2 - 6x - 5).$$

Assigning values to x, and computing the corresponding values of y, the results are as follows:

$x =$	-2	-1	0	1	2	3	4	5	6	7	8
$y =$	$\frac{11}{4}$	$\frac{1}{2}$	$-\frac{5}{4}$	$-\frac{5}{2}$	$-\frac{13}{4}$	$-\frac{7}{2}$	$-\frac{13}{4}$	$-\frac{5}{2}$	$-\frac{5}{4}$	$\frac{1}{2}$	$\frac{11}{4}$
point	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}

Plotting the points thus determined it is evident that they do not lie on a straight line as was the case in example 1. If through the points thus located a curve be sketched smoothly the result is as shown in Fig. 15.

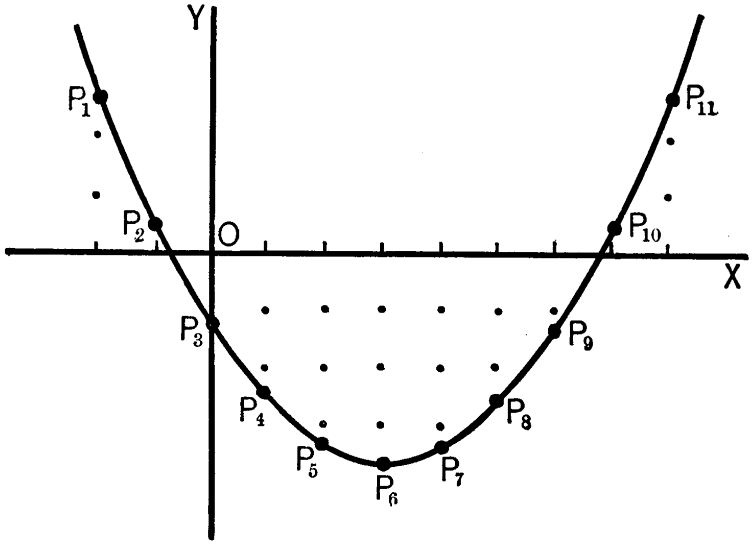

FIG. 15

3. Construct the graph of $y = x^3 - 6x^2 + 11x - 3$.

$x =$	-1	0	1	2	3	4	5
$y =$	-21	-3	3	3	3	9	27
point	*	P_1	P_2	P_3	P_4	P_5	*

* The points $(-1, -21)$ and $(5, 27)$ are omitted because the numerical values of y are too large to go on the diagram conveniently.

The five points P_1, P_2, $\cdots$, P_5 are shown by heavy dots in Fig. 16 (a). From these alone the true shape of the graph is uncertain, and some intermediate points must be found. When

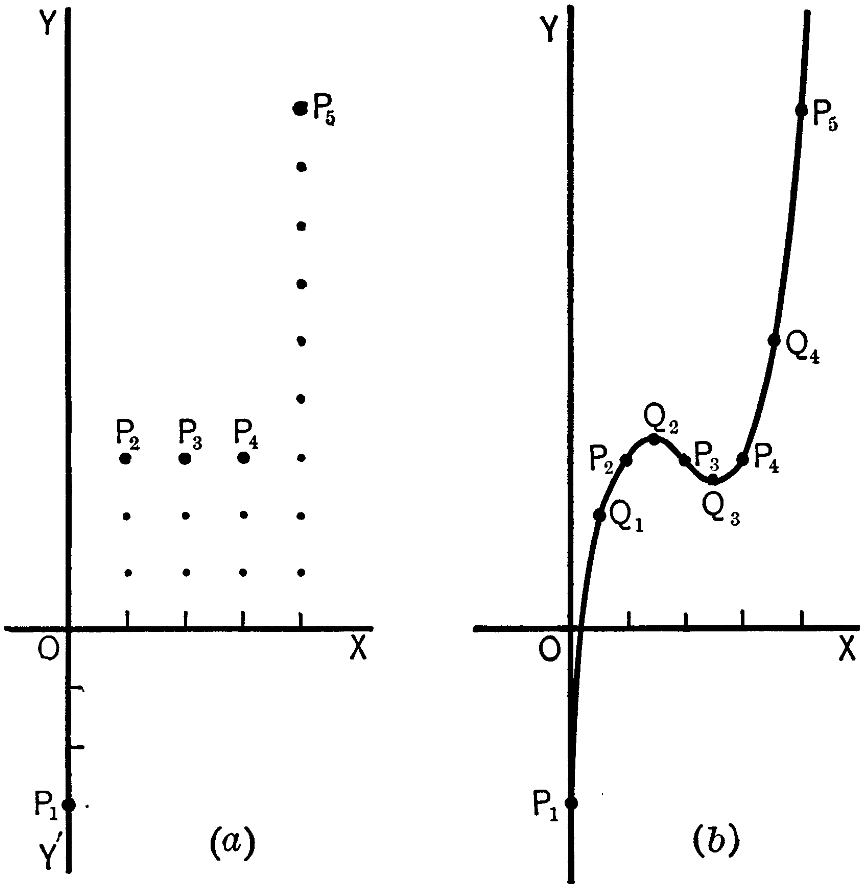

FIG. 16

the additional points $Q_1(0.7, 2.1)$, $Q_2(\frac{3}{2}, \frac{27}{8})$, $Q_3(\frac{5}{2}, \frac{21}{8})$, $Q_4(\frac{7}{2}, \frac{39}{8})$ are plotted in Fig. 16 (b), and the curve drawn through them all, the result is as shown in the diagram.

If still greater accuracy is required, especially in that portion of the curve between P_2 and P_4, more values of x must be taken between 1 and 3, and a larger scale must be used. Thus:

$x = 1.2$	1.4	1.75	2.25	2.6	2.8
$y = 3.29$	3.38	3.23	2.77	2.62	2.71
point R_1	R_2	R_3	R_4	R_5	R_6

These with the points P_2, P_3, P_4, Q_2, Q_3 already found are plotted on a larger scale in Fig. 17 and the corresponding portion

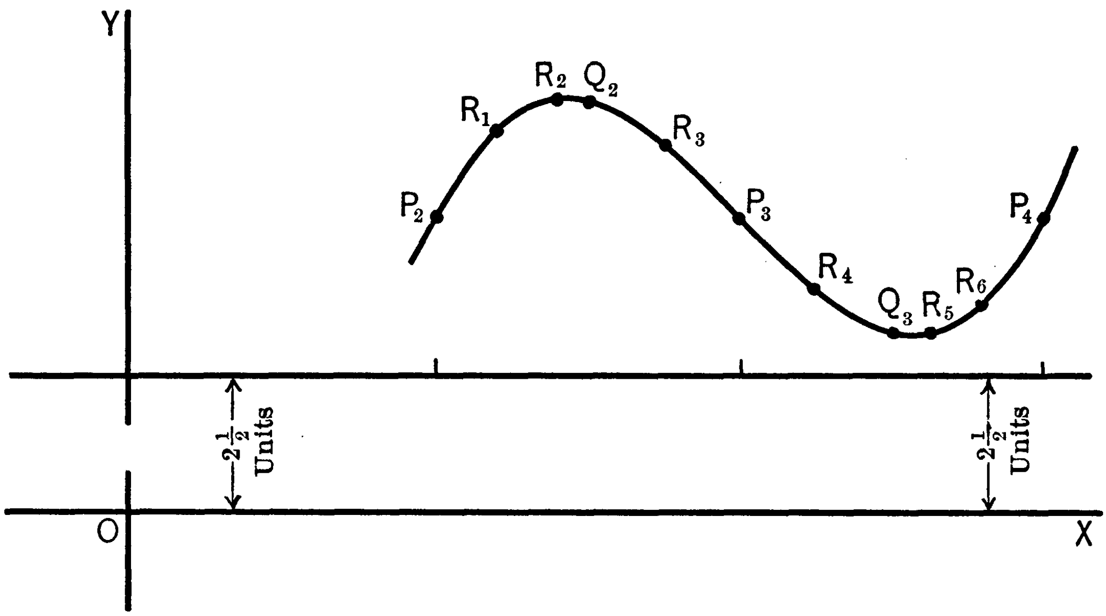

FIG. 17

of the curve sketched through them.

EXERCISES. 1. Construct the graph of each of the following equations

(*a*) $y = 2x + 3$, (*c*) $x - 2y - 3 = 0$, (*e*) $y = x$,
(*b*) $3y = 4x + 5$, (*d*) $x - 2y + 3 = 0$, (*f*) $y = -x$.

2. Construct the graph of each of the following equations

(*a*) $y = 4x^2 + 7$, (*d*) $y = x^3$,
(*b*) $x^2 + 4y - 2x - 3 = 0$, (*e*) $y = x^3 - 6x^2 + 11x - 6$,
(*c*) $y^2 = x - 2$, (*f*) $y = x^4 - x^2 + 2$.

3. Construct on one diagram the graphs of $3x + 4y - 5 = 0$ and $3x + 4y + 2 = 0$. What relation do these graphs bear to each other?

4. Construct on one diagram the graphs of $4y = x^2$ and $4y = x^2 + 8$. In what respect do these two graphs differ?

5. Construct on one diagram the graphs of $4y = x^2$ and $8y = x^2$. In what respect do these two graphs differ?

11. Graphical representation of equations, continued. The preceding examples show that the construction of the graph of an equation in two variables x and y consists fundamentally in finding pairs of corresponding values of the variables which satisfy the equation. Points having these coordinates are then marked on the diagram, and the curve or graph is drawn through them. Care must be taken to use enough values to give the curve completely and correctly. The actual labor involved may often be considerably reduced by a careful examination of the form of the equation before any numerical values are substituted in it. The following additional examples will illustrate some of the ways in which this can be done.

1. Construct the graph of $4x^2 + 9y^2 = 36$.

It is often well to begin by examining the equation to see whether the curve passes through the origin, and by finding where it cuts the coordinate axes. In this case the curve does not pass through the origin, because the coordinates (0, 0) do not satisfy the equation.

To find where the curve cuts the X-axis, make $y = 0$ in the equation and solve for x, because at each point where the curve cuts the X-axis the ordinate is zero. Doing this in the equation given above we have $x^2 = 9$, or $x = \pm 3$. Hence the curve cuts the X-axis at two points A, A', three units on either side

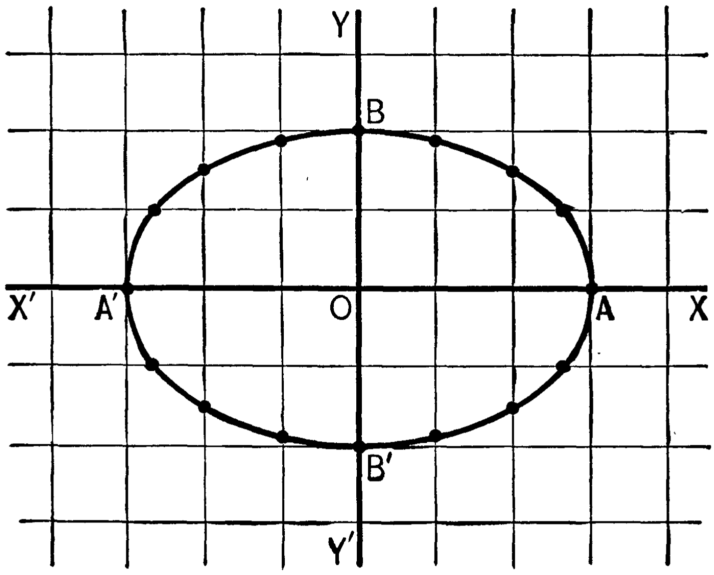

FIG. 18

of the origin. By the same reasoning, making $x = 0$, $y = \pm 2$, and hence the curve cuts the Y-axis at B, B', two units above and below the origin.

If the equation be solved for y,

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}, \qquad (i)$$

from which the following conclusions are drawn:

(*a*) For every value of x there are two equal values of y with opposite signs. Hence the curve is symmetrical with respect to the X-axis.

(*b*) Since x occurs only in the second degree, equal positive and negative values of x lead to the same values of y. Hence the curve is also symmetrical with respect to the Y-axis.

(*c*) If $x^2 > 9$, y is imaginary. Hence the only values of x which make y real lie between -3 and $+3$ and the curve does not extend beyond these limits.

If the equation be solved for x, the result is

$$x = \pm \tfrac{3}{2}\sqrt{4 - y^2}, \qquad (ii)$$

from which, in addition to (*a*), (*b*), (*c*) above, it appears that if $y^2 > 4$, x is imaginary, and hence the curve is included between $y = -2$ and $y = +2$.

To construct the curve some pairs of corresponding values of x and y must be computed. For this purpose either (*i*) or (*ii*) may be used. From the former

$x =$	-3	-2.7	-2	-1	0	$+1$	$+2$	$+2.7$	$+3$
$y =$	0	± 0.87	± 1.49	± 1.89	± 2	± 1.89	± 1.49	± 0.87	0

and the resulting curve is drawn in Fig. 18.

It may be asked what will happen if a value of x greater than 3 or less than -3 be substituted in the equation. It has been shown that the curve does not extend beyond these limits, yet it is evident that any value we please may be substituted for either x or y. Let $x = 4$ in (i), then the resulting values of y are $\pm \frac{2}{3}\sqrt{-7}$, a pair of imaginary numbers. Hence the corresponding points cannot be constructed. Nevertheless, since the two pairs of coordinates $(4, +\frac{2}{3}\sqrt{-7})$, $(4, -\frac{2}{3}\sqrt{-7})$ both satisfy the equation they are said to be the coordinates of two **imaginary points** on the curve.

2. Construct the graph of $x^2 - y^2 = a^2$.

GENERAL DEDUCTIONS. The curve does not pass through the origin; it cuts the X-axis at $x = +a$, and $x = -a$; it does not cut the Y-axis, because when $x = 0$, y is imaginary.

Since x and y are involved only in the second power the curve is symmetrical with respect to both axes.

Since $y = \pm\sqrt{x^2 - a^2}$, all values of x between $-a$ and $+a$ make y imaginary, hence the curve does not exist in the part of the plane between the two lines parallel to $Y'Y$ at a distance a on either side of it.

Since $x = \pm\sqrt{a^2 + y^2}$, all values of y lead to real values of x.

The curve therefore consists of two distinct portions, one lying on the positive side of $x = a$, and the other on the negative side of $x = -a$.

This equation differs in one important respect from those treated up to this point, in that it contains a general constant a^2. Moreover the equation $x^2 - y^2 = a^2$ is homogeneous (See A, I, (c), p. vii) in the three letters x, y, a, and therefore since x and y represent lengths, a is a length also. When an equation contains a general constant, and is homogeneous in x, y and the constant, the best practice in plotting the graph is to assume some arbitrary length for the constant, and then take the values of x or y as multiples of the constant. Thus if $x = 2a$, $y = \pm a\sqrt{3}$, and if $y = \frac{1}{2}a$, $x = \pm\frac{1}{2}a\sqrt{5}$. It is easily seen that with different assumed lengths for the constant the resulting figures will be alike in every respect except size, for the effect of a change in the length assumed for the constant will be merely to change all linear distances in the figure proportionally.

Taking the equation in the form $x = \pm \sqrt{a^2 + y^2}$, the following table of values is computed.

$y =$	0	$\frac{1}{2}a$	a	$\frac{3}{2}a$	$2a$	$3a$	etc.
$x =$	$\pm a$	$\pm 1.12a$	$\pm 1.41a$	$\pm 1.80a$	$\pm 2.24a$	$\pm 3.16a$	

For negative values of y the resulting values of x are the same as for the corresponding positive values of y. The portion of

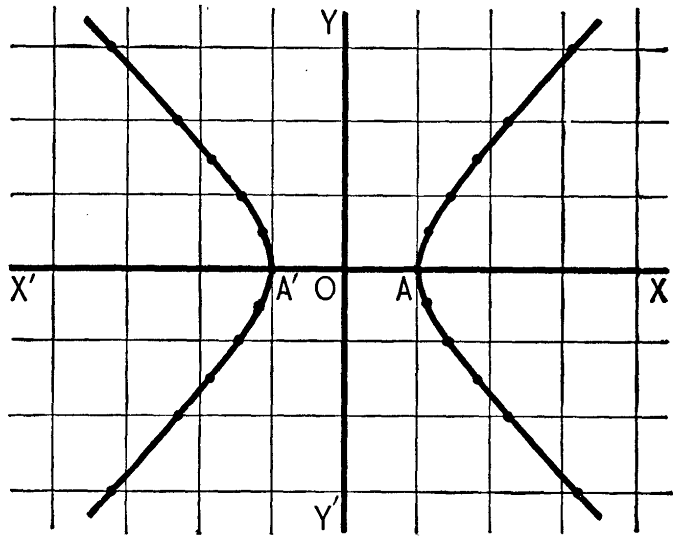

Fig. 19

the curve between $y = -3a$, and $y = +3a$ is drawn in Fig. 19.

3. Plot the graph of

$$y = \frac{3cx}{cx^2 + 2}.$$

This equation like that of example 2 contains a general constant c, but the equation is not homogeneous in the constant and variables. This will perhaps be more clearly seen if the equation be cleared of fractions, $cx^2y + 2y - 3cx = 0$. Hence c is not a length and some numerical value must be assigned to it in order to plot the graph. Let $c = \frac{5}{2}$, then the equation becomes $5x^2y + 4y - 15x = 0$.

Putting $x = 0$ the only resulting value of y is $y = 0$, hence the graph passes through the origin and does not have any other intersection with the Y-axis. Putting $y = 0$, the equation gives $x = 0$, showing that the graph cuts the X-axis only at the origin. It should be noted, however, that as the numerical value of x increases indefinitely, the value of y approaches zero as a limit. To show this solve the equation for y, thus

$$y = \frac{15x}{5x^2 + 4}. \qquad (i)$$

Then dividing numerator and denominator by x

$$y = \frac{15}{5x + \frac{4}{x}}. \qquad (ii)$$

From (ii) it is clear that as the numerical value of x increases indefinitely y approaches zero as a limit. Changing the sign of x, but not its value, changes only the sign of y. Hence the graph is symmetrical with respect to the origin. Using (i) the following table of values for x and y is computed.

$x = -5$	-4	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
$y = -\frac{75}{129}$	$-\frac{5}{7}$	$-\frac{45}{49}$	$-\frac{5}{4}$	$-\frac{90}{61}$	$-\frac{5}{3}$	$-\frac{10}{7}$	0 etc.
$= -0.58$	-0.71	-0.92	-1.25	-1.48	-1.67	-1.43	0

The resulting graph is drawn in Fig. 20.

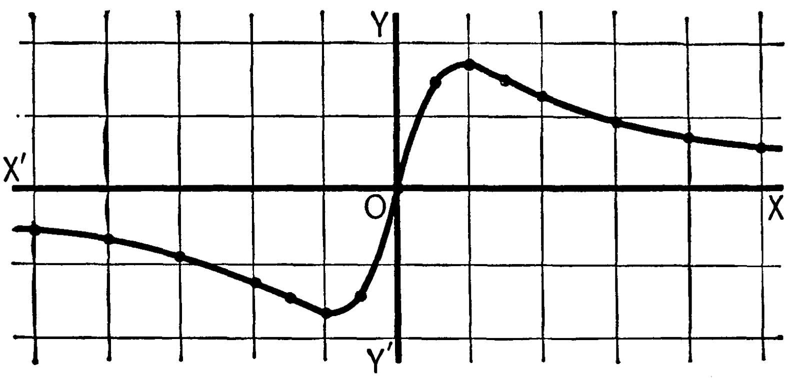

FIG. 20

It will be a valuable exercise to assume different values for c in this equation and construct the corresponding curves, noting the changes thus produced. These will be especially conspicuous if both positive and negative values of c are taken. The student will do well, however, not to consider negative values of c until after he has studied example 4, immediately following.

4. Plot the graph of

$$y = \frac{x(x-2)}{(x+2)(x-1)}.$$

Making $x = 0$, the equation gives $y = 0$, showing that the graph passes through the origin.

Making $y = 0$, the equation gives $x(x-2) = 0$, or $x = 0$ and $x = 2$, showing that the graph cuts the X-axis at (2, 0) as well as at the origin. Thus two points O and A are located on the curve.

For values of x which make one or three of the four factors x, $x - 2$, $x + 2$, $x - 1$, negative, y will be negative, otherwise y will be positive. For all values of $x < -2$ these four factors are all negative, hence y is positive. To the left of $M'M$ therefore the curve lies above $X'X$. If $-2 < x < 0$ (read "x is between -2 and 0") $x + 2$ is positive, but the other three factors are negative, hence y is negative, and the curve lies below $X'X$ between $M'M$ and $Y'Y$. If $0 < x < 1$, y is positive, because two factors x and $x + 2$

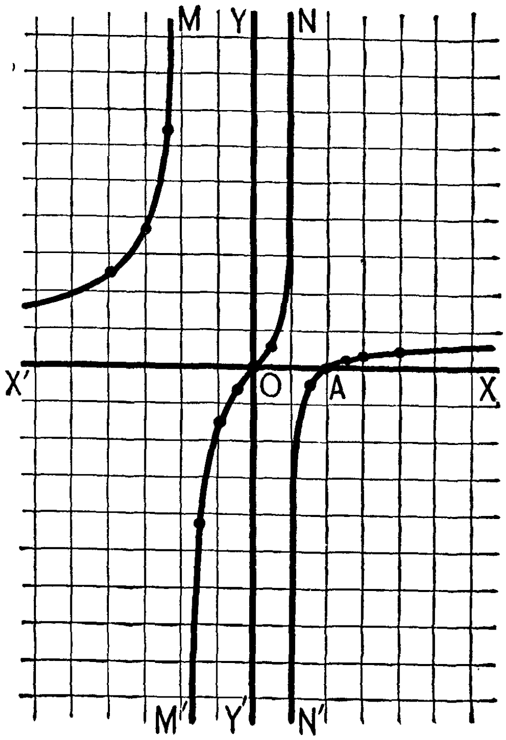

FIG. 21

are positive and the other two negative, hence between $Y'Y$ and $N'N$ the curve lies above OX. Similarly if $1 < x < 2$, y is negative, so that between $N'N$ and A the curve is below OX, and if $x > 2$, y is positive, and hence to the right of A the curve is above OX.

Special attention must be given to values of x in the neighborhood of -2 and $+1$. As x approaches these values, from either side, the corresponding numerical values of y increase without limit, and when $x = -2$ or $+1$ the denominator of the fraction vanishes, and hence the value of y is undefined. This is customarily expressed under such circumstances by writing $y = \infty$.

Without any further information the curve can now be roughly sketched in, but in order to construct it with accuracy the coordinates of a few points must be computed.

$x = -4$	-3	$-2\frac{1}{2}$	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{3}{2}$	$+2\frac{1}{2}$	$+3$	$+4$
$y = \frac{12}{5}$	$\frac{15}{4}$	$\frac{45}{7}$	$-\frac{21}{5}$	$-\frac{3}{2}$	$-\frac{5}{9}$	$\frac{3}{5}$	$-\frac{3}{7}$	$\frac{5}{27}$	$\frac{3}{10}$	$\frac{4}{9}$
$= 2.4$	3.75	6.43	-4.2	-1.5	-0.56	0.6	-0.43	0.18	0.3	0.44

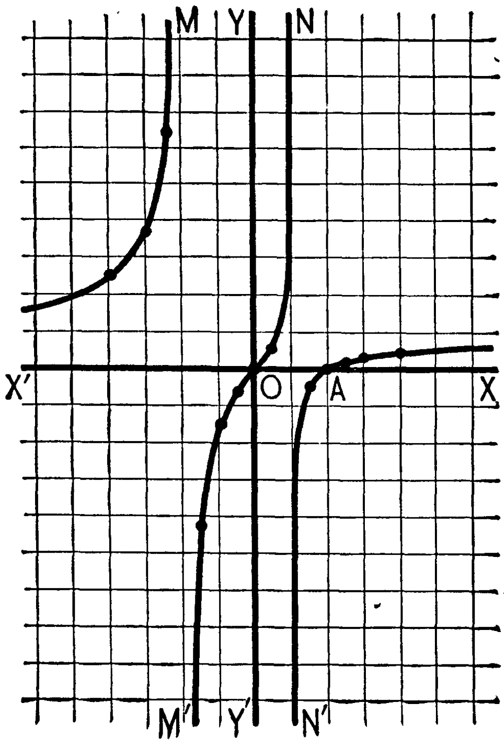

FIG. 21

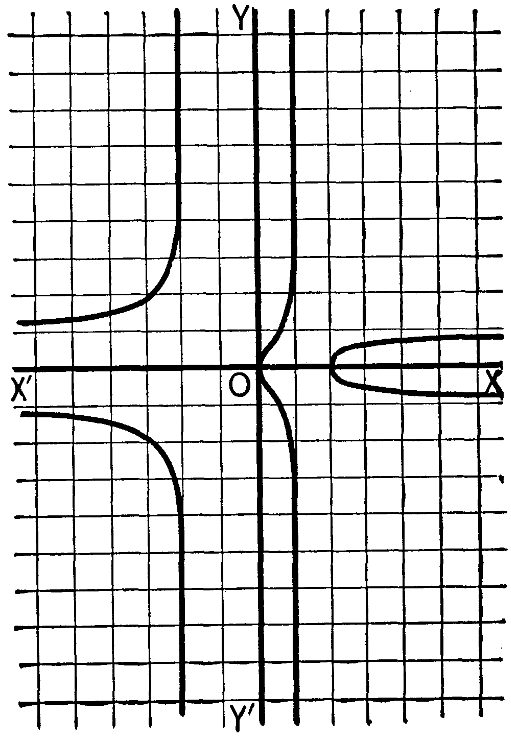

FIG. 22

The part of the curve between these limits is drawn in Fig. 21.

5. Let the student show that the curve in Fig. 22 is the graph of

$$y^2 = \frac{x(x - 2)}{(x + 2)(x - 1)}.$$

12. The fundamental principle of analytic geometry.

From the examples discussed in Arts. 10, 11, it is clear that the graph of an equation has the nature of a locus. It is in fact the locus of points whose coordinates satisfy the equation. Hence the expressions *graph of an equation*, and *locus of an equation* will be used interchangeably in this book.

Definition. *The* ***graph*** *or* ***locus*** *of an equation in two variables is that straight line or curve upon which lies every point whose coordinates are pairs of values of the variables which satisfy the equation; and the coordinates of every point on which satisfy the equation when substituted for the variables.*

This definition is the foundation upon which the whole structure of analytic geometry is built. The measure of the student's apprehension of the principle here set forth will be the measure of his success in mastering the subject.

13. Practical Suggestions.—In following the discussion of the examples in Arts. 10, 11, the student will have seen that the work does not consist exclusively in determining definite corresponding pairs of values of x and y, but that it is desirable by careful examination of the equation to learn as much as possible about the general nature of the graph before undertaking the systematic computation of pairs of values of x and y. Attention was directed in some of the examples to definite methods of procedure by which this can be done. These consist in determining whether the locus passes through the origin, where it cuts the axes, whether it has symmetry with respect to either axis, or with respect to the origin, the range of values of one variable (if any) which make the other variable imaginary, and the values of one variable (if any) which make the other infinite.

Before undertaking to substitute numerical values for x and y in the equation it is usually best to solve the equation for one of the variables, choosing the one for which this can be done more easily. It will sometimes happen that the equation cannot be solved for either variable. For example, $x^3 + y^3 - 3xy = 0$. Values of x or y may then be substituted in the equation as it stands, and the resulting equation solved for the other variable.

From what has been said it may be concluded that no general rules of procedure can be formulated which apply to all, or even to a majority of cases. The student must depend to a great extent upon his own ingenuity, and precisely for this reason he will find the plotting of loci, or curve tracing as it is called, a valuable exercise, and one which will abundantly repay considerable effort to secure some mastery of it.

EXERCISES. 1. Construct the graph of each of the following equations

(*a*) $xy = 4$, (*b*) $4x^2 + y^2 = 16$, (*c*) $4x^2 - y^2 = -36$,

(*d*) $x^2 + y^2 = 25a^2$, (*e*) $ay^2 = x^3$, (*f*) $xy = c + ax + y$,

(*g*) $y^3 + x^3 = 8$, (*h*) $y^4 + x^4 = 16$, (*i*) $x^3 + y^3 - 4x + 4 = 0$,

(*j*) $y = \dfrac{a^3}{x^2 + a^2}$, (*k*) $y^2 = \dfrac{a^4}{x^2 + 4a^2}$, (*l*) $y^2 = \dfrac{x^2 - 4x}{x + 1}$,

2. Construct on one diagram the graphs of the following equations, and note the differences in the graphs due to the differences in sign in the equations $x^2 + y^2 = 25$, $x^2 - y^2 = 25$, $-x^2 + y^2 = 25$.

3. Construct on one diagram the graphs of the following equations, and note the differences in the graphs due to differences in the values of the numerical coefficients $x^2 + y^2 = 36$, $x^2 + 4y^2 = 36$, $4x^2 + y^2 = 36$, $x^2 + 4y^2 = 25$.

14. Intersection of loci.—Since the graph of an equation is the locus of points whose coordinates satisfy the equation, it follows that, if two loci are plotted on the same figure, the coordinates of the points where they intersect must satisfy both equations. Hence

RULE. *To determine the coordinates of the points of intersection of two loci solve their equations simultaneously for x and y.*

EXAMPLES. 1. Find the intersection of the loci of $3x - 7y + 2 = 0$ and $x + 2y - 6 = 0$.

Solving the equations for x and y the results are $x = \frac{38}{13}$, $y = \frac{20}{13}$, the coordinates of the required point.

Construct the loci and verify the results from the figure.

2. Determine the intersections of the loci of $x^2 + y^2 = 25$ and $3x + 4y = 25$.

From the second equation $y = \frac{1}{4}(25 - 3x)$, and substituting this value in the first the result is

$$x^2 + \tfrac{1}{16}(25 - 3x)^2 = 25,$$

which reduces to

$$x^2 - 6x + 9 = 0, \qquad \text{or} \qquad (x - 3)^2 = 0,$$

a quadratic with *two equal roots*, $x = 3$.

Substituting $x = 3$ in $x^2 + y^2 = 25$ gives $y = \pm 4$, and making the same substitution in $3x + 4y = 25$ gives $y = 4$ only. The latter is therefore the only value of y which satisfies *both* equations when $x = 3$. Each of the two coincident values $x = 3$ leads therefore to the same value of y, viz., $y = 4$, and the loci are said to meet in two coincident points (3, 4).

Draw the figure and show that the line $3x + 4y = 25$ is tangent to the circle $x^2 + y^2 = 25$ at (3, 4).

15. Solution of equations by intersection of loci.—The approximate solution of a pair of simultaneous equations in two variables can be obtained graphically from the loci of the equations. This method is useful when the algebraic solution cannot be conveniently performed.

For example, let it be required to solve for x and y the equations $x^2 + y^2 - 7x + 2y - 3 = 0$ and $xy + 2x - 4y - 2 = 0$.

The loci of these two equations are drawn in Fig. 23. The points of intersection are P_1 and P_2, whose coordinates are found by measurement to be approximately (2.8, 2.9) and (– 0.4, – 0.6). From the principles of algebra it is evident that there are two other pairs of values of x and y, but since the loci intersect visibly at P_1 and P_2 only, the other solutions are imaginary.

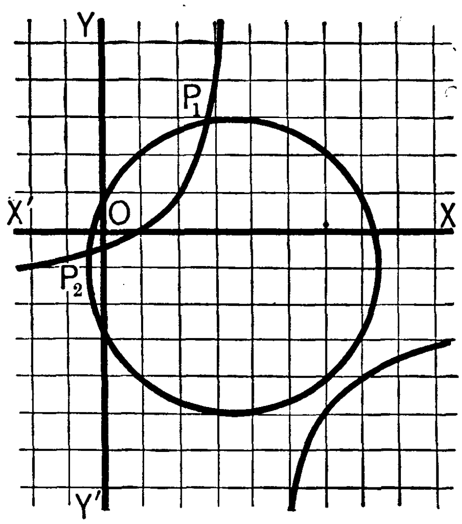

FIG. 23

16. Imaginaries in analytic geometry. — In several of the preceding discussions of this Chapter reference has been made to imaginary values of x or y. See Ex. 1, p. 21, Ex. 2, p. 22, and the end of Art. 15. These serve to show that imaginary points can exist on real loci, which is only a way of stating in geometric language that in general an equation in two variables can be satisfied by imaginary values of the variables. There are also equations whose loci are wholly imaginary, or which contain at most only a limited number of real points.

DEFINITION. *An equation which is satisfied by only imaginary values of the variables, or by at most a finite number of real values, is called the equation of an* ***imaginary locus.***

Three simple types of equations fall under this classification.

(*a*) *Equations one or more of whose coefficients are imaginary numbers.* For example

$$y^2 - (2 + i)x + (3 - 2i)y + 2 = 0. \quad [i = \sqrt{-1}]$$

Putting any real or imaginary value for one variable in this equation and solving for the other variable, the result will in general be imaginary. If, however, the equation be written in the form

$$y^2 - 2x + 3y + 2 - (x + 2y)i = 0$$

it will evidently be satisfied for values of x and y which make

$$y^2 - 2x + 3y + 2 = 0, \qquad \text{and} \qquad x + 2y = 0$$

simultaneously.* Solving these two equations therefore, for x and y, we find the two pairs of real values $x = 7 + \sqrt{41}$, $y = -\frac{7}{2} - \frac{1}{2}\sqrt{41}$, and $x = 7 - \sqrt{41}$, $y = -\frac{7}{2} + \frac{1}{2}\sqrt{41}$, both of which satisfy the original equation, and therefore determine two real points on the imaginary locus. Moreover, these are the only pairs of real values of x and y which can satisfy the given equation.

(*b*) *Equations all of whose coefficients are real, but which are not satisfied by any real values of the variables.*

Equations of this type consist of, or can be reduced to an even power of a variable term, or the sum of two or more such even powers, equated to a negative constant. For example

$$(x - 2)^2 + y^2 = -3.$$

As the square of a real number is always positive it is clear that an equation of this type cannot be satisfied by any real values of x and y.

(*c*) *Equations all of whose coefficients are real, but which can be satisfied by only a limited number of pairs of real values of x and y.*

Equations of this type can be reduced to a sum of even powers of variable terms equated to zero. For example $x^2 + y^2 = 0$ is satisfied by the single pair of real values $x = 0$, $y = 0$. Similarly

$$(x^2 - y^2)^2 + (2x - y + 4)^2 = 0$$

is satisfied by those values of x and y which make $x^2 - y^2 = 0$ and $2x - y + 4 = 0$ simultaneously. Solving these two equations for x and y, we find $x = -4$, $y = -4$, and $x = -\frac{4}{3}$, $y = \frac{4}{3}$, which determine the only two real points on the locus.

* See IV (*c*) p. viii.

EXERCISES. 1. Find from the equations the coordinates of the points of intersection of the graphs of each of the following pairs of equations. Check the results graphically.

(*a*) $x - 2y + 5 = 0, \quad x^2 + y^2 = 25;$ Ans. (3, 4), (− 5, 0).
(*b*) $2y^2 = x, \quad 4x^2 + 9y^2 = 25;$ Ans. (2, ± 1).
(*c*) $16y - 15x + 99 = 0, \quad x^2 - 4y^2 = 16;$

2. Show by finding the coordinates of their points of intersection that the graphs of the following pairs of equations are tangent to each other. Check graphically.

(*a*) $x^2 + y^2 = 25, \quad 3x + 4y - 25 = 0;$
(*b*) $4x^2 + 9y^2 = 25, \quad 8x + 9y + 25 = 0;$
(*c*) $xy = 4, \quad x + y - 4 = 0;$

3. Find approximate values graphically for the coordinates of the points of intersection of the graphs of the following pairs of equations

(*a*) $x^2 - 4y^2 = 4, \quad y = \dfrac{8}{4 + x^2};$ (*c*) $y^2 = \dfrac{x(x-1)}{x^2 - 4}, \quad 8y = x^3.$
(*b*) $x^2 + y^2 = 25, \quad y^2 - x^3 = 0;$

4. Find the coordinates of the real points in each of the graphs whose equations are

(*a*) $y^2 + (3 + i)x + (7i - 4)y - 3 - 23i = 0;$
Ans. (2, 3), (− 131, 22).
(*b*) $(2 + i)x + (3i - 4)y + 10 - 5i = 0;$ Ans. (− 1, 2).
(*c*) $(1 + i)x^2 + (1 - i)y^2 - 13 + 5i = 0.$ Ans. (± 2, ± 3).

5. Show that there is but one real point on each of the graphs whose equations are

(*a*) $(3 + \sqrt{-4})x + (5\sqrt{-1} - 4)y - 6 + 19\sqrt{-1} = 0;$
(*b*) $x^2 + i(y^2 + x^2) + 2ixy - 4x + 4 = 0;$
(*c*) $x^2 + y^2 - 4x + 6y + 13 = 0.$

6. Show that there is no real point on the graph of any of the following equations

(*a*) $2x^2 + y^2 + ix - 2(2 + i)y + 40 - 4i = 0;$
(*b*) $x^2 + y^2 + 4 = 0;$ (*c*) $x^2 + 4y^2 - 2x + 5 = 0.$

7. Show graphically that the roots of the following pairs of equations are imaginary

(a) $x^2 + y^2 = 4, \qquad x + y = 4;$
(b) $y^2 - 4x + 4 = 0, \qquad y^2 + 4x + 4 = 0;$

17. Functions.—The notation $f(x, y)$ is often used to denote in general an expression containing the variables x and y. It is read: "function of x and y," or "f of x and y." Hence $f(x, y) = 0$ is a short way of representing in algebraic language any equation in the two variables x and y all the terms of which have been transposed to the first member. In a specific problem or discussion $f(x, y)$ may stand for an expression of definite form, and when so used it represents the same form of expression throughout the discussion. Thus in one discussion we may have $f(x, y) \equiv x^2 + y^2 - a^2$, in another $f(x, y) \equiv 3x + 2y - 6$, etc.

If $f(x, y) \equiv x^2 - 3y^2 + 6x - 4$, then in the same discussion $f(a, b)$ means the expression obtained by substituting a for x, and b for y, hence $f(a, b) \equiv a^2 - 3b^2 + 6a - 4$. Similarly
$f(2, 3) \equiv 2^2 - 3(3)^2 + 6(2) - 4 = 4 - 27 + 12 - 4 = -15,$
$f(1, 0) \equiv 1^2 - 3(0)^2 + 6(1) - 4 = 3, \qquad f(0, 0) = -4,$ etc.

18. Composite loci.

Theorem. *If the expression $f(x, y)$ is factorable, the locus of the equation $f(x, y) = 0$ consists of as many distinct lines and curves as there are variable factors of $f(x, y)$.*

Let $f(x, y)$ consist of three factors, $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$. Then the equation $f(x, y) = 0$ can be written in the form

$$f_1(x, y) \cdot f_2(x, y) \cdot f_3(x, y) = 0. \qquad (i)$$

Any values of x and y which satisfy the equation

$$f_1(x, y) = 0$$

will also satisfy equation (i), and similarly with reference to $f_2(x, y) = 0$ and $f_3(x, y) = 0$. Hence all points on the three

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separate loci of

$$f_1(x, y) = 0, \qquad f_2(x, y) = 0, \qquad f_3(x, y) = 0$$

will also be points on the locus of $f(x, y) = 0$.

The same argument holds no matter what number of factors $f(x, y)$ may have. The theorem is therefore proved.

If $f(x, y)$ has equal factors, the corresponding component of the complete locus is counted as many times as the factor occurs.

DEFINITION. *A locus which is thus made up of separate parts is called a* ***composite locus.***

19. Equations containing only one variable.—Equations which contain only one variable explicitly, for example $2x - 3 = 0$, may always be interpreted as containing the other variable also with coefficient zero. For all values of y in the equation $2x - 3 = 0$, or $2x + 0y - 3 = 0$, x has the same value, $x = \frac{3}{2}$. Hence the locus consists of a straight line parallel to the Y-axis and $1\frac{1}{2}$ units on the positive side, because this line contains all points having the abscissa $\frac{3}{2}$, and any ordinate whatsoever.

Similarly $y^2 - 5y - 6 = 0$, or $(y - 6)(y + 1) = 0$ is the equation of the two lines parallel to the X-axis, one six units above and the other one unit below this axis.

EXERCISES. 1. Construct the complete locus of each of the following equations

(a) $x^2 = y^2$,
(b) $x^2 - 4xy + 4y^2 - 4 = 0$,
(c) $x^3 + xy^2 - 4x = 0$,
(d) $x^4 - y^4 = 0$,
(e) $xy - 3x + 5y - 15 = 0$,
(f) $x^3 - 6x^2 + 11x - 6 = 0$.

2. Show that the locus of the equation $Ax^2 + Bx + C = 0$ is a pair of parallel lines, a pair of coincident lines, or a pair of imaginary lines, according as $B^2 - 4AC$ is positive, zero, or negative.

4. What is the form of the equation of a line parallel to the X-axis? the Y-axis?

5. What is the equation of the X-axis? the Y-axis?

6. Find for each of the following sets of equations a single equation

whose locus will be the combination of the loci of the separate equations.

(a) $x = y,\quad x = -y.$ Ans. $x^2 - y^2 = 0.$

(b) $x = 0,\quad x = 4,\quad x = 6.$ Ans. $x^3 - 10x^2 + 24x = 0.$

(c) $x = y,\quad x^2 + y^2 = 4,$ (d) $x^2 + y^2 = 8,\quad xy + 4 = 0.$

20. Determination of the equation of a given locus.— So far in this chapter we have discussed the construction of the loci of given equations. The inverse problem of finding the equation from the locus, or from its definition, is a vital part of the subject.

EXAMPLES. 1. A point moves in a plane so as to be always equally distant from the points (2, 6), (4, − 2). Find the equation of the locus of the moving point.

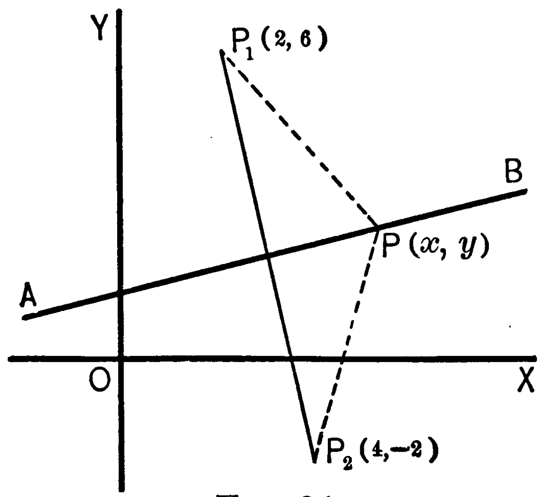

FIG. 24

Let $P_1 = (2, 6)$, and $P_2 = (4, -2)$, then we know from elementary geometry that the required locus is AB, the perpendicular bisector of P_1P_2. To determine the equation of AB take any representative point $P(x, y)$ on it. Then by (3), p. 6, the distance

$$PP_1 = \sqrt{(x-2)^2 + (y-6)^2},$$

and

$$PP_2 = \sqrt{(x-4)^2 + (y+2)^2}.$$

By the conditions stated these are equal, hence

$$\sqrt{(x-2)^2 + (y-6)^2} = \sqrt{(x-4)^2 + (y+2)^2} \qquad (i)$$

which is the required equation.

Equation (*i*) may be simplified. Thus squaring and expanding, it becomes

$$x^2 - 4x + 4 + y^2 - 12y + 36 = x^2 - 8x + 16 + y^2 + 4y + 4,$$

which reduces to $\qquad x - 4y + 5 = 0.$

3. A point moves in a plane so that the sum of the squares of its distances from two fixed points in the plane is constant. What locus will it describe?

The problem is stated without reference to any particular lines as axes of coordinates. Any two lines in the plane may be used for this purpose. The *form* of the resulting equation will depend upon the choice of axes, but the *geometric properties* expressed by the equation will be the same in all cases. It is best to choose the axes so that the resulting equation will be as simple as possible.

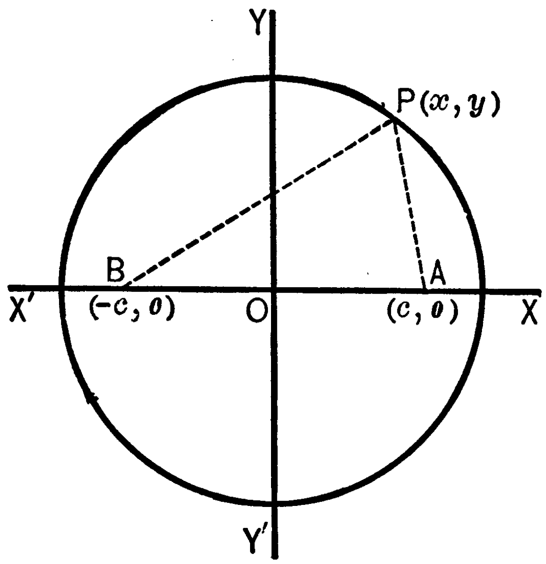

FIG. 25

In this problem take the line joining the two given points as X-axis, with the origin half way between them. Hence let $A(c, 0)$ and $B(-c, 0)$ be the two given points, and $P(x, y)$ any point on the required locus. Then since it is given that $\overline{AP}^2 + \overline{BP}^2 = k^2$, where k^2 is a constant, we have by (3) p. 6

$$(x - c)^2 + y^2 + (x + c)^2 + y^2 = k^2,$$

which reduces to $$x^2 + y^2 = \tfrac{1}{2}k^2 - c^2.$$

If this be plotted for definite values of k^2 and c it will be found that the locus is a circle with center half way between the given points.

EXERCISES. In each of the following exercises find the equation of the locus of a point which moves in a plane according to the conditions stated:

1. So as to be always equally distant from the origin and the point (2, 4). Ans. $x + 2y = 5$.

2. So that the sum of its distances from $(-2, 0)$ and $(2, 0)$ is always equal to 6. Ans. $5x^2 + 9y^2 = 45$.

3. So that its distance from the origin is always equal to the slope of the line joining it to the origin. Ans. $y^2(1 - x^2) = x^4$.

4. So that its distance from a fixed point is twice its distance from a fixed line.

5. So that its distance from the point $(2, 4)$ is twice its distance from the point $(-1, 4)$. Ans. $x^2 + y^2 + 4x - 8y + 16 = 0$.

EXERCISES ON CHAPTER II

Normal Exercises

1. Construct the graph of each of the following equations

(a) $x^2 - y^2 = 4a^2$,
(b) $y^2 = x(x^2 - 9)$,
(c) $y = \dfrac{x(x-4)}{x^2-4}$,
(d) $y = ax + 4$,
(e) $x^2y^2 - 4 = 0$,
(f) $x^2 - 6x + 5 = 0$,
(g) $y = 4$.

2. Find from the equations the coordinates of the points of intersection of the graphs of each of the following pairs of equations

(a) $x^2 + y^2 = 10$, $x^2 + y^2 - 10x = 0$; Ans. $(1, \pm 3)$.
(b) $x^2 - y^2 = 0$, $y + 3x - 6 = 0$. Ans. $(3, -3)$, $(\frac{3}{2}, \frac{3}{2})$.

3. Find from their graphs approximate values of the roots of the following pairs of simultaneous equations

(a) $y(x^2 + 5) = 8$, $y = x^4 - 4x^2 + 2$;
(b) $x^2y + xy^2 = 4$, $8x^2 - y^3 = 0$.

4. Determine for each of the following pairs of equations the coordinates of the points of intersection of their graphs and hence show that the graphs are tangent to each other

(a) $y^2 = 4x$, $2x + 4y + 8 = 0$;
(b) $x^2 - y^2 = 4$, $x^2 + y^2 + 2x - 8 = 0$.

5. Find the coordinates of the real points on the graphs of each of the following equations

(a) $(2 + 3i)x + (4 - i)y - 6i = 0$, Ans. $(\frac{12}{7}, -\frac{6}{7})$.
(b) $(1 + i)x + (2 - 3i)y - 6 - i = 0$, Ans. $(4, 1)$.
(c) $x^2 + 2y^2 - 4x - 12y + 22 = 0$, Ans. $(2, 3)$.
(d) $4x^2 + y^2 + 4y + 4 = 0$. Ans. $(0, -2)$.

6. Show that there are no real points on the graph of any of the following equations

(*a*) $x^2 + y^2 + 4 = 0$, (*b*) $2x^2 + 3y^2 + 4x - 12y + 16 = 0$,
(*c*) $(4 + i)x^2 + 3y^2 + (3 + 4i)x + 4 + 5i = 0$,
(*d*) $x^2 + 2y^2 + ix - 3iy + 6 - 7i = 0$.

7. A point moves in a plane so that the ratio of its distances from the points (2, 4) and (− 3, 7) is always as 1 to 3. Find the equation of its locus. *Ans.* $4x^2 + 4y^2 - 21x - 29y + 61 = 0$.

8. Find the equation of the locus of a point which moves in a plane so that the slope of the line joining it to the point (2, − 4) is always equal to the slope of the line joining it to the point (3, − 5).

Ans. $x + y + 2 = 0$.

General Exercises

9. Construct the graphs of the following equations

(*a*) $x = 2$, (*f*) $x^2 - 4y^2 = 36$, (*k*) $x^3 - y = 0$,
(*b*) $x = 2y$, (*g*) $x^2 - 4y = 36$, (*l*) $yx = a^2$,
(*c*) $x = 2y + 2$, (*h*) $x^2 - 4y^2 = 0$, (*m*) $yx^2 = a^3$,
(*d*) $x^2 + y^2 = 36$, (*i*) $x^2 - 4 = 0$, (*n*) $y^2(x^2 - 4) = x^4$.
(*e*) $x^2 + 4y^2 = 36$, (*j*) $y^2(x^2 - 3x + 2) = x(x + 4)$,

10. Construct on one diagram the graphs of the following equations

$x + y = 2, \quad x^2 + y^2 = 4, \quad x^3 + y^3 = 8, \quad x^4 + y^4 = 16.$

11. Construct on one diagram the graphs of the following equations

$y = 4, \quad y + \frac{1}{2}x = 4, \quad y + x = 4, \quad y + 2x = 4.$

12. Construct on one diagram the graphs of the equation $y = x + b$, for b equal to 0, 2, 4, and − 2 respectively. In what respect do these graphs differ from each other?

13. Construct on one diagram the graphs of the equation $xy^n = 1$, for n equal to − 2, − 1, − ½, 0, ½, 1, and 2 respectively. What is common to all these graphs? What properties are common to the graphs with negative values of n? What for positive values of n?

14. Construct the graph of the equation $y^2 = (x - a)(x - 1)(x - 4)$ (*i*) for $a = -1$, (*ii*) for $a = 0$, (*iii*) for $a = +1$.

15. Find the value of A so that there will be only one real point on the graph of the equation $x^2 - 4x + 4y^2 - 16y + A = 0$. *Ans.* $A = 20$.

16. For what range of values of A will the graph of the equation of exercise 15 be entirely imaginary? Ans. $A > 20$.

17. Find the coordinates of the real points of the graph of each of the following equations

(a) $x^2 + y^2 - 4x + 4 = 0$, Ans. (2, 0).

(b) $(1 + i)x + (1 - i)y = 10$, Ans. (5, 5).

(c) $(1 + 2i)x^2 + (8i - 4)xy + (4 + 2i)y^2 = 0$, Ans. (0, 0).

18. Determine which of the graphs of the following equations are tangent to each other $5x - 4y = 9$, $x^2 - y^2 = 9$, $x^2 + y^2 = 9$, $x^2 + y^2 - 12x + 10y + 20 = 0$.

19. A point moves so that the difference of its distances from the points (2, − 4) and (2, 4) is always equal to 6. Determine the equation of its locus, and construct the locus from the equation.

Ans. $9x^2 - 7y^2 - 36x + 99 = 0$.

20. A point moves so that the slope of the line joining it to the point (0, 2) is twice the slope of the line joining it to the point (0, − 2). Determine the equation of its locus, and construct the locus from the equation. Ans. $y = -6$.

21. $A = (2, 4)$ and $B = (-1, 3)$ are fixed points. The variable point P moves so that the area of the triangle ABP is always equal to 6. Find the equation of its locus, and construct the locus from the equation.

Ans. $x - 3y - 2 = 0$, and $x - 3y - 10 = 0$.

22. A point moves so that the slope of the line joining it to the point (2, 4) is always equal to the square of its distance from the Y-axis. Find the equation of its locus and construct the locus from the equation.

Ans. $y = x^3 - 2x^2 + 4$.

23. Let $A = (0, -2)$, and let AP, any line through A, cut the X-axis at M. If P moves so that MP always equals 3, find the equation of the locus of P, and construct the locus.

Ans. $x^2y^2 + (y^2 - 9)(y + 2)^2 = 0$.

24. A point moves so that its distance from the origin is always equal to four times the cosine of the angle which the line joining it to the origin makes with the positive extension of the X-axis. Find the equation of its locus, and construct the locus.

Ans. $x^2 + y^2 - 4x = 0$.

CHAPTER III

THE STRAIGHT LINE

21. Conditions which determine a straight line.—In plane geometry a straight line is said to be *determined* when its position in the plane is definitely fixed. Elementary geometry teaches that a straight line is thus determined by *two conditions,* properly chosen. For example, a straight line is determined when two points on it are given. In this chapter several standard forms of the equation of the straight line, determined by prescribed sets of conditions, will be derived.

22. Problem.— *To derive the equation of a straight line in terms of the coordinates of two given points on the line.*

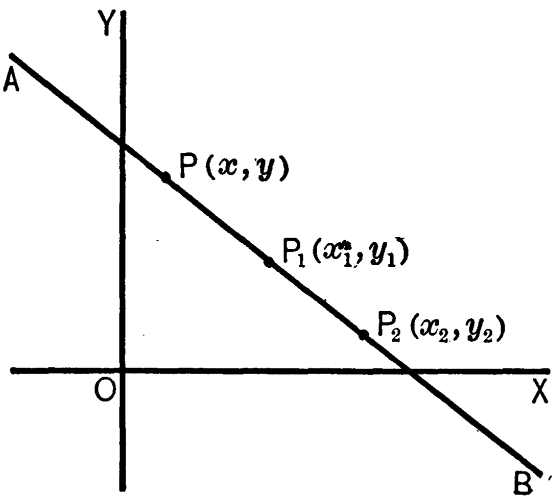

FIG. 26

Let AB be the straight line determined by the two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and let $P(x, y)$ be a representative point on AB. Since P, P_1, P_2 are all on the same line AB, the slopes of PP_1 and of P_1P_2 are equal. Hence by (4), p. 7, the required equation is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, \tag{1}$$

which may also be written in the form

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \tag{2}$$