ANALYTIC GEOMETRY

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PREFACE

In adding another text-book on analytic geometry to the large list of such books now available, the authors feel that a statement of the reasons for this one is desirable. In compiling the work they have endeavored to keep constantly in view the needs of that large class of students in our colleges and technical schools who, though in a way sympathetic toward mathematical study, are yet lacking in that quick appreciation of the mathematical point of view which is a characteristic of the so-called mathematical mind. The assimilation of new ideas, or even of new combinations and applications of ideas already familiar, in which the study of analytic geometry especially abounds, is not easy even for the trained mind, and for the student approaching the subject for the first time is the source of most of his difficulties. The attempt here is to make this approach by easy gradations so that the student may not lose himself on the way, but may feel sure of his ground as he advances. To try to gain this end by the lack of rigorous demonstration would be to lose sight of an important object of such study, which, rightly viewed, is to assist the student to an acquaintance with correct principles of mathematical reasoning and the accompanying scientific attitude of mind. The aim has been therefore to preserve mathematical rigor throughout.

As to details, much of the book necessarily follows well established lines of procedure. The experienced teacher will find, however, here and there, some departure from these lines which it is hoped will be approved. Owing to the increasing use of the imaginary, and its growing importance to the student of both pure and applied mathematics, some elementary discussion of imaginary elements in geometry has been included, which the authors believe will be of value in accustoming the student to look upon the complex number as a useful member of the number system. The lists of exercises have been prepared with especial care. Throughout the text short lists have been inserted where needed for the immediate illustration of principles or methods. At the conclusion of each chapter longer lists are inserted. These are divided into two parts, "normal exercises," which are direct applications of the text of the chapter, and "general exercises," which are designed to give the student opportunity of testing his grasp of the subject as a whole, and of its underlying principles. There will be found in each set of "normal exercises" one or two examples illustrating each separate point of the theory developed in the corresponding portion of the text, so far as it can be illustrated by exercises. The student may therefore be assured that he has made application of all of the theory when he shall have worked *all* the normal exercises.

While the book is to be regarded as a text-book of plane analytic geometry, the concluding chapter is devoted to solid geometry. Only the barest outline of some of the fundamental principles of this subject is included, enough to enable the student, when he studies calculus, and wishes to apply its principles to problems involving solids and surfaces, to feel that he is not on entirely new ground.

UNIVERSITY OF PENNSYLVANIA,
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FORMULAS, TABLES, ETC., FOR REFERENCE

A few important definitions, formulas, and tables, from elementary algebra and trigonometry, which will be needed in the study of this book, are placed here for convenience of reference.

A. Algebraic definitions and formulas

I. Definitions

(*a*) The **degree of a term** in specified letters is the number of times these letters occur as factors in that term. Thus ab^2c^3d is of the seventh degree in a, b, c, and d; $a^2x^2y^2$ is of the fourth degree in x and y.

(*b*) The **degree of a polynomial** in specified letters is defined as that of the term of highest degree in those letters. Thus $ax^3 + bx^2y + cx^2 + dy + e$ is of the third degree in x and y.

(*c*) A polynomial is said to be **homogeneous** in specified letters when each of its terms is of the same degree in those letters. Thus $ax^2 + 5xy - b^2y^2$ is homogeneous and of the second degree in x and y.

(*d*) In an equation containing one or more variables the term which contains no variable factors is called the **absolute term.** Thus in $ax^2 + by^2 - a^2c^2 = 0$, where x and y are the variables, the absolute term is $-a^2c^2$. In $x + y = 0$, the absolute term is zero.

(*e*) A **root of an equation** is a value of the variable which, when substituted for the variable, satisfies the equation.

II. The Quadratic Equation

(*a*) The roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

(*b*) The roots of the quadratic (1) are

$$\left.\begin{array}{ll} \text{real and unequal} & \text{if } b^2 > 4ac, \\ \text{real and equal} & \text{if } b^2 = 4ac, \\ \text{imaginary} & \text{if } b^2 < 4ac. \end{array}\right\} \qquad (3)$$

(*c*) The sum of the roots of the quadratic (1) is

$$r_1 + r_2 = -\frac{b}{a}. \tag{4}$$

(*d*) The product of the roots of the quadratic (1) is

$$r_1 r_2 = \frac{c}{a}. \tag{5}$$

(*e*) If the coefficient a in the quadratic (1) varies and approaches zero, the coefficients b and c remaining constant, one root of the equation increases without limit.

III. Factors of $Ax^2 + Hxy + By^2$

A homogeneous expression of the second degree in x and y,

$$Ax^2 + Hxy + By^2, \tag{6}$$

can always be factored. The factors are conveniently expressed thus

$$\frac{1}{4A}[2Ax + (H + \sqrt{H^2 - 4AB})y][2Ax + (H - \sqrt{H^2 - 4AB})y]. \tag{7}$$

IV. Imaginaries

(*a*) $\sqrt{-1}$ is called the imaginary unit, and is usually represented by the letter i.* Thus $x + iy$ means $x + y\sqrt{-1}$.

(*b*) An expression containing both real and imaginary terms is called a complex expression.

(*c*) If two complex expressions are equal, the real parts of the two expressions are equal, and the imaginary parts are equal. Thus

$$\left.\begin{array}{lll} \text{if } x + iy = a + ib, & \text{then } x = a, & \text{and } y = b, \\ \text{if } x + iy = 0, & \text{then } x = 0, & \text{and } y = 0. \end{array}\right\} \tag{8}$$

V. Logarithms

(*a*) If $y = a^x$, x is called the logarithm of y to the base a. This equation may therefore be written in the form $x = \log_a y$.

(*b*) In the case of the so-called common logarithms, the logarithms used in computation, the base is 10.

* In electrical theory j is used instead of i to represent $\sqrt{-1}$.

(*c*) Logarithms used algebraically are usually Napierian, or natural logarithms. The base of this system of logarithms is the incommensurable number $e = 2.7182818285\cdots$, called the Napierian base, and defined by the infinite series

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \cdots. \tag{9}$$

(*d*) When in algebraic work a logarithm is written without any expressed base, e. g., $\log(1 + x^2)$, the base is understood to be e.

B. Trigonometric formulas

If l is the length, and r the radius of the arc subtending the angle of θ radians,

$$l = r\theta. \tag{10}$$

$$180° = \pi \text{ radians}. \tag{11}$$

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}. \tag{12}$$

$$\sin^2\theta + \cos^2\theta = 1. \tag{13}$$

$$\sec^2\theta = 1 + \tan^2\theta. \tag{14}$$

$$\csc^2\theta = 1 + \cot^2\theta. \tag{15}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}. \tag{16}$$

If $\tan\theta = \frac{m}{n}$, then

$$\sin\theta = \pm\frac{m}{\sqrt{m^2 + n^2}}, \qquad \cos\theta = \pm\frac{n}{\sqrt{m^2 + n^2}}. \tag{17}$$

$$\cos\theta = \sin(90° \pm \theta), \qquad \cot\theta = \begin{cases} \tan(90° - \theta), \\ -\tan(90° + \theta). \end{cases} \tag{18}$$

$$\sin(180° - \theta) = \sin\theta, \qquad \cos(180° - \theta) = -\cos\theta,$$
$$\tan(180° - \theta) = -\tan\theta. \tag{19}$$

$$\sin(180° + \theta) = -\sin\theta, \qquad \cos(180° + \theta) = -\cos\theta,$$
$$\tan(180° + \theta) = \tan\theta. \tag{20}$$

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta. \tag{21}$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y. \qquad (22)$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \qquad (23)$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}. \qquad (24)$$

$$\sin 2\theta = 2 \sin \theta \cos \theta. \qquad (25)$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta. \qquad (26)$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \qquad (27)$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta, \qquad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta. \qquad (28)$$

$$1 - \cos \theta = 2 \sin^2 \tfrac{1}{2}\theta, \qquad 1 + \cos \theta = 2 \cos^2 \tfrac{1}{2}\theta. \qquad (29)$$

$$\tan \tfrac{1}{2}\theta = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}. \qquad (30)$$

In any plane triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad (31)$$

$$a^2 = b^2 + c^2 - 2bc \cos A, \text{ etc.} \qquad (32)$$

Signs of the Trigonometric Functions

	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
$\sin\theta$ $\csc\theta$	+	+	−	−
$\cos\theta$ $\sec\theta$	+	−	−	+
$\tan\theta$ $\cot\theta$	+	−	+	−

C. Values of the trigonometric functions of angles

I. *Certain exact values of these functions*

Angle in Degrees	Angle in Radians	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	∞	1	∞
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	$\frac{1}{2}\pi$	1	0	∞	0	∞	1

II. *Decimal values of the trigonometric functions for each 5° of the quadrant*

Angle in Degrees	Angle in Radians	sin	cos	tan	cot	sec	csc
0°	0	0.000	1.000	0.000	∞	1.000	∞
5°	0.087	0.087	0.996	0.087	11.430	1.004	11.474
10°	$0.17\bar{5}$	0.174	$0.98\bar{5}$	0.176	5.671	1.015	5.759
15°	0.262	0.259	0.966	0.268	3.732	1.035	3.864
20°	0.349	0.342	0.940	0.364	2.747	1.064	2.924
25°	0.436	0.423	0.906	0.466	$2.14\bar{5}$	1.103	2.366
30°	0.524	0.500	0.866	0.577	1.732	$1.15\bar{5}$	2.000
35°	0.611	0.574	0.819	0.700	1.428	1.221	1.743
40°	0.698	0.643	0.766	0.839	1.192	1.305	1.556
45°	0.785	0.707	0.707	1.000	1.000	1.414	1.414
50°	0.873	0.766	0.643	1.192	0.839	1.556	1.305
55°	0.960	0.819	0.574	1.428	0.700	1.743	1.221
60°	1.047	0.866	0.500	1.732	0.577	2.000	$1.15\bar{5}$
65°	1.134	0.906	0.423	$2.14\bar{5}$	0.466	2.366	1.103
70°	1.222	0.940	0.342	2.747	0.364	2.924	1.064
75°	1.309	0.966	0.259	3.732	0.268	3.864	1.035
80°	1.396	$0.98\bar{5}$	0.174	5.671	0.176	5.759	1.015
85°	1.484	0.996	0.087	11.430	0.087	11.474	1.004
90°	1.571	1.000	0.000	∞	0.000	∞	1.000

D. Miscellaneous tables

I. *Degrees and Radians*

1° = 0.0174533 rad., 1′ = 0.0002909 rad., 1″ = 0.0000048 rad.
1 rad. = 57°.29578 = 57° 17′ 44″.8 = 3437′.75 = 206264″.8

0.1 rad. = 5° 43′ 46″.5	0.4 rad. = 22° 55′ 5″.9	0.7 rad. = 40° 6′ 25″.4
0.2 " = 11° 27′ 33″.0	0.5 " = 28° 38′ 52″.4	0.8 " = 45° 50′ 11″.8
0.3 " = 17° 11′ 19″.4	0.6 " = 34° 22′ 38″.9	0.9 " = 51° 33′ 58″.3

II. *Mantissas of the common logarithms of numbers*

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	$41\bar{5}$	431	447	462
3	477	491	505	519	531	544	556	568	580	591
4	602	613	623	633	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	929	934	940	944	949
9	954	959	964	968	973	978	982	987	991	996

III. *Napierian logarithms of numbers*

N	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	$-\infty$	−2.30	−1.61	−1.20	−0.92	−0.69	−0.51	−0.36	−0.22	−0.11
1	0.00	0.10	0.18	0.26	0.34	0.41	0.47	0.53	0.59	0.64
2	0.69	0.74	0.79	0.83	0.88	0.92	0.96	0.99	1.03	1.06
3	1.10	1.13	1.16	1.19	1.22	1.25	1.28	1.31	1.34	1.36
4	1.39	1.41	1.44	1.46	1.48	1.50	1.53	$1.5\bar{5}$	1.57	1.59
5	1.61	1.63	$1.6\bar{5}$	1.67	1.69	1.70	1.72	1.74	1.76	1.77
6	1.79	1.81	1.82	1.84	1.86	1.87	1.89	1.90	1.92	1.93
7	$1.9\bar{5}$	1.96	1.97	1.99	2.00	2.01	2.03	2.04	2.05	2.07
8	2.08	2.09	2.10	2.12	2.13	2.14	2.15	2.16	2.17	2.19
9	2.20	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29
10	2.30	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39

IV. *Positive powers of e (Napierian anti-logarithms)*

Exp.	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.00	1.11	1.22	$1.3\bar{5}$	1.49	$1.6\bar{5}$	1.82	2.01	2.23	2.46
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	$6.0\bar{5}$	6.69
2	7.39	8.17	9.03	9.97	11.0	12.2	$13.\bar{5}$	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	$81.\bar{5}$	90.0	$99.\bar{5}$	110.	122.	134.
5	148.	164.	181.	200.	221.	$24\bar{5}.$	270.	299.	330.	365.
6	403.	446.	493.	$54\bar{5}.$	602.	665.	735.	812.	898.	992.

V. *Negative powers of e (Napierian anti-logarithms)*

Exp.	0	−.1	−.2	−.3	−.4	−.5	−.6	−.7	−.8	−.9
−5	0.01	0.01	0.01	0.00						
−4	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
−3	$0.0\bar{5}$	$0.0\bar{5}$	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02
−2	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.07	0.06	0.06
−1	0.37	0.33	0.30	0.27	$0.2\bar{5}$	0.22	0.20	0.18	0.17	$0.1\bar{5}$
0	1.00	0.90	0.82	0.74	0.67	0.61	$0.5\bar{5}$	$0.5\bar{0}$	$0.4\bar{5}$	0.41

VI. *Square roots of numbers*

N	0	1	2	3	4	5	6	7	8	9
0	000	1.00	1.41	1.73	2.00	2.24	$2.4\bar{5}$	$2.6\bar{5}$	2.83	3.00
1	3.16	3.32	3.46	3.61	3.74	3.87	4.00	4.12	4.24	4.36
2	4.47	4.58	4.69	4.80	4.90	5.00	5.10	5.20	5.29	5.39
3	5.48	5.57	5.66	5.74	5.83	5.92	6.00	6.08	6.16	6.24
4	6.32	6.40	6.48	6.56	6.63	6.71	6.78	6.86	6.93	7.00
5	7.07	7.14	7.21	7.28	$7.3\bar{5}$	7.42	7.48	$7.5\bar{5}$	7.62	7.68
6	$7.7\bar{5}$	7.81	7.87	7.94	8.00	8.06	8.12	8.19	$8.2\bar{5}$	8.31
7	8.37	8.43	8.49	8.54	8.60	8.66	8.72	8.77	8.83	8.89
8	8.94	9.00	9.06	9.11	9.17	9.22	9.27	9.33	9.38	9.43
9	9.49	9.54	9.59	9.64	9.70	$9.7\bar{5}$	9.80	$9.8\bar{5}$	9.90	$9.9\bar{5}$

he Greek alphabet

Α, α, alpha	Ι, ι, iota	Ρ, ρ, rho
Β, β, beta	Κ, κ, kappa	Σ, σ, ς, sigma
Γ, γ, gamma	Λ, λ, lambda	Τ, τ, tau
Δ, δ, delta	Μ, μ, mu	Υ, υ, upsilon
Ε, ε, epsilon	Ν, ν, nu	Φ, φ, phi
Ζ, ζ, zeta	Ξ, ξ, xi	Χ, χ, chi
Η, η, eta	Ο, ο, omicron	Ψ, ψ, psi
Θ, θ, ϑ, theta	Π, π, pi	Ω, ω, omega.

θ

ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COORDINATES

1. Theorem. *Let $X'OX$, $Y'OY$ be two fixed straight lines in a plane, then the position of any point in the plane is determined by its distance from $Y'Y$ measured parallel to $X'X$, and its distance from $X'X$ measured parallel to $Y'Y$.*

Let P be a point in the plane of $X'X$, $Y'Y$, whose distance to the right of $Y'Y$, measured parallel to $X'X$ is a, and whose distance above $X'X$, measured parallel to $Y'Y$ is b. Then the position of P in the plane is determined. Because P must lie in the line MN, parallel to $Y'Y$ at the distance a (as defined above) from it on the right, and also in the line RS, parallel to $X'X$ at the distance b from it on the upper side. Hence P must be at the intersection of MN and RS.

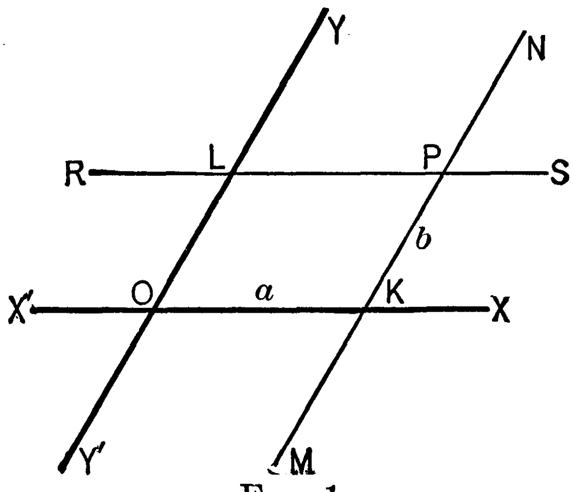

Fig. 1

Definitions. The two lengths or distances which determine the position of a point, as a and b determine P in the preceding demonstration, are called the **coordinates of the point.**

The fixed lines in the plane to which the position of the point is referred are called the **axes of coordinates,** and their intersection O is called the origin of coordinates, or simply the

origin. The axis $X'X$ is called the **axis of abscissas,** or the **X-axis,** and $Y'Y$ is called the **axis of ordinates,** or the **Y-axis.**

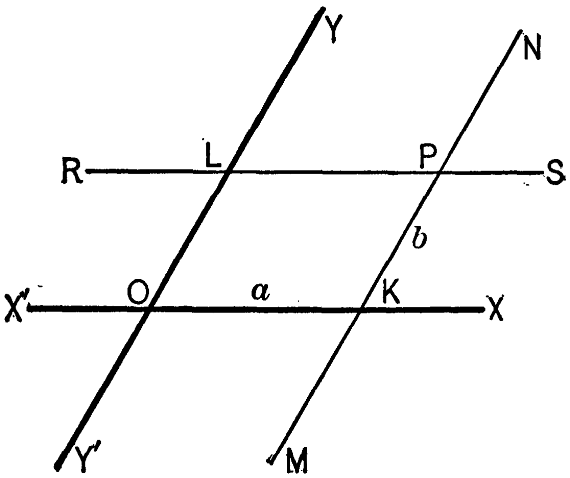

FIG. 1

The two coordinates a and b of the point P, Fig. 1, are called respectively the **abscissa** and the **ordinate** of the point. That is:

The abscissa of a point is its distance from the Y-axis, measured parallel to the X-axis.

The ordinate of a point is its distance from the X-axis, measured parallel to the Y-axis.

The letter x is customarily used to designate the abscissa of a point, and the letter y to designate the ordinate. So also the expression "the x of a point" is often used, meaning its abscissa, or "the y of a point," meaning its ordinate.

The coordinate axes are usually taken at right angles to each other, and in this book they will always be considered as so situated unless the contrary is stated. When the axes are not at right angles the coordinate system is said to be oblique.

NOTE. The term *Cartesian coordinates* is applied to the system of coordinates which has just been described because it is essentially the system invented by the French philosopher and mathematician Descartes (1596–1650). Other coordinate systems are often used, one of which, the polar system, will be described in a later chapter of this book.

2. Signs of the coordinates. Notation. In order to be able to locate a point when its coordinates are given we must know not only the lengths of its coordinates but also the direction (right or left, up or down) in which they are to be measured. Thus, Fig. 2, if $NO = OM = a$, and $RO = OS = b$, there are four points P_1, P_2, P_3, P_4 at the distance a from $Y'Y$ and the distance b from $X'X$. To avoid this ambiguity we adopt the usual convention of distinguishing between lengths measured in opposite directions by $+$ and $-$ signs. Then, Fig. 3,

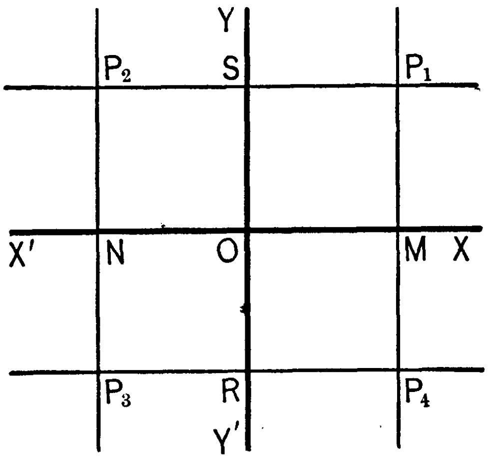

FIG. 2

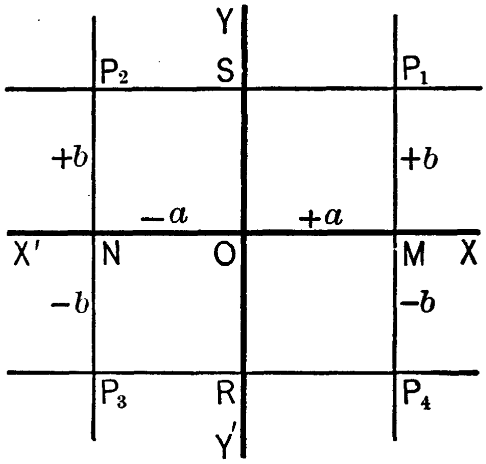

FIG. 3

if OM is called $+ a$, ON is $- a$; and if OS is $+ b$, OR is $- b$. Hence the coordinates of P_1 are $x = + a$, $y = + b$, those of P_2 are $x = - a$, $y = + b$, those of P_3 are $x = - a$, $y = - b$, and those of P_4 are $x = + a$, $y = - b$.

In actual practice when an abscissa or ordinate is $+$ the sign is usually not written.

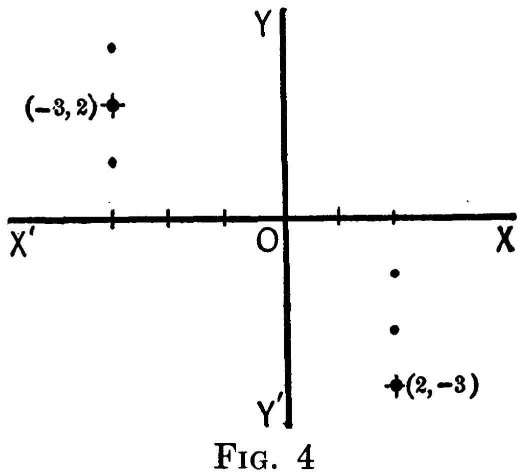

FIG. 4

In designating the position of a point by its coordinates the notation (x, y) is used. That is, the coordinates are written in parenthesis with a comma separating them, and the abscissa is invariably written first. Thus the point $(2, - 3)$ means the point whose abscissa is 2 and ordinate $- 3$, the point $(- 3, 2)$ is the point whose abscissa is $- 3$ and ordinate 2. See Fig. 4.

EXERCISES. 1. Draw a pair of coordinate axes and, using any convenient unit of length, locate the following points (1, 2), (5, −2), (3, 3), (− 3, 1), (− 2, 0), (0, 4), (− 4, − 2), (3, 0), (0, − 2), (0, 0).

2. If a point is on the X-axis what is its ordinate?
3. If a point is on the Y-axis what is its abscissa?
4. What are the coordinates of the origin?
5. If the abscissa of a point is 0, on what line will it lie?
6. If the ordinate of a point is 0, on what line will it lie?

3. Directed lengths. By a directed length is meant a length measured in a given direction. As has been explained the coordinates of a point are always directed lengths. The length AB means the distance from A to B in magnitude and direction.

I. If A and B are two points, then

length AB = − length BA.

FIG. 5

That is, the magnitude element of the length is the same, whether we think of AB or of BA, but the direction element in the one case is opposite to that in the other.

II. THEOREM. *If three points A, B, C are arranged in any order on a straight line then $AC = AB + BC$.*

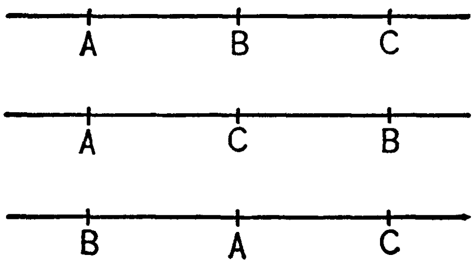

FIG. 6

In the upper line, Fig. 6, the three segments AC, AB, BC are all measured in the same direction, and AC is composed of the other two. Hence $AC = AB + BC$.

In the second line of the figure $AC = AB - CB$, but from I, $-CB = +BC$, hence $AC = AB + BC$.

In the third line $AC = BC - BA = AB + BC$, as before.

Other possible arrangements can be similarly treated.

III. Problem. *To express the distance from one point to another, when the distances of these two points from a third point on the same line are known.*

Let the distances AB and AC, Fig. 6, be known in both length and direction, and let the distance from B to C be required. It is then only necessary to express the length of the required segment in terms of the given segments as in II, being careful to write the letters in the proper order in each case. Since it is here required to find the distance *from B to C* we write BC, and then, according to II,

$$BC = BA + AC,$$

and hence

$$BC = -AB + AC. \tag{1}$$

Had the question been to find the distance from C to B, the same procedure would lead to the equation

$$CB = CA + AB,$$

or

$$CB = -AC + AB. \tag{2}$$

The student should examine this discussion in connection with each arrangement of points in Fig. 6 and satisfy himself that the truth of the result is independent of the order of the points.

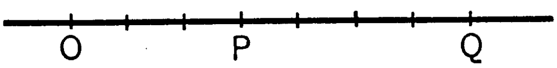

Fig. 7

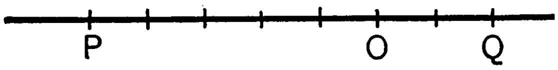

Fig. 8

Illustrations. In Fig. 7 let $OP = 3$, and $OQ = 7$, then

$$PQ = PO + OQ = -3 + 7 = 4.$$

Again in Fig. 8 suppose $OP = -5$, and $OQ = 2$, then

$$PQ = PO + OQ = -(-5) + 2 = 7.$$

Similarly with the same data

$$QP = QO + OP = -2 + (-5) = -7.$$

4. Distance between two points.

PROBLEM. *To find the length of a line joining two points whose coordinates are given, the axes being rectangular.*

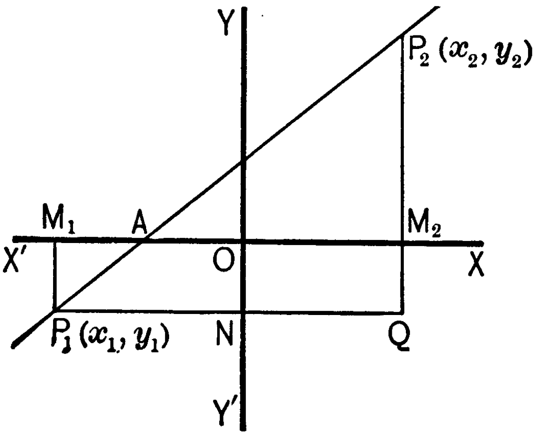

FIG. 9

Let it be required to express the length of P_1P_2, in terms of the given coordinates (x_1, y_1), (x_2, y_2) of these two points. That is we have given $OM_1 = x_1$, $M_1P_1 = y_1$, $OM_2 = x_2$, $M_2P_2 = y_2$.

Draw P_1Q parallel to OX to meet M_2P_2 (extended in this case) at Q. Since P_1QP_2 is a right angle,

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2. \qquad (i)$$

But

$$P_1Q = P_1N + NQ = -OM_1 + OM_2 = -x_1 + x_2,$$

and

$$QP_2 = QM_2 + M_2P_2 = -M_1P_1 + M_2P_2 = -y_1 + y_2.$$

Substituting these values in (*i*) the result is

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

or since $(x_2 - x_1)^2 = (x_1 - x_2)^2$, and $(y_2 - y_1)^2 = (y_1 - y_2)^2$,

$$\boldsymbol{P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}. \qquad (3)$$

Note carefully that in indicating a line-segment, simply as a length, it is to be written in the positive direction, as P_1Q; QP_2 in the foregoing demonstration; but in indicating a line-segment as one of the coordinates of a point it must be taken as starting *from* the axis and going *to* the point. Thus, above, OM_1 (not M_1O) is x_1, and M_1P_1 (not P_1M_1) is y_1.

5. Slope of a line.

I. DEFINITION. *The tangent of the angle which a line makes with the X-axis is called the* **slope** *of the line.*

Of the four angles formed by the intersection of a line with the X-axis that one is taken as the "slope" angle which lies between the positive extension of the X-axis and the upper extension of the line. Thus the slope of AB is $\tan \theta_1$ and the slope of CD is $\tan \theta_2$.

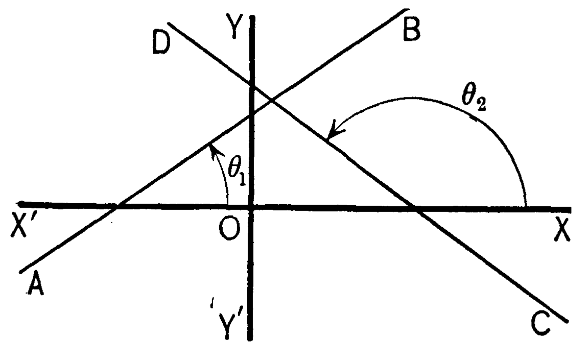

Fig. 10

II. Problem. *To find the slope of a line in terms of the coordinates of two points on the line.*

The slope of the line determined by the points P_1, P_2 Fig. 9 is $\tan XAP_2 = \tan QP_1P_2$. Now

$$\tan QP_1P_2 = \frac{QP_2}{P_1Q},$$

and as shown in Art. 4, $QP_2 = -y_1 + y_2$, and $P_1Q = -x_1 + x_2$,

$$\therefore \quad \tan QP_1P_2 = m = \frac{-y_1 + y_2}{-x_1 + x_2} = \frac{y_1 - y_2}{x_1 - x_2}, \qquad (4)$$

where m is used to designate the slope.

Since in deriving (3) and (4) the principles of directed lengths have been strictly followed the results are true for all positions of the points P_1 and P_2.

6. Division of a segment of a line in a given ratio.

Problem. *To find the coordinates of the point which divides a given segment of a line in a given ratio.*

Some preliminary remarks are necessary.

(*a*) By a given segment of a line is meant a segment the coordinates of whose extremities are known.

(*b*) A segment may be divided internally or externally. Thus in Fig. 11 (*a*) the point P divides the segment AB internally in the ratio $AP : PB$; and in Fig. 11 (*b*) the point P

(*a*) (*b*)

FIG. 11

divides the segment AB externally in the ratio $AP : PB$.

(*c*) When the division is *internal* the ratio is *positive*, because the two segments AP, PB, which are the terms of the ratio, are measured in the same direction, and hence have the same sign; but when the division is *external* the ratio is negative because in that case the two segments AP, PB are measured in opposite directions.

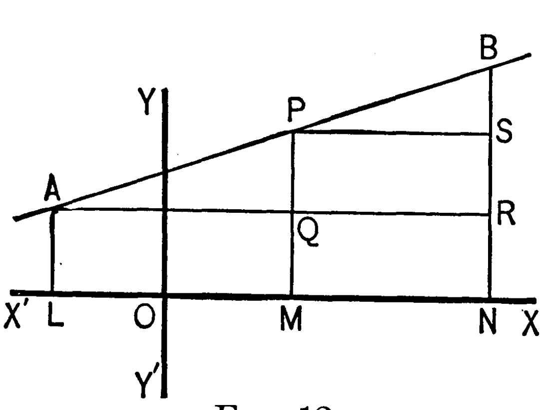

FIG. 12

Our problem then is to find the coordinates of P, having given the coordinates of A and B, and the ratio $AP : PB$.

Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $AP : PB = h : k$. Draw LA, MP, NB parallel to OY, and PS, AR parallel to OX. Then

$$\frac{AQ}{PS} = \frac{AP}{PB} = \frac{h}{k}, \tag{i}$$

$$\frac{QP}{SB} = \frac{AP}{PB} = \frac{h}{k}. \tag{ii}$$

But

$$\frac{AQ}{PS} = \frac{LM}{MN} = \frac{LO + OM}{MO + ON} = \frac{-x_1 + x}{-x + x_2}, \tag{iii}$$

$$\frac{QP}{SB} = \frac{QM + MP}{SN + NB} = \frac{AL + MP}{PM + NB} = \frac{-y_1 + y}{-y + y_2}. \tag{iv}$$

Hence, substituting from (*iii*) in (*i*) and from (*iv*) in (*ii*)

$$\frac{-x_1 + x}{-x + x_2} = \frac{h}{k}, \qquad \frac{-y_1 + y}{-y + y_2} = \frac{h}{k}. \tag{v}$$

Solving the first of (v) for x,

$$-kx_1 + kx = -hx + hx_2,$$

or

$$(h+k)x = hx_2 + kx_1,$$

and

$$x = \frac{hx_2 + kx_1}{h+k}.$$

Similarly

$$y = \frac{hy_2 + ky_1}{h+k}. \qquad (5)$$

It should be noted that equations (5) are true whether the axes are rectangular or oblique. Also that when P divides AB externally, the ratio being then negative, it is immaterial, in applying (5), whether the negative sign is attached to h or k.

Important Special Case. The case of bisection is especially important. In this case $h : k = 1 : 1$, and substituting $h = k = 1$ in (5) we have

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}, \qquad (6)$$

which are the coordinates of the point bisecting the line joining (x_1, y_1) and (x_2, y_2).

Exercises. 1. Find the length of the sides of the triangle whose vertices are the points (2, 3), (– 2, 0), (1, – 4). Ans. 5, 5, $5\sqrt{2}$.

2. Find the length of the sides and diagonals of the quadrilateral whose vertices are the points (5, 2), (2, 6), (– 3, 4), (– 1, – 1).

Ans. Sides 5, $\sqrt{29}$, $\sqrt{29}$, $3\sqrt{5}$; diagonals $\sqrt{68}$, $\sqrt{58}$.

3. Find the middle points of the sides of the triangle whose vertices are (5, 2), (– 1, 6), (1, – 4), and find the lengths of the medians of the triangle. Ans. $\sqrt{26}$, $\sqrt{65}$, $\sqrt{65}$.

4. What form does formula (3) take if one of the two given points is the origin?

5. Prove that the quadrilateral whose vertices are (4, 2), (– 2, – 1), (0, – 4), (6, – 1) is a parallelogram.

6. Find the coordinates of both points of trisection of the line joining the two points (1, 6), (4, – 2). Ans. $(2, \frac{10}{3})$, $(3, \frac{2}{3})$.

7. $A = (-2, 1)$, $B = (4, 3)$, and P divides AB externally so that $AP : PB = -2 : 5$. Find the coordinates of P. Ans. $(-6, -\frac{1}{3})$.

8. Trisect each median of the triangle in Ex. 3, so that the longer segment in each case is adjacent to the corresponding vertex. What does the result show?

9. Extend the line joining $P_1 = (-1, 3)$, $P_2 = (3, 5)$ each way a distance equal to P_1P_2, and find the coordinates of the two points thus determined. Ans. $(7, 7)$, $(-5, 1)$.

10. A parallelogram has two opposite vertices at $(1, 2)$ and $(4, 1)$, and a third vertex at the origin. Find the coordinates of the fourth vertex. Ans. $(5, 3)$.

7. Area of a triangle. Let the vertices of a triangle be

$$P_1 = (x_1, y_1),$$
$$P_2 = (x_2, y_2),$$
$$P_3 = (x_3, y_3).$$

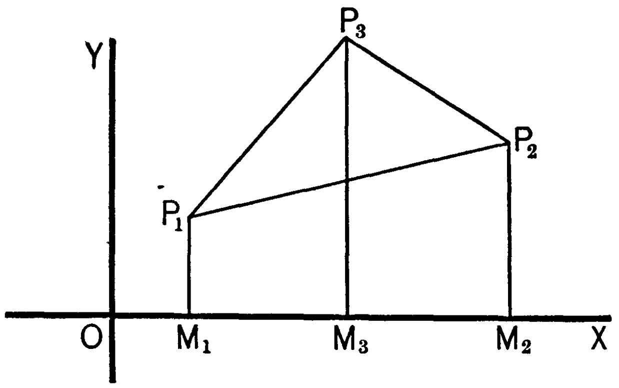

FIG. 13

Then area $P_1P_2P_3 =$

$$M_1M_3P_3P_1 + M_3M_2P_2P_3 - M_1M_2P_2P_1. \quad (i)$$

Since each of the areas on the right in (i) is a trapezoid we have

$$\text{area } M_1M_3P_3P_1 = \tfrac{1}{2}M_1M_3(M_1P_1 + M_3P_3) = \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1),$$
$$\text{area } M_3M_2P_2P_3 = \tfrac{1}{2}M_3M_2(M_3P_3 + M_2P_2) = \tfrac{1}{2}(x_2 - x_3)(y_2 + y_3),$$
$$\text{area } M_1M_2P_2P_1 = \tfrac{1}{2}M_1M_2(M_1P_1 + M_2P_2) = \tfrac{1}{2}(x_2 - x_1)(y_1 + y_2).$$

Hence substituting in (i)

$$\text{area } P_1P_2P_3 = \tfrac{1}{2}[(x_3 - x_1)(y_3 + y_1) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)],$$

from which, expanding and collecting terms,

$$\textbf{Area } \boldsymbol{P_1P_2P_3} = \tfrac{1}{2}[\boldsymbol{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)}]. \quad (7)$$

Formula (7) may also be written thus:

$$\text{area } P_1P_2P_3 = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]. \quad (8)$$

When the coordinates of the vertices of a given triangle are substituted in (7) or (8) the resulting number expressing the area of this triangle will be positive or negative according to the order in which the points are taken. The numerical value obtained, irrespective of sign, will always be the required area.

EXAMPLE. Find the area of the triangle whose vertices are $(-1, 2)$, $(3, -1)$, $(-4, 2)$.

Substituting the given coordinates in (7) the result is

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}[-(-1-2) + 3(2-2) - 4(2+1)]\\ &= \tfrac{1}{2}(3 + 0 - 12) = -4\tfrac{1}{2}.\end{aligned}$$

The required area is therefore $4\tfrac{1}{2}$.

The student acquainted with the determinant notation will recognize that formula (8) can be written

$$\text{area } P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}. \quad (9)$$

8. Area of any polygon. Since any polygon can be divided into triangles by diagonals drawn from one of the vertices, the area of such a figure can be determined by finding the areas of its several parts by means of (7), (8) or (9), and combining the results.

EXERCISES. 1. Determine the areas of the following triangles

(*a*) $(2, 6)$, $(4, -1)$, $(0, 0)$.
(*b*) $(3, -2)$, $(-4, 1)$, $(0, 5)$.
(*c*) $(1, -3)$, $(4, 5)$, $(-2, 3)$.
(*d*) $(0, 3)$, $(4, -5)$, $(-3, -2)$.
(*e*) $(5, -1)$, $(-1, -5)$, $(2, 4)$.
(*f*) $(3, -2)$, $(0, 4)$, $(5, 1)$.

Ans. (*a*) 13, (*b*) 20, etc.

2. Determine the area of the quadrilateral whose vertices are $(1, 4)$, $(5, -2)$, $(0, -3)$, $(-2, 0)$. Ans. $25\tfrac{1}{2}$.

3. Determine the area of the quadrilateral whose vertices are $(5, 2)$, $(2, -5)$, $(-5, 3)$, $(0, 1)$. Ans. 29.

EXERCISES* ON CHAPTER I

Normal Exercises

1. If the abscissa of a point is 4 on what line will it lie?

2. If the abscissa of a point is equal to its ordinate on what line will it lie?

3. What is the distance of the point (2, 4) from the origin?

4. How far is the point (− 3, − 5) from the point (− 1, 2)?

5. What is the slope of the line joining the point (2, 4) to the origin?

6. What is the slope of the line joining the points (2, − 3) and (− 4, − 2)?

7. Find the coordinates of the middle points of the sides of the triangle whose vertices are the points (0, 0), (− 2, − 4), (− 2, 3).

8. If the line from (− 2, 3) to (4, − 1) be extended through the latter point until its length is three times its original length, what are the coordinates of the end point? Ans. (16, − 9).

9. Find the area of the triangle of exercise 7.

10. Find the area of the quadrilateral whose vertices are (− 1, 3), (− 2, 3), (− 3, − 3), and (3, − 4). Ans. 22.

General Exercises

11. A straight line joining two points is bisected by the origin. One of the points is (2, 3). What is the other?

12. Determine which of the following sets of points are points of the same straight line

(*a*) (2, 4), (− 1, 0), (5, 8); (*c*) (2, 4), (3, 5), (5, 7), (0, 2);
(*b*) (− 1, 2), (0, − 1), (2, 4); (*d*) (− 1, 1), (2, − 1), (0, 0), (− 4, 3).

13. Show that (− 1, 2), (2, − 2), (4, 7), and (9, 5) are the vertices of a trapezoid.

* The exercises at the end of each chapter are divided into two groups, "Normal Exercises," and "General Exercises." Those given under the first heading require for their solution only the direct application of the methods or formulas developed in the chapter. Under the second heading are given other exercises of like character together with more varied problems requiring indirect, combined, or extended applications of the chapter's results.

14. Show that $(-1, 2)$, $(3, -1)$, $(4, 7)$, and $(8, 4)$ are the vertices of a parallelogram.

15. Show that $(2, 4)$, $(7, 3)$, $(-2, -3)$, $(-1, 2)$ are the vertices of an isosceles trapezoid.

16. Determine which of the following sets of points are the vertices of a parallelogram

(*a*) $(2, 5)$, $(0, 0)$, $(-1, -2)$, $(2, -4)$; (*b*) $(1, 2)$, $(7, 5)$, $(3, 3)$, $(5, 4)$.

17. Show that the middle points of the sides of the quadrilateral whose vertices are the points $(1, -2)$, $(2, 4)$, $(-1, -4)$, and $(3, -5)$ are the vertices of a parallelogram.

18. Find the coordinates of the fourth vertex of each of the parallelograms three of whose vertices are the points $(-1, 2)$, $(-3, 4)$, $(2, 1)$. Ans. $(0, 3)$, $(-6, 5)$, $(4, -1)$.

19. Find the lengths of the sides of the triangle whose vertices are the points $(2, 1)$, $(5, 5)$, and $(-5, 0)$. Ans. 5, $5\sqrt{2}$, $5\sqrt{5}$.

20. Show that the points $(-1, 3)$, $(-2, -4)$, $(6, -4)$, $(2, -6)$, and $(7, -1)$ are all on one circle whose center is the point $(2, -1)$.

21. Show that the diagonals of the quadrilateral whose vertices are the points $(0, 3)$, $(3, 5)$, $(5, 2)$, and $(2, 0)$ are equal. Show that this quadrilateral is a rectangle.

22. Find the point on the X-axis equally distant from the points $(2, 4)$ and $(-2, 6)$. Ans. $(-\frac{5}{2}, 0)$.

23. Find the point on the Y-axis equally distant from the points $(2, 4)$ and $(6, 5)$. Ans. $(0, 20\frac{1}{2})$.

24. Find the point equally distant from the points $(-2, -4)$, $(6, -4)$, and $(-1, 3)$. Ans. $(2, -1)$.

25. Show that the points $(-1, 2)$, $(4, -3)$, $(5, 3)$ are the vertices of an isosceles triangle.

26. Find the third vertex of one of the equilateral triangles two of whose vertices are the points $(1, 2)$, $(4, 6)$. Ans. $(\frac{5}{2} + 2\sqrt{3},\ 4 - \frac{3}{2}\sqrt{3})$.

27. The coordinates of the points A and B are $(-1, 2)$ and $(4, -2)$ respectively. P is a point on AB produced, and is twice as far from A as it is from B. Find its coordinates. Ans. $(9, -6)$.

28. The point $(4, 7)$ is on the line joining $(-1, 2)$ to the point B, and is one third as far from $(-1, 2)$ as from B. Find the coordinates of B. Ans. $(19, 22)$ or $(-11, -8)$.

29. Find the area of the triangles the coordinates of whose vertices are:

(*a*) (0, 0), (0, 4), (4, 0); (*b*) (− 1, − 4), (− 5, − 5), (− 2, − 1).

30. Determine two values of x so that the area of the triangle whose vertices are the points $(x, 2)$, $(2x, 1)$, and (4, 5) is one half the area of the triangle whose vertices are the points $(x, -1)$, (2, 3), and (5, 2). Ans. $-\frac{22}{3}$, $\frac{6}{5}$.

31. Show that the area of the triangle whose vertices are (− 3, 5), (3, 7), (5, − 1) is four times the area of the triangle whose vertices are the mid-points of its sides.

32. Find the area of the quadrilateral whose vertices are (4, 1), (− 1, 2), (− 2, − 3), and (1, 3). Ans. $16\frac{1}{2}$.

33. Find the area of the polygon whose vertices are (2, 4), (5, 1), (4, − 3), (1, − 5), (− 2, − 3), (− 2, 4); first by combining the areas of the triangles into which it is divided by diagonals from one vertex, and secondly, by combining the areas of the triangles formed by joining the vertices to the origin.

34. Find the coordinates of the middle points of each of the diagonals of the parallelogram whose vertices are (0, 0), $(a, 0)$, (b, c), and $(a + b, c)$, and thus prove that the diagonals of a parallelogram bisect each other.

NOTE. Exercises 35 to 40 inclusive are propositions in elementary plane geometry which are to be proved by the use of coordinates in a manner similar in general to that indicated in exercise 34.

35. Prove that the diagonals of a rectangle are equal to each other.

36. Prove that the mid-point of the hypotenuse of a right triangle is equally distant from each of the vertices.

37. The line joining the mid-points of two sides of a triangle is parallel to the third side and is half the length of the third side.

38. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

39. The lines joining the mid-points of opposite edges of any quadrilateral bisect each other.

40. The medians of a triangle intersect in one point which trisects each of them.