A Note on going from F to $\mathsf{F}^{\mathsf{Curry}}$ and back *

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We work out in detail the transfer process between second order polymorphic lambda calculus à la Church $\langle \mathsf{F}, \rhd \rangle$ and its version $\langle \mathsf{F}^{\mathsf{Curry}}, \rhd_{\mathsf{C}} \rangle$ as a type assignment system. We will prove that if one of these systems strongly normalizes then also the other. Moreover we conclude subject reduction of $\mathsf{F}^{\mathsf{Curry}}$ from that of F.

Definition 1 (Erase Functions). *The functions* $e, e' : Term(\mathsf{F}) \to Term(\mathsf{F}^{\mathsf{Curry}})$ *which erase types, are defined as follows:*

$$\begin{array}{rlrl}
e(x) &:= x & e'(x) &:= x \\
e(c) &:= c & e'(c) &:= c \\
e(\lambda x^{\sigma}.M) &:= \lambda x.e(M) & e'(\lambda x^{\sigma}.M) &:= \lambda x.e'(M) \\
e(MN) &:= e(M)e(N) & e'(MN) &:= e'(M)e'(N) \\
e(\Lambda\alpha.M) &:= e(M) & e'(\Lambda\alpha.M) &:= (\lambda y.y)e'(M) \\
e(M\sigma) &:= e(M) & e'(M\sigma) &:= e'(M)
\end{array}$$

Note that the only difference between e and e' is the case for universal abstractions.

Proposition 1 (Properties of e, e').

1. $e(M[x := N]) \equiv e(M)[x := e(N)]$.
2. $e'(M[x := N]) \equiv e'(M)[x := e'(N)]$.
3. $e(M[\alpha := \sigma]) = e(M)$.
4. $e'(M[\alpha := \sigma]) = e'(M)$.
5. *If* $M \to_{\beta} N$ *then* $e(M) \to_{\beta}^{=} e(N)$,[1]
6. *If* $M \to_{\beta} N$ *then* $e'(M) \to_{\beta} e'(N)$.

Proof. Induction on M and $\to_{\beta}$ respectively. Qed

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[1] $\to_{\beta}^{=}$ is the reflexive closure of $\to_{\beta}$.

Lemma 1 (From Church to Curry). *If* $\Sigma \rhd M : \sigma$ *then* $\Sigma \rhd_{\mathsf{C}} e(M) : \sigma$ *and* $\Sigma \rhd_{\mathsf{C}} e'(M) : \sigma$.

Proof. Induction on $\rhd$. Qed

Recall that the sentence *"* F ($\mathsf{F}^{\mathsf{Curry}}$) *is strongly normalizing"* means that whenever we have a typing $\Sigma \rhd M : \sigma$ ($\Sigma \rhd_{\mathsf{C}} M : \sigma$) the raw term M in it strongly normalizes.

Corollary 1. *The strong normalisation of* F *is inherited from that of* $\mathsf{F}^{\mathsf{Curry}}$.

Proof. Using e' in lemma 1 and by prop. 1 part 6. Qed

In other words, if $\mathsf{F}^{\mathsf{Curry}}$ is strongly normalizing then F strongly normalizes.

Lemma 2 (From Curry to Church). *If* $\Sigma \rhd_{\mathsf{C}} M : \sigma$ *then there exists a term* N *such that* $\Sigma \rhd N : \sigma$ *and* $e(N) \equiv M$.

Proof. Induction on $\rhd_{\mathsf{C}}$. Qed

In this way derivations in $\mathsf{F}^{\mathsf{Curry}}$ are represented by fully typed terms of F with respect to the given context.

Lemma 3. *If* $\Sigma \rhd M : \sigma \to \rho$ *and* $e(M) \equiv \lambda x.N$ *then there exists a term* M' *such that* $e(M') \equiv N$ *and*

$$M \to^*_{\beta_\forall} \lambda x^\sigma.M'$$

Proof. (SKETCH) Induction on the weight $w(e'(M))$ of the term $e'(M)$. $e(M) \equiv \lambda x.N$ implies that M cannot be a variable or application and $\Sigma \rhd M : \sigma \to \rho$ implies that M cannot be a universal abstraction.
If M is an abstraction, say $M \equiv \lambda x^\sigma.L$ then we are done with $M' := L$.
Now assume that M is a universal application. Therefore $M \equiv L\tau$ with $\Sigma \rhd L : \forall\alpha\theta$. This implies that L cannot be an abstraction. Moreover we have $e(L) \equiv e(M) \equiv \lambda x.N$ which implies that L is neither a variable nor an application. Therefore L is either a universal abstraction or a universal application. In the first case we have $L \equiv \Lambda\alpha.L_1$ with $\Sigma \rhd L_1 : \theta$ and we have $M \equiv L\tau \equiv (\Lambda\alpha.L_1)\tau$. In the second case we have $L \equiv L_1\tau_1$ with $\Sigma \rhd L_1 : \forall\beta_1\theta_1$ and $\forall\alpha\theta \equiv \theta_1[\beta_1 := \tau_1]$ and $M \equiv L\tau \equiv L_1\tau_1\tau$. Again L_1 can only be a universal application or a universal abstraction. In the first case we have $L_1 \equiv \Lambda\beta_1.L_2$ which implies $M \equiv (\Lambda\beta_1.L_2)\tau_1\tau$. In the

second case we have $L_1 \equiv L_2\tau_2$ and $M \equiv L_2\tau_2\tau_1\tau$. This process ends after finitely many steps, and at the end we obtain

$$M \equiv (\Lambda\gamma.L_0)\tau_0\vec{\tau}.$$

Next observe that $e(L_0[\gamma := \tau_0]\vec{\tau}) \equiv e(L_0) \equiv e(M) \equiv \lambda x.N$ and $M \equiv (\Lambda\gamma.L_0)\tau_0\vec{\tau} \to_{\beta_\forall} L_0[\gamma := \tau_0]\vec{\tau}$ which by subject reduction of F leads to $\Sigma \rhd L_0[\gamma := \tau_0]\vec{\tau} : \sigma \to \rho$.
We have $w(e'(L_0[\gamma := \tau_0]\vec{\tau})) = w(e'(L_0)) < w((\lambda y.y)e'(L_0)) = w(e'(M))$. Therefore by IH we conclude $L_0[\gamma := \tau_0]\vec{\tau} \to^*_{\beta_\forall} \lambda x^\sigma.M'$ for some M' such that $e(M') \equiv N$. This implies $M \to^*_{\beta_\forall} \lambda x^\sigma.M'$.

Qed

Lemma 4. *Let M be a term such that $\Sigma \rhd M : \rho$. If $e(M) \to_\beta N$ then there exists a term N' such that $N \equiv e(N')$ and $M \to^+_\beta N'$.*

Proof. Induction on $\rhd$.
The only interesting case is $M \equiv M_1M_2$ with $\Sigma \rhd M_1 : \sigma \to \rho, \Sigma \rhd M_2 : \sigma$ and $e(M) \equiv e(M_1)e(M_2) \to_\beta N$ where $e(M_1) \equiv \lambda x.N_1$ and $N \equiv N_1[x := e(M_2)]$. We have $\Sigma \rhd M_1 : \sigma \to \rho$ and $e(M_1) \equiv \lambda x.N_1$ therefore by lemma 3 there exists M_1' such that $e(M_1') \equiv N_1$ and $M_1 \to^*_{\beta_\forall} \lambda x^\sigma.M_1'$. Next observe that $M \equiv M_1M_2 \to^*_\beta (\lambda x^\sigma.M_1')M_2 \mapsto_\beta M_1'[x := M_2]$. Therefore $M \to^+_\beta M_1'[x := M_2]$. On the other hand $e(M_1'[x := M_2]) \equiv e(M_1')[x := e(M_2)]$ (by prop. 1, part 1) and $e(M_1')[x := e(M_2)] \equiv N_1[x := e(M_2)] \equiv N$. Hence we set $N' := M_1'[x := M_2]$, and the case is finished. Qed

Observe that this lemma is false for $\to_\eta$. Take $\Sigma := \{z : \alpha \to \forall\gamma.\beta\}, M := \lambda x^\alpha.(zx)\gamma$ and $N := z$. Then we have $\Sigma \rhd M : \alpha \to \beta$ and $e(M) \equiv \lambda x.e((zx)\gamma) \equiv \lambda x.e(zx) \equiv \lambda x.e(z)e(x) \equiv \lambda x.zx$. Therefore $e(M) \equiv \lambda x.zx \to_\eta z \equiv N$. But M is an η-normal term. Therefore there is no term N' with $M \to^+_\eta N'$.
As M is also in $\beta\eta$-normal form, the lemma is also false for $\to_{\beta\eta}$.

Corollary 2 (Subject Reduction for $\mathsf{F}^{\mathsf{Curry}}$).
If $\Sigma \rhd_{\mathsf{C}} M : \sigma$ and $M \to_\beta N$ then $\Sigma \rhd_{\mathsf{C}} N : \sigma$.

Proof. Assume $\Sigma \rhd_{\mathsf{C}} M : \sigma$ and $M \to_\beta N$. By lemma 2 there is a term M' such that $e(M') \equiv M$ and $\Sigma \rhd M' : \sigma$ and by lemma 4, as $e(M') \to_\beta N$, we obtain a term N' with $N \equiv e(N')$ and $M' \to^+_\beta N'$. $\Sigma \rhd M' : \sigma$ and $M' \to^+_\beta N'$ implies, by subject reduction of F, $\Sigma \rhd N' : \sigma$. Finally lemma 1 yields $\Sigma \rhd_{\mathsf{C}} e(N') : \sigma$. That is $\Sigma \rhd_{\mathsf{C}} N : \sigma$. Qed

Corollary 3. *The strong normalisation of* $\mathsf{F}^{\mathsf{Curry}}$ *is inherited from that of* F.

Proof. Assume $\Sigma \rhd_{\mathsf{C}} M : \sigma$ and an infinite reduction sequence

$$M \equiv M_0 \to_\beta M_1 \to_\beta \ldots \to_\beta M_i \to_\beta \ldots$$

We will construct an infinite sequence $(N_i)_{i\in\mathbb{N}}$ in F with $\Sigma \rhd N_i : \sigma$, $e(N_i) \equiv M_i$ and $N_i \to_\beta^+ N_{i+1}$ thus contradicting strong normalisation of F:

By lemma 2 we get a term N_0 such that $\Sigma \rhd N_0 : \sigma$ and $e(N_0) \equiv M_0$. Assume that N_i with $e(N_i) \equiv M_i$ and $\Sigma \rhd N_i : \sigma$ is defined. By the main hypothesis we also have $e(N_i) \to_\beta M_{i+1}$. Therefore, by lemma 4 there exists a term N_{i+1} such that $e(N_{i+1}) \equiv M_{i+1}$ and $N_i \to_\beta^+ N_{i+1}$. Finally by subject reduction of F we conclude $\Sigma \rhd N_{i+1} : \sigma$.

Qed

In [GHR93,vBLRU97] Cubes of Curry-Systems are presented and compared with Barendregt's Cube of Church-Systems. [GHR93] develops a concept of isomorphism between both classes of systems. It is proved that the four systems without dependent types are isomorphic to their respective Church-Systems. Moreover in [vBLRU97] Strong Normalisation of Dependent Type Systems is reduced to that of systems without dependencies by a function that erases dependencies. That article also presents a new cube of Curry-Systems in which every system is isomorphic to the corresponding one in Barendregt's Cube.

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References

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