

Geometría Analítica II

TAREA-EXAMEN 2

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NOMBRE: _____

En la copias adjuntas, encontrará la sección de ejercicios a resolver, entre los cuales se pide resuelva 2, 3, 5, 7, 9, 10, 12, 14, 15, 16, 19, 20, 21, 23 y 25.

Nota: Argumente adecuadamente su respuesta; no serán tomadas en cuenta observaciones o señalamientos que realicen, sin su debida justificación.



THE SPHERE AND THE CIRCLE

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1. The sphere

A SPHERE is a surface traced by a point whose distance from a fixed point, the CENTRE, has a constant magnitude, the RADIUS. For centre (α, β, γ) and radius r , the equation is

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2.$$

In vector notation, if \mathbf{x} is the position vector of a point on a sphere of centre α and radius r , then

$$(\mathbf{x}-\alpha)^2 = r^2 \quad \text{or} \quad |\mathbf{x}-\alpha| = r.$$

NOTE. The GENERAL QUADRATIC FORM in the variables x, y, z contains three parts:

(i) a quadratic expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

(ii) a linear expression

$$2ux + 2vy + 2wz,$$

(iii) a constant d .

It is thus

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d.$$

The equation of the sphere, on expansion, is

$$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z + (\alpha^2 + \beta^2 + \gamma^2 - r^2) = 0.$$

Hence *necessary conditions for the general quadratic equation to represent a sphere are*

$$a = b = c,$$

$$f = g = h = 0.$$

Conversely, *these conditions are also sufficient, provided also that $a \neq 0$ and that*

$$u^2 + v^2 + w^2 - da > 0.$$

For the equation may then be written

$$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0,$$

or, after division by (non-zero) a ,

$$\left(x + \frac{u}{a}\right)^2 + \left(y + \frac{v}{a}\right)^2 + \left(z + \frac{w}{a}\right)^2 = \frac{u^2 + v^2 + w^2}{a^2} - \frac{d}{a}.$$

If the right-hand side is positive, this equation ensures that the variable point (x, y, z) is at constant distance

$$\sqrt{\{(u^2 + v^2 + w^2 - da)/a^2\}}$$

from the fixed point $(-u/a, -v/a, -w/a)$.

In practice the value of a is usually taken to be unity. We have then shown that the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

represents a sphere of centre $(-u, -v, -w)$ and radius

$$+\sqrt{(u^2 + v^2 + w^2 - d)}.$$

The equation of the sphere of centre the origin and radius a is

$$x^2 + y^2 + z^2 = a^2,$$

or, in vector notation, $\mathbf{x}^2 = a^2$.

It may be remarked that *the point $P(x_1, y_1, z_1)$ lies inside the sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

if $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d < 0$

and outside if

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d > 0.$$

The point lies inside if its distance from the centre is less than the radius; that is, if (squaring)

$$(x_1 + u)^2 + (y_1 + v)^2 + (z_1 + w)^2 < u^2 + v^2 + w^2 - d,$$

or $x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d < 0$.

The 'outside' test is proved similarly.

2. The sphere with a given diameter

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be two given points and $P(x, y, z)$ a variable point of the sphere on AB as diameter. The angle APB is a right angle, so that the direction ratios

$$(x - x_1, y - y_1, z - z_1), \quad (x - x_2, y - y_2, z - z_2)$$

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$$(x_1 + \lambda x_2)$$

$$+ 2(1 - \lambda)$$

represent perpendicular lines. Hence the equation of the sphere on (x_1, y_1, z_1) , (x_2, y_2, z_2) as diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0.$$

In vector notation, this is

$$(\mathbf{x}-\mathbf{x}_1) \cdot (\mathbf{x}-\mathbf{x}_2) = 0.$$

3. Joachimstal's ratio equation

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be two given points. The coordinates of the point dividing the segment \overrightarrow{PQ} in the (positive or negative) ratio $\lambda:1$ are (p. 17)

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda} \right).$$

To find a quadratic equation for the two values of λ for which this point lies on the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

The following notation, which is typical of much that will occur later, helps to make the statements more concise. Write

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d,$$

$$S_1 \equiv x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d,$$

$$S_{11} \equiv x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d,$$

$$S_{12} \equiv x_1 x_2 + y_1 y_2 + z_1 z_2 + u(x_1 + x_2) + v(y_1 + y_2) + w(z_1 + z_2) + d.$$

Then

$$S_{12} \equiv S_{21}.$$

The equation of the sphere is thus

$$S = 0,$$

and the conditions for P , Q to lie on it are

$$S_{11} = 0, \quad S_{22} = 0$$

respectively.

The point

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda} \right)$$

lies on the sphere if, on substituting its coordinates and then multiplying by $(1 + \lambda)^2$,

$$(x_1 + \lambda x_2)^2 + (y_1 + \lambda y_2)^2 + (z_1 + \lambda z_2)^2 + 2(1 + \lambda)\{u(x_1 + \lambda x_2) + v(y_1 + \lambda y_2) + w(z_1 + \lambda z_2)\} + d(1 + \lambda)^2 = 0.$$

Arrange in powers of λ and use the notation just defined:

$$S_{22}\lambda^2 + 2S_{12}\lambda + S_{11} = 0.$$

This is the quadratic equation whose roots serve to determine the two points where the line meets the sphere.

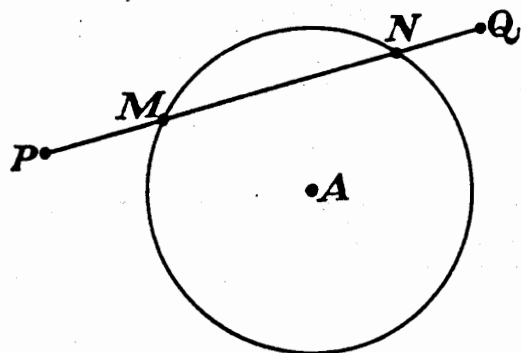


FIG. 36

The two points do not have real existence unless the roots of the quadratic are real; that is, unless

$$S_{12}^2 - S_{11}S_{22} \geq 0.$$

When real, they are denoted by the letters M, N (Fig. 36).

In vector notation, the point

$$\frac{\mathbf{x}_1 + \lambda \mathbf{x}_2}{1 + \lambda}$$

lies on the sphere if

$$(\mathbf{x} - \boldsymbol{\alpha})^2 = r^2$$

$$\{(\mathbf{x}_2 - \boldsymbol{\alpha})^2 - r^2\}\lambda^2 + 2\{(\mathbf{x}_1 - \boldsymbol{\alpha}) \cdot (\mathbf{x}_2 - \boldsymbol{\alpha}) - r^2\}\lambda + \{(\mathbf{x}_1 - \boldsymbol{\alpha})^2 - r^2\} = 0.$$

It is, however, very doubtful whether the language of vectors is of much use here.

4. Tangency

A straight line is said to be a **TANGENT** to a sphere at a point L (or to **TOUCH** it at L) if it meets the sphere at L and at no other point.

(i) **THE TANGENT PLANE.** In the notation of § 3, let the point $P(x_1, y_1, z_1)$ be chosen to lie on the sphere, so that

$$S_{11} = 0.$$

The equation for λ has thus one root zero; that is, one of the two points of intersection of the line and the sphere is at P . If, in addition, the line is chosen to be a tangent at P , there can be (by definition) no root other than zero, so that the second root of the quadratic equation is also zero. Hence $S_{12} = 0$. Thus the condition for $Q(x_2, y_2, z_2)$ to lie on a tangent line at $P(x_1, y_1, z_1)$ is $S_{12} = 0$.

As Q varies the equation satisfies x, y, z on any line

$$S_1 \equiv x_1^2$$

or

$$(x_1 + \dots)$$

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(ii) **THE TANGENT PLANE.** § 3, the line PQ is a tangent so that J for this is

If, then, that the line Q satisfies the variables

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As Q varies subject to this condition, its coordinates satisfy the equation found by replacing x_2, y_2, z_2 by current coordinates x, y, z , namely $S_1 = 0$. Thus the coordinates of any point on any line touching the sphere at $P(x_1, y_1, z_1)$ satisfy the equation

$$S_1 \equiv x_1x + y_1y + z_1z + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0,$$

or

$$(x_1+u)x + (y_1+v)y + (z_1+w)z + (ux_1+vy_1+wz_1+d) = 0.$$

This is the equation of a plane, called the TANGENT PLANE at P to the sphere. It contains all the tangent lines through P .

COROLLARY. The direction ratios of the normal to the tangent plane at P are (p. 21)

$$(x_1+u, y_1+v, z_1+w),$$

and these (p. 19) are also the direction ratios of the radius joining P to the centre $(-u, -v, -w)$. Hence the tangent plane at P is perpendicular to the radius through P .

(ii) THE TANGENT CONE. Suppose next that, in the work of § 3, the points P, Q do not lie on the sphere, but that the line PQ is a tangent. The two points M, N (Fig. 36) then coincide, so that Joachimstal's equation has equal roots. The condition for this is

$$S_{11}S_{22} = S_{12}^2.$$

If, then, P is regarded as given, while Q moves in such a way that the line PQ always touches the sphere, the coordinates of Q satisfy the relation found by replacing x_2, y_2, z_2 by the current variables x, y, z , namely

$$S_{11}S = S_1^2.$$

The locus of Q is called the TANGENT CONE from P to the sphere, so that the coordinates (x, y, z) of any point Q on the tangent cone from $P(x_1, y_1, z_1)$ satisfy the equation

$$S_{11}S = S_1^2.$$

For example, the tangent cone to the sphere

$$x^2 + y^2 + z^2 = a^2$$

is $(x_1^2 + y_1^2 + z_1^2 - a^2)(x^2 + y^2 + z^2 - a^2) = (x_1x + y_1y + z_1z - a^2)^2$;

and the tangent cone from the origin to the general sphere is

$$d(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) = (ux + vy + wz + d)^2,$$

or $(u^2 - d)x^2 + (v^2 - d)y^2 + (w^2 - d)z^2 + 2vwy + 2wuzx + 2uvxy = 0$.

5. Pole and polar; harmonic separation

The interpretation of the relation $S_{12} = 0$ in terms of tangency at $P(x_1, y_1, z_1)$ when P is on the sphere ($S_{11} = 0$) suggests consideration of the relation $S_{12} = 0$ under the more general condition $S_{11} \neq 0$.

Joachimstal's equation

$$S_{22}\lambda^2 + 2S_{12}\lambda + S_{11} = 0$$

becomes, under the conditions $S_{12} = 0$, $S_{11} \neq 0$,

$$S_{22}\lambda^2 + S_{11} = 0,$$

and then the two values† of λ are equal in magnitude but opposite in sign. By definition of λ , the two values are (p. 17)

$$\vec{PM}/\vec{MQ}, \quad \vec{PN}/\vec{NQ},$$

so that M and N divide \vec{PQ} internally (λ positive) and externally (λ negative) in the same ratio.

DEFINITION. *Four points P, Q, M, N such that*

$$\vec{PM}/\vec{MQ} = -\vec{PN}/\vec{NQ},$$

(so that M and N divide \vec{PQ} internally and externally in the same ratio) are said to form a HARMONIC RANGE. The points M, N are called HARMONIC CONJUGATES with respect to P, Q .

COROLLARY. *If M, N are harmonic conjugates with respect to P, Q , then P, Q are also harmonic conjugates with respect to M, N :*

For the relation

$$\vec{PM}/\vec{MQ} = -\vec{PN}/\vec{NQ}$$

is also

$$\vec{PM}/\vec{PN} = -\vec{MQ}/\vec{NQ},$$

or

$$\vec{MP}/\vec{PN} = -\vec{MQ}/\vec{QN}.$$

We return to the main problem. Since the relation $S_{12} = 0$ gives $\vec{PM}/\vec{MQ} = -\vec{PN}/\vec{NQ}$, it follows that the points P, Q are

† The (real) values of λ exist only if S_{11}, S_{22} have opposite signs; that is (p. 100), if one of the points P, Q is inside the sphere and the other outside. But we do not wish to emphasize this aspect unduly.

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separated harmonically by the two points in which the line PQ meets the sphere. Two points, such as P, Q , related to the sphere in this way are said to be CONJUGATE with respect to it. Thus the condition for the two points $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ to be conjugate with respect to the sphere $S = 0$ is

$$S_{12} = 0.$$

Suppose now that the point P is regarded as given. Then those points Q such that P, Q are conjugate with respect to the sphere lie in the plane given by the equation

$$S_1 = 0.$$

This plane is called the POLAR PLANE of P with respect to the sphere; also P is the POLE of its polar plane.

Note that, if P lies on the sphere, then the polar plane of P , given by the equation $S_1 = 0$, is (p. 103) the tangent plane at P .

The relation (p. 101)

$$S_{12} = S_{21}$$

shows that, if the polar plane of P passes through Q (so that $S_{12} = 0$), then the polar plane of Q passes through P (since $S_{21} = 0$).

6. The segment theorem; diameters

The work of this section is very similar to that of § 3 (p. 101), but it deals with distances and directions instead of ratios.

Let $P(x_1, y_1, z_1)$ be a given point and (l, m, n) the direction cosines of a line through P . The point $Q(x, y, z)$ on this line, such that $\vec{PQ} = r$, satisfies the relations

$$x = x_1 + lr, \quad y = y_1 + mr,$$

$$z = z_1 + nr.$$

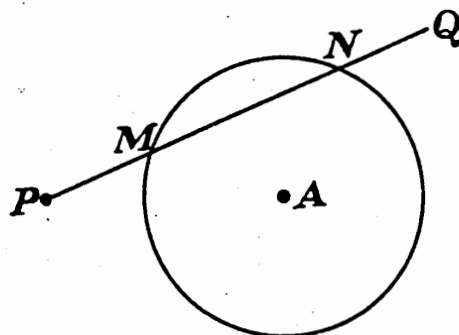


FIG. 37

The line cuts the sphere in two points M, N . To prove that the two values of r corresponding to M, N satisfy the equation

$$r^2 + 2r\{(x_1 + u)l + (y_1 + v)m + (z_1 + w)n\} + S_{11} = 0.$$

Substitute the values of x, y, z into the equation of the sphere; thus

$$(lr+x_1)^2+\dots+2u(lr+x_1)+\dots+d=0.$$

Arrange in powers of r , remembering that $l^2+m^2+n^2=1$; thus

$$r^2+2r\{(x_1+u)l+(y_1+v)m+(z_1+w)n\}+S_{11}=0.$$

This equation is called the r -EQUATION of the point $P(x_1, y_1, z_1)$ and the direction (l, m, n) for the sphere S .

In vector notation, the point $\mathbf{x}_1+r\mathbf{l}$ lies on the sphere

$$(\mathbf{x}-\boldsymbol{\alpha})^2=k^2, \quad \text{or} \quad |\mathbf{x}-\boldsymbol{\alpha}|=k,$$

if

$$r^2+2\mathbf{l} \cdot (\mathbf{x}_1-\boldsymbol{\alpha})r+\{(\mathbf{x}_1-\boldsymbol{\alpha})^2-k^2\}=0.$$

THE RECTANGLE THEOREMS. *It is important to remember that the formulae now to be given pre-suppose that S is expressed in a form such that the coefficients of x^2, y^2, z^2 are unity.*

(i) To prove that, if a variable line through a fixed point $P(x_1, y_1, z_1)$ meets a given sphere in points M, N , then $\vec{PM} \cdot \vec{PN}$ is constant.

The r -equation for the fixed point (x_1, y_1, z_1) and (variable) direction (l, m, n) is

$$r^2+\{\dots\}r+S_{11}=0,$$

so that, if $\vec{PM} \equiv r_1, \vec{PN} \equiv r_2$, the product of the roots is given by the formula

$$r_1 r_2 = S_{11};$$

thus

$$\vec{PM} \cdot \vec{PN} = S_{11}.$$

But the right-hand side is independent of l, m, n and is therefore constant.

Note that, if P is outside the sphere, \vec{PM}, \vec{PN} have the same signs, so that their product is positive; if P is inside the sphere, \vec{PM}, \vec{PN} have opposite signs so that their product is negative. Hence the point P lies outside or inside the sphere according as S_{11} is positive or negative. (Compare p. 100.)

(ii) To prove that, if $P(x_1, y_1, z_1)$ lies outside the sphere $S=0$, then the length t of a tangent from P to the sphere is given by the formula

$$t^2 = S_{11}.$$

This is merely the formula $r_1 r_2 = S_{11}$ when $r_1 = r_2 = t$.

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DIAMETERS. Suppose next that P is the middle point of a chord MN whose direction is (l, m, n) . Then $\vec{PM} = -\vec{PN}$, and so the r -equation of P for that direction has roots which are equal and opposite. The coefficient of r thus vanishes, so that

$$l(x_1+u)+m(y_1+v)+n(z_1+w)=0.$$

Hence, replacing x_1, y_1, z_1 by current variables x, y, z , the middle points of chords in the given direction (l, m, n) all lie in the plane

$$lx+my+nz+(lu+mv+nw)=0.$$

This plane passes through the centre $(-u, -v, -w)$ of the sphere and is perpendicular to the given direction (l, m, n) .

The condition

$$l(x_1+u)+m(y_1+v)+n(z_1+w)=0$$

is satisfied for all values of l, m, n when $P(x_1, y_1, z_1)$ is at the centre $(-u, -v, -w)$ of the sphere; that is, all chords through the centre of the sphere are bisected there.

There are many problems in which it is convenient to take as a starting-point a circle of given centre drawn on the sphere. The following theorem is useful:

To prove that the equation of the plane cutting the sphere $S=0$ in a circle of centre $P(x_1, y_1, z_1)$ is

$$S_1 = S_{11}.$$

The centre of the sphere is $A(-u, -v, -w)$, so that the direction ratios of AP are

$$(x_1+u, y_1+v, z_1+w).$$

Hence the plane, being perpendicular to AP and passing through P , is

$$(x_1+u)(x-x_1)+(y_1+v)(y-y_1)+(z_1+w)(z-z_1)=0,$$

or

$$(x_1+u)x+(y_1+v)y+(z_1+w)z$$

$$= (x_1+u)x_1+(y_1+v)y_1+(z_1+w)z_1,$$

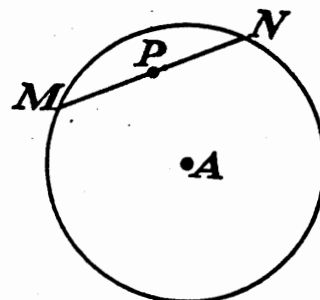


FIG. 38

or, adding $ux_1 + vy_1 + wz_1 + d$ to each side,

$$S_1 = S_{11}.$$

ILLUSTRATION. To prove that the centre of a circle cut on the sphere S by a plane through the given point $P(x_1, y_1, z_1)$ lies on the sphere whose equation is

$$S = S_1.$$

Suppose that the centre of a typical circle is $Q(x_2, y_2, z_2)$. Then the equation of the plane of the circle is

$$S_2 = S_{22}.$$

This passes through P if

$$S_{12} = S_{22}.$$

The locus of Q , found by replacing x_2, y_2, z_2 by current coordinates x, y, z , is therefore

$$S_1 = S.$$

7. Orthogonal spheres

DEFINITION. Two intersecting spheres

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

are called **ORTHOGONAL** (cutting AT RIGHT ANGLES) if the tangent planes at a common point $P(x_1, y_1, z_1)$ are perpendicular.

To prove that the condition for S, S' to be orthogonal is

$$2uu' + 2vv' + 2ww' = d + d'.$$

The tangent planes

$$(x_1 + u)x + (y_1 + v)y + (z_1 + w)z + \dots = 0,$$

$$(x_1 + u')x + (y_1 + v')y + (z_1 + w')z + \dots = 0$$

are perpendicular if and only if

$$(x_1 + u)(x_1 + u') + (y_1 + v)(y_1 + v') + (z_1 + w)(z_1 + w') = 0,$$

or

$$x_1^2 + y_1^2 + z_1^2 + (u + u')x_1 + (v + v')y_1 + (w + w')z_1 + uu' + vv' + ww' = 0.$$

Since P lies on each sphere,

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0,$$

$$x_1^2 + y_1^2 + z_1^2 + 2u'x_1 + 2v'y_1 + 2w'z_1 + d' = 0.$$

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Rearranging,

$$-(x_1 + u)u'$$

Add and divide by 2:

$$x_1^2 + y_1^2 + z_1^2 + (u+u')x_1 + (v+v')y_1 + (w+w')z_1 + \frac{1}{2}(d+d') = 0.$$

The preceding equation of condition thus gives

$$uu' + vv' + ww' = \frac{1}{2}(d+d').$$

COROLLARIES. (i) *The tangent planes to two orthogonal spheres are perpendicular at every common point. The condition*

$$2uu' + 2vv' + 2ww' = d+d'$$

is, in fact, independent of the point $P(x_1, y_1, z_1)$ from which the argument started.

(ii) *If two orthogonal spheres, of radii a, b , have their centres distant k apart, then*

$$k^2 = a^2 + b^2.$$

For

$$a^2 = u^2 + v^2 + w^2 - d, \quad b^2 = u'^2 + v'^2 + w'^2 - d',$$

$$k^2 = (u-u')^2 + (v-v')^2 + (w-w')^2,$$

so that

$$\begin{aligned} a^2 + b^2 - k^2 &= 2uu' + 2vv' + 2ww' - d - d' \\ &= 0. \end{aligned}$$

(iii) *If two spheres are orthogonal, then the centre of either lies in the tangent plane to the other at any common point.*

The tangent plane to S at $P(x_1, y_1, z_1)$ contains the centre $(-u', -v', -w')$ of S' if

$$-(x_1+u)u' - (y_1+v)v' - (z_1+w)w' + ux_1 + vy_1 + wz_1 + d = 0,$$

where

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0,$$

$$x_1^2 + y_1^2 + z_1^2 + 2u'x_1 + 2v'y_1 + 2w'z_1 + d' = 0.$$

Subtract the last two equations:

$$2(u-u')x_1 + 2(v-v')y_1 + 2(w-w')z_1 + d - d' = 0.$$

Subtract from this the orthogonality relation

$$2uu' + 2vv' + 2ww' - d - d' = 0$$

and divide by 2:

$$(u-u')x_1 + (v-v')y_1 + (w-w')z_1 - uu' - vv' - ww' + d = 0.$$

Rearranging, this is the required condition

$$-(x_1+u)u' - (y_1+v)v' - (z_1+w)w' + ux_1 + vy_1 + wz_1 + d = 0.$$

NOTE. The results of these Corollaries are otherwise obvious, and could have been used as a basis for the discussion. The treatment actually used does, however, lay greater emphasis on the root conception of orthogonality.

8. Pairs of spheres; circles

Let the equations of two given spheres be

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0,$$

of centres $A(-u, -v, -w)$, $A'(-u', -v', -w')$.

Consider the equation

$$S - kS' = 0.$$

When $k = 1$, this represents the plane

$$2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0,$$

and we regard this case temporarily as excluded. When $k \neq 1$, the equation, after division by $1 - k$, is

$$S_k \equiv x^2 + y^2 + z^2 + 2u_k x + 2v_k y + 2w_k z + d_k = 0,$$

where

$$u_k = \frac{u - ku'}{1 - k}, \quad v_k = \frac{v - kv'}{1 - k}, \quad w_k = \frac{w - kw'}{1 - k}, \quad d_k = \frac{d - kd'}{1 - k}.$$

This equation represents a *sphere* of centre $B(-u_k, -v_k, -w_k)$.

Now the equation $S - kS' = 0$ is satisfied whenever,

$$S = 0, \quad S' = 0$$

simultaneously. Hence the sphere S_k (and the plane when $k = 1$) passes through all the points, if any, common to the two given spheres S, S' .

In particular, the curve common to two intersecting spheres lies entirely in a plane. The intersection is therefore a CIRCLE.

Observe carefully that *two equations are necessary to specify a circle in space*. A natural choice would be the equations of the plane containing the circle and of a sphere through it. Alternatively, the equations of two spheres might be selected, the equation of the plane, if required, being obtained by the process just described. If the equations $U = 0, V = 0$ represent either a plane and a sphere or two spheres, then all spheres

through the
 $U - kV = 0$

ILLUSTRATION
origin and

$$x^2 + y^2 + z^2 = r^2$$

The equation

$$x^2 + y^2 + z^2 = r^2$$

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NOTE: To determine
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 $(-3, -4, 0)$

from the plane

But

and so the

Finally,

cuts the sphere

through their circle of intersection are given by the equation $U - kV = 0$ for appropriate values of k .

ILLUSTRATION. *To find the equation of the sphere through the origin and the circle*

$$x^2 + y^2 + z^2 + 6x + 8y + 10 = 0, \quad 4x + 3y + 2z + 5 = 0.$$

The equation of any sphere through the circle is

$$x^2 + y^2 + z^2 + 6x + 8y + 10 - k(4x + 3y + 2z + 5) = 0,$$

and it passes through the origin if

$$10 - 5k = 0,$$

or

$$k = 2.$$

Hence the equation is

$$x^2 + y^2 + z^2 - 2x + 2y - 4z = 0.$$

NOTE: The effective existence of the circle depends on whether the plane cuts the sphere or not. A simple test to settle this point is that *the sphere cuts the plane provided that the distance of its centre from the plane is less than its radius.*

Consider, for example, the sphere

$$x^2 + y^2 + z^2 + 6x + 8y + 10 = 0$$

and the plane $4x + 3y + 2z + 5 = 0$

of the preceding Illustration. The centre of the sphere is $(-3, -4, 0)$, which is at a distance

$$\frac{-12 - 12 + 5}{\pm\sqrt{29}} = \frac{19}{\sqrt{29}}$$

from the plane. Also the radius of the sphere is

$$\sqrt{(9 + 16 - 10)} = \sqrt{15}.$$

But

$$19/\sqrt{29} < \sqrt{15},$$

and so the plane cuts the sphere.

Finally, *to find the radius of the circle in which the plane*

$$lx + my + nz + p = 0$$

cuts the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0:$$

If a is the radius of the circle, r the radius of the sphere, and b the distance of the centre $(-u, -v, -w)$ from the plane, then

$$\begin{aligned} a^2 &= r^2 - b^2 \\ &= (u^2 + v^2 + w^2 - d) - \left\{ \frac{(-lu - mv - nw + p)^2}{\sqrt{(l^2 + m^2 + n^2)}} \right\} \\ &= \frac{(l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d) - (lu + mv + nw - p)^2}{(l^2 + m^2 + n^2)}. \end{aligned}$$

COROLLARY. *The condition for the plane to touch the sphere is*

$$(l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d) = (lu + mv + nw - p)^2,$$

and the condition for the plane to intersect the sphere in a circle is

$$(l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d) > (lu + mv + nw - p)^2.$$

9. The radical plane

There is another way of interpreting the equation

$$S - kS' = 0$$

considered in § 8. Denote by t, t' the lengths of the tangents from the point $P(x, y, z)$ to the spheres

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0,$$

so that (p. 106) $t^2 = S, \quad t'^2 = S'.$

Then the locus of a point which moves so that

$$t = mt'$$

is the sphere (plane if $m = 1$)

$$S = m^2 S'.$$

The plane is called the **RADICAL PLANE** of the two given spheres, and the system of spheres defined by the equation

$$S = m^2 S'$$

for varying m is called a **COAXAL SYSTEM**. Each two spheres selected from the coaxal system have the plane $S = S'$ as their radical plane.

The above statement gives the definition of the radical plane in its most graphic form, but it needs modifying if, for example, there are values of x, y, z for which S is negative (compare

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10. Coaxal

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p. 100). To meet this difficulty, define the POWER of a point $P(x_1, y_1, z_1)$ with respect to a sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

to be the function

$$S_{11} \equiv x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d.$$

Then the RADICAL PLANE of two spheres S, S' is the locus of a point whose powers with respect to the spheres are equal.

The result may be extended. The RADICAL LINE of three spheres S, S', S'' is the locus of a point whose powers with respect to the three spheres are equal. The locus is given by the two equations

$$S = S' = S'',$$

and is a straight line. [For example, it is the line of intersection of the two planes $S - S' = 0, S - S'' = 0$, which, in the general case, are not parallel.]

Similarly the RADICAL CENTRE of four spheres S, S', S'', S''' is that point (unique for general positions of the spheres, with which alone we concern ourselves) whose powers with respect to the four spheres are equal. The point is given by the three equations

$$S = S' = S'' = S'''.$$

It follows easily (compare p. 109) that the sphere with centre any point (i) on the radical plane of two spheres, (ii) on the radical line of three, or (iii) at the radical centre of four, and having its radius equal to the tangents from the point to the spheres, cuts orthogonally each of the two, three, or four spheres.

10. Coaxal system; simplified equation

The radical plane of the two spheres

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

$$\text{is } 2(u - u')x + 2(v - v')y + 2(w - w')z + (d - d') = 0,$$

and the direction cosines of its normals are

$$(u - u', v - v', w - w').$$

Hence the radical plane of two spheres is perpendicular to their line of centres.

If, then, the line of centres is taken to be the axis $y = z = 0$, the equations of the spheres appear in the simpler form

$$S \equiv x^2 + y^2 + z^2 + 2ux + d = 0,$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + d' = 0,$$

and the radical plane is

$$2(u - u')x + (d - d') = 0.$$

Suppose, further, that the origin is chosen to be that point where the line of centres meets the radical plane; then

$$d - d' = 0.$$

Hence the equations of the two given spheres may be reduced to the simplified form

$$S \equiv x^2 + y^2 + z^2 + 2ux + d = 0,$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + d = 0.$$

The spheres of the coaxial system are then

$$x^2 + y^2 + z^2 + \frac{2(u - \lambda u')}{1 - \lambda}x + d = 0$$

for varying λ . Thus, writing

$$\frac{u - \lambda u'}{1 - \lambda} \equiv \mu,$$

the equation for the spheres of a coaxial system may be expressed in the form

$$x^2 + y^2 + z^2 + 2\mu x + d = 0$$

for varying μ .

The radical plane is $x = 0$.

This plane meets the spheres of the system, if at all, in the circle

$$x = 0, \quad y^2 + z^2 + d = 0.$$

The circle exists if d is negative, but not if it is positive.

We confine our attention now to non-intersecting spheres, for which d is positive; say $d = a^2$. Then the spheres are

$$x^2 + y^2 + z^2 + 2\mu x + a^2 = 0.$$

In particular, the two 'spheres' given by $\mu = -a$ and $\mu = +a$ are

$$(x \mp a)^2 + y^2 + z^2 = 0,$$

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and so reduce to the two points $(a, 0, 0)$, $(-a, 0, 0)$. They are called the **LIMITING POINTS** of the coaxal system.

Consider next any sphere (if existing) which cuts *each* sphere of the coaxal system orthogonally. Such a sphere has equation

$$x^2 + y^2 + z^2 + 2px + 2qy + 2rz + w = 0,$$

where, for orthogonality (p. 108),

$$2p\mu - a^2 - w = 0.$$

For this to be true for *all* values of μ , we need the relations

$$p = 0, \quad w = -a^2,$$

so that the equation of a typical sphere is

$$x^2 + y^2 + z^2 + 2qy + 2rz - a^2 = 0.$$

But this sphere passes through the two limiting points $(\pm a, 0, 0)$. Hence *every sphere cutting the spheres of a coaxal system orthogonally passes through the limiting points.*

MISCELLANEOUS EXAMPLES

1. Find the equation of the sphere whose centre is the point $(2, 2, 1)$ and which touches the plane $3x + 4y + 12z = 0$.

The plane $z = h$ cuts the sphere in a circle. Prove that the radius of the circle is $\sqrt{\{(3-h)(1+h)\}}$, and deduce the equations of those tangent planes to the sphere which are parallel to the plane $z = 0$.

2. Prove that the two circles

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, \quad 5y + 6z + 1 = 0$$

$$\text{and} \quad x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, \quad x + 2y - 7z = 0$$

lie on the same sphere, and find its equation.

3. Find the equation of the sphere whose centre is the point $(1, 2, 3)$ and which touches the plane given by the equation $3x + 2y + z + 4 = 0$.

Find also the radius of the circle in which the sphere is cut by the plane $x + y + z = 0$.

$$4. \text{ The line } \frac{x+2}{3} = \frac{y+1}{4} = \frac{z-8}{-5}$$

intersects the sphere

$$x^2 + y^2 + z^2 - 2x - 6y + 4z - 11 = 0$$

in the points P_1, P_2 . Find the coordinates of P_1, P_2 , and obtain also the equations of the line that passes through the centre of the sphere and through the mid-point of P_1P_2 .

5. Find the equation of the sphere with centre $(3, 0, 8)$ which cuts off a chord of length 16 units on the line

$$2x + y - z = 7, \quad 4x - 4y - 5z = 29.$$

6. Find the equations of the straight line through the point $P(18, 23, -3)$ and the centre C of the sphere

$$S \equiv x^2 + y^2 + z^2 + 8x - 6y + 14z + 38 = 0.$$

Hence, or otherwise, find the equations of the spheres which have their centres on the line PC , pass through P , and touch S .

7. A point P moves on the surface of a sphere

$$x^2 + y^2 + z^2 + 2x - 4y + 1 = 0$$

in such a way that its distance from the point $U(2, 1, -3)$ is always 3. Find the equation of the plane in which P always lies.

The line UP cuts the sphere again in Q . Find the equation of the plane in which Q always lies, and the distance between these two planes.

8. Find the equation of the sphere whose centre is the origin and whose radius is 5 units.

Find the range of values of λ for which the plane

$$3x + 4y + 12z = \lambda$$

cuts the sphere, and find the radius of the circle of intersection when $\lambda = 39$.

9. Find the centre and radius of the circle in which the spheres

$$x^2 + y^2 + z^2 - 8x - 10y - 4z - 15 = 0,$$

$$x^2 + y^2 + z^2 + 2x + 10y + 6z + 5 = 0$$

intersect, and obtain the equation of the sphere on which this circle is a great circle.

10. A sphere passes through the points $(4, 3, -2)$, $(-1, -1, 1)$, $(3, 0, -2)$, $(2, 3, 2)$. Find its equation.

Find the centre and radius of the section of the sphere by the plane $x - y = 0$ and the equations of the projection of this section on to the plane $x = 0$.

11. A sphere has its centre at the point $(0, -2, 1)$, and it touches the plane which passes through the point $(1, 1, 0)$ and the line

$$\frac{x-1}{1} = \frac{y+1}{4} = \frac{z+1}{1}.$$

Find the radius of the sphere and its point of contact with the plane.

12. Find the equation of the sphere through the points $(1, -2, 0)$, $(0, -2, -1)$, $(1, -1, -1)$, $(1, -3, -1)$, and show that the plane

$$x + y - z = 0$$

passes through the centre of the sphere.

13. Find the equation of the sphere with centre $(1, 2, 3)$ and radius 5.

Show that the plane $3x + 4y + 12z = 86$ cuts the sphere in a circle of radius 4, and find the equation of the parallel plane at the same distance from the centre but on the opposite side.

14. Points A, B, C, D have coordinates $(3, 5, 2)$, $(1, 3, 0)$, $(3, 4, 1)$, $(-1, 6, -1)$ respectively. Find the points in which the straight line CD meets the sphere of which AB is a diameter.

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15. Show that the spheres

$$x^2 + y^2 + z^2 - 2x - 2z - 2 = 0,$$

$$x^2 + y^2 + z^2 - 8x - 4y - 2z + 20 = 0$$

do not intersect.

Obtain conditions for the plane

$$lx + my + nz = d,$$

where

$$l^2 + m^2 + n^2 = 1,$$

to touch both spheres. Deduce that all such planes pass through one or other of two fixed points collinear with the centres of the spheres, and find the coordinates of these points.

16. Find the equation of the sphere through the origin O and the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

If U is the centre of this sphere, show that the sphere on OU as diameter passes through the mid-points of the six edges of the tetrahedron $OABC$.

17. A is the point $(0, 0, 1)$, P is a point of the sphere of unit radius and centre the origin O , and Q is a point of the plane $z = a - 1$, where $-1 < a < 1$. If AP and OP are perpendicular to PQ and OQ respectively, show that the positions of P are confined to a certain circle on the sphere and those of Q to the region exterior to the circle

$$x^2 + y^2 = a^2(1 - a)/(1 + a)$$

in the plane.

18. Find the values of d for which the plane

$$3x - 2y + z = d$$

touches the sphere

$$x^2 + y^2 + z^2 - 2x - 4y + 2z - 8 = 0,$$

and obtain the coordinates of the points of contact.

19. The points A, B, C, D have coordinates $(5, -3, 2)$, $(6, -2, 2)$, $(5, -2, 3)$, $(6, -3, 3)$ respectively. Show that spheres may be centred on these points so that each sphere touches the three others externally.

A plane (not intersecting the sphere about A) is laid in contact with the spheres about B, C, D . Find its distance from A , and its equation.

Find the equation of the sphere through all four points.

20. Find the condition for the plane

$$lx + my + nz = p$$

to cut the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz - c = 0$$

in a (real) circle.

Prove that the plane $x + 2y - z = 4$

cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$

in a circle of unit radius, and find the equation of the sphere which has this circle as one of its great circles.

21. Prove that the tangent lines from the origin of coordinates to the sphere

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2$$

are the generators of the cone given by the equation

$$(a^2 + b^2 + c^2 - k^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2.$$

22. Find the plane, the centre, and the radius of the circle common to the two spheres

$$x^2 + y^2 + z^2 - 4z + 1 = 0,$$

$$x^2 + y^2 + z^2 - 4x - 2y - 1 = 0.$$

23. Find the length of the chord cut on the line

$$x - 2y + 3 = 0, \quad 2x - 2y - z + 5 = 0$$

by the sphere

$$x^2 + y^2 + z^2 - 2x + 3y - 16 = 0.$$

24. Find the centre and radius of the circle common to the two spheres

$$x^2 + y^2 + z^2 - 3y - 5z - 2 = 0,$$

$$x^2 + y^2 + z^2 - 4x - 5y - 7z + 12 = 0.$$

25. If \mathbf{n} is a unit vector, show that the condition for the plane $\mathbf{n} \cdot \mathbf{r} = p$ to touch the sphere $(\mathbf{r} - \mathbf{c})^2 = a^2$ is

$$(p - \mathbf{n} \cdot \mathbf{c})^2 = a^2.$$

A cone has its vertex at the origin and consists of tangents to a sphere of radius a and centre \mathbf{b} . Show that the position vectors \mathbf{r} of points on the cone satisfy the equation

$$(\mathbf{r} \cdot \mathbf{b})^2 = (b^2 - a^2)r^2.$$

26. Three planes have equations

$$\mathbf{r} \cdot \mathbf{l} = 0, \quad \mathbf{r} \cdot \mathbf{m} = 0, \quad \mathbf{r} \cdot \mathbf{n} = 0,$$

where $\mathbf{l}, \mathbf{m}, \mathbf{n}$ are unit vectors. Give the conditions for a vector \mathbf{p} to be equally inclined to $\mathbf{l}, \mathbf{m}, \mathbf{n}$.

Find \mathbf{p} when $\mathbf{l}, \mathbf{m}, \mathbf{n}$ point in the directions $(1, 2, 2), (2, 3, 6), (0, 3, 4)$. Deduce that there is a cone of semi-vertical angle $\cos^{-1}(1/\sqrt{26})$ touching all three planes, and give the vector equation of this cone.

27. The position vector \mathbf{x} of a point P at time t satisfies the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{w} \wedge \mathbf{x},$$

where \mathbf{w} is a fixed vector. Show that P lies on a fixed sphere and also in a fixed plane.

Deduce that P moves on a circle, and show that it describes the circle with constant speed.

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